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Contribution à la stabilité de Lyapunov non-régulière des inclusions différentielles avec opérateurs monotones maximaux

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Chapter 1

Introduction

1.1 General introduction

The main objective of this work is the study in Hilbert spaces setting of nonsmooth
Lyapunov functions and pairs associated to dynamical systems represented as first-
order differential inclusions of the following form

$$\dot{x}(t; x_0) \in F(x(t; x_0)), \ \ t \in [0, T), \ x(0; x_0) = x_0, \tag{1.1}$$

for appropriate initial conditions $x_0 \in H$, and different types of multifunctions
$F : H \rightrightarrows H$ defined on a real Hilbert space H. Namely, we provide primal and
dual explicit criteria for a pair of given lower semi-continuous extended real-valued
functions $V, W : H \to \mathbb{R} \cup \{+\infty\}$, and a nonnegative real number a, to be an a-
Lypunov pair associated to differential inclusion (1.1); that is,

$$e^{at} V(x(t; x_0)) + \int_0^t W(x(\tau; x_0)) d\tau \leq V(x_0),$$

for all $t \geq 0$ and all x_0 closed to points of definition of multifunction F. In this
way, our analysis allows the initial condition x_0 to be possibly a point where F is
not well-defined. The inequality above may hold for at least one solution of (1.1),
in which case the pair (V, W) is referred to as a weak a-Lyapunov pair, or for
all solutions of (1.1), and in this case we say that (V, W) is a strong a-Lyapunov
pair. The objective of this thesis fits within the main spirit of Lyapunov's non-
direct approach to the stability of differential equations, since that we provide
criteria for a-Lypunov pairs, which only depend on the involved data, represented
by F, and which do not require an apriori knowledge of the solutions. When

$W \equiv 0$ and $a \equiv 0$, one recovers the classical concept of Lyapunov functions $V : H \to \mathbb{R} \cup \{+\infty\}$, that satisfy

$$V(x(t; x_0)) \leq V(x_0) \text{ for all } t \geq 0.$$

Lyapunov functions are fundamental for the study of different stability concepts of dynamical systems, including Lyapunov stability, asymptotic stability, exponential stability and so on. From the mechanical point of view, Lyapunov functions are interpreted as energy-like functions whose decreasingness along trajectories of the systems drives the system to its equilibrium state. They are also important for control theory where they play a crucial role within the theory of Hamilton-Jacobi equations.

The main novelty of this work resides in the consideration of nonsmooth data functions, namely function V which is allowed to be nondifferentiable, even may having extended real-values. The nonsmoothness is handled by the use of general subdifferentials and deep techniques from nonsmooth and variational analysis. We also allow F to be a very general multifunction, so that to include the cases of maximal monotone operators, of Cusco (convex upper semi-continuous, nonempty compact valued) mappings, or both.

We are also concerned with the investigation of explicit criteria of the so-called invariant sets associated to differential inclusion (1.1); that is, sets $S \subset H$ such that

$$x_0 \in S \implies x(t; x_0) \in S \text{ for all } t \geq 0.$$

As for Lyapunov pairs, when the relation above is satisfied for all solutions, the set S is said to be a strong invariant set, and a weak invariant or viable when such a relation is satisfied for at least one solution of (1.1). Invariant sets and Lyapunov pairs or functions associated to general differential inclusions/equations of the form of (1.1) have been the subject of extensive research during the last decades; namely, in relation with differential inclusions involving Cusco mappings in their right-hand side (see, e.g., [11]), or (possibly unbounded) maximal monotone operators (see, e.g., [11, 14, 21, 70]).

The problem of a-Lyapunov pairs will be investigated in different setting, relying on the nature of the multifunction F governing the dynamical system in (1.1). Due to these various situations depending on the right-hand side F, the scope of this work covers different topics of analysis and optimization theory, including the theory of maximal monotone operators, differential inclusions and

equations, nonsmooth and variational analysis, and stability theory.

The need of more explicit conditions for a-Lyapunov pairs and invariant sets, depending only on the data F and the Lyapunov candidate functions and invariant candidates sets, is important for many reasons. For example, inclusion (1.1) above is sometimes evoked as a companion tool to analyze other differential inclusions, in which case the operator F may not be known explicitly, and the access to its semi-group (in the case of maximal monotone operators) can be more complicate. In [5] we investigated the existence of solutions to a differential inclusion governed by the normal cone to a prox-regular set [73], by rewriting it in the form of (1.10) with A being some intrinsic maximal monotone operator to this prox-regular set. Such an operator A is not known explicitly but it processes enough information in order to check the invariance of the involved prox-regular set with respect to (1.1). This was sufficient to get the desired existence results; see Section 4.4 of Chapter 4.

Invariant sets are also referred to, in the wide literature, as viable sets [11–13], and are of crucial use in many domains, as in economic, renewable resources, biology, diseases propagation, control processes of species and so on. Lyapunov pairs and functions are used extensively in dynamic systems and control theory, among many other applications; see, e.g., [1, 22].

Characterizations of Lyapunov pairs for the system (1.1) have been studied in the case of maximal monotone operators by Pazy [70], and next, extended to single-valued Lipschitz perturbations of maximal monotone operators by Carja and Montreanu [26], Kocan and Soravia [54, 55], Adly, Hantoute and Théra [7, 8], among many other contributions. Pazy [70] proved some sufficient criterias for Lyapunov pairs in the homogeneous case ($f \equiv 0$), by taking into account that the solution has the following explicit form:

$$x(t; x_0) = \lim_{n \to \infty} (\mathrm{I} + \lambda_n A)^{-k_n} (x_0) \tag{1.2}$$

whenever $\lambda_n k_n \to t$ as $n \to \infty$ (see [37]). Observe that since the operator A is maximal monotone, this expression makes sense, actually $(\mathrm{I} + \lambda_n A)^{-1}$ is a well-defined single-valued and Lipschitz mapping.

In the case of single-valued Lipschitz perturbations of maximal monotone operators, Carja and Motreanu [26] proved a characterization of Lyapunov pairs for (1.1) in the Banach spaces setting where A is a multi-valued m-accretive operator. The characterizations of [26] rely on the flow invariance and the contingent derivative associated to the operator A. Kocan and Soravia [54, 55]

provided another characterization using nonlinear unbounded Hamilton-Jacobi partial equations, whose viscosity solutions turn to be Lyapunov functions. These two approaches use the semigroup generated by the operator A. In Adly, Hantoute and Théra [7, 8], the authors provided a characterization which does not involve the semi-group generated by A. The case of Cusco mappings was treated for example in [30–32], where the authors use Euler approximations to provide criteria for strong and weak invariance in terms of the associated Hamiltonian. All these results will be reviewed at the end of this chapter.

1.1.1 An overview of Lyapunov methods

This work goes in the spirit of the nondirect Lyapouv's method to approach stability problems of complex dynamical systems, whose solutions are not easily accessible or that the associated calculation are expensive way. Roughly speaking, in front of lack of explicit information on the solutions of (1.1), the original Lyapunov's idea to check wether a given dynamical system is stable, consists of looking for an associated nonnegative real-valued function V, hopefully regular, which is "strongly" continuous in the sense that $x \to \theta$ iff $V(x) \to 0$, and such for each trajectory $x(\cdot; x_0)$ of (1.1)

$$t \mapsto V(x(t; x_0)) \text{ is non-increasing.} \tag{1.3}$$

The existence of such a function easily ensures the stability of the system at its equilibrium point θ; that is, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $y \in \mathrm{B}(\theta, \delta)$ and every solution $x(\cdot; y)$ of (1.1), we have that $x(t; y) \in \mathrm{B}(\theta, \varepsilon)$ for all $t \geq 0$. Since we do not dispose of explicit calculus of the solutions of (1.1), a natural question is then to find accurate criteria depending only on V, and which guarantee the validity of relation (1.3).

Suppose for instance that our setting is finite-dimensional, $H = \mathbb{R}^n$, and that (1.1) is a classical differential equation; that is, $F = f$ with f being continuous. If V is smooth, it can be easily verified that the following condition written by means of the derivative of V, V', or equivalently, the gradient of V,

$$\langle V'(x), f(x) \rangle = \langle \nabla V(x), f(x) \rangle \leq 0, \quad \text{for all } x \in \mathbb{R}^n, \tag{1.4}$$

leads to relation (1.3). Indeed, in this case, function $x(\cdot; x_0)$ is C^1 and by the classical chain rule we obtain that

$$\frac{d}{dt} V(x(t; x_0)) = \langle \nabla V(x(t; x_0)), f(x(t; x_0)) \rangle \leq 0,$$

which means that $V(x(\cdot; x_0))$ is non-increasing.

By arguing similarly as above, the use of a-Lyapunov's pairs instead of functions allows analyzing other concepts of stability as asymptotic and exponential stability.

Now, to extend this analysis to differential inclusion (1.1) one has to handle the following difficulties:

- The solution $x(\cdot; x_0)$ of (1.1) may not be sufficiently smooth; for instance, it is generally only continuous when $F = -A$, with A a maximal monotone

operator, and x_0 in the closure of the domain of A. In this case, the solutions of (1.1) are understood in the weak sense. Even under existence of strong solutions that are absolutely continuous, all what can be expected is that they are differentiable almost everywhere.

- The domain of F does not need to be closed, nor the values of F are necessarily bounded or even nonempty. This makes the scope of the equation above going beyond the differential inclusions treated in [8, 7, 13, 14, 16], where the right-hand side is generally represented by a Cusco set-valued mapping (in particular, with nonempty and weak*-compact multi-valued operator). The monotonicity assumption of F will compensate the lack of compacity in our differential inclusion (1.1), while the maximality of this operator guarantees, among other properties, the existence and the regularity of solutions. These two facts are also essential when checking the invariance of closed sets.

- The Lyapunov candidate functions are generally only lower semi-continuous, while the solution $x(\cdot; x_0)$ is only continuous, or at most absolutely continuous. Thus, one need to use tools of nonsmooth analysis, like general subdifferentials, calculus subdiferential rules, mean value theorems and so on, in order to provide criteria written by means of first-order approximations of the Lyapunov functions and invariant sets.

In this thesis, we follow the last ideas above and try to find characterizations in the line of (1.4) when V is only an extended-real-valued lower semi-continuous function. The main tools that allow us to overcome the difficulties listed above come from general subdifferentials theory and techniques of variational analysis.

1.1.2 Our contributions

Our contributions are listed below:

A- Lyapunov stability of differential inclusions with Lipschitz perturbations of maximal monotone operators

We study differential inclusion (1.1) when the right-hand side F is given by

$$F = f - A,$$

with $A : H \rightrightarrows H$ being a maximal monotone operator defined on the real Hilbert space H, and $f : H \to H$ a Lipschitz perturbation. Hence, the right-hand

side may be empty, non-compact or even unbounded, and possibly non upper-semi-continuous. Also, the initial condition x_0 may be not in the domain of definition of operator A, giving rise to more general concept of solutions called weak solutions. This model includes and covers many typical partial differential equations, as well as control problems dealing with differential equations of the form $\dot{x}(t; x_0) = f(x(t; x_0))$. A typical example of maximal monotone operators is the Fenchel subdifferential mapping of lower semi-continuous convex proper functions, namely, the normal cone to closed convex sets. However, it is known that there are maximal monotone operators which are not necessarily of subdifferential's type. The problem of the existence of solutions was completely solved since the sixties (see, e.g. [11, 14, 21, 70]) and many important results have been done, regarding the regularity of solutions. We investigate in this thesis different primal and dual criteria for closed invariant sets and lower semi-continuous (extended real-valued) a-Lyapunov pairs with respect to differential inclusion (5.7), now written as

$$\dot{x}(t; x_0) \in f(x(t; x_0)) - A(x(t; x_0)), \;\; t \in [0, T), \;\; x(0; x_0) = x_0 \in \overline{\mathrm{dom}A}. \quad (1.5)$$

We provide the following sharp criteria for the associated invariant closed sets $S \subset \overline{\mathrm{dom}A \cap S}$, using proximal normal cones: for every $x \in S \cap \mathrm{dom}A$ there exists a large enough $m > 0$ such that for all $y \in S$ closed to x it holds

$$\sup_{\xi \in \mathrm{N}^P_{S_m}(y)} \min_{y^* \in A(y) \cap \mathrm{B}(\theta, m)} \langle \xi, f(y) - y^* \rangle \leq 0, \quad (1.6)$$

where S_m is some appropriate subset of S, which reflects in some sense the farness of the values of A from the origin θ. This criterion leads to more explicit invariance criteria in many natural situations. For instance, we have proved that when the minimal norm mapping A° is locally bounded, each one of the following conditions (i)-(ii)-(iii), provides a characterization for a closed set $S \subset \overline{\mathrm{dom}A}$ to be invariant for (1.5):

(i) For every $x \in S \cap \mathrm{dom}A$

$$f(x) - \Pi_{A(x)}(f(x)) \in \mathrm{T}^B_S(x),$$

where $\mathrm{T}^B_S(x)$ is the Bouligand tangent cone and $\Pi_{A(x)}$ is the orthogonal projection onto $A(x)$; in particular, when A is a normal cone mapping, $A = \mathrm{N}_C$ for some closed convex set $C \subset H$, this last relation is also equivalent to $\Pi_{\mathrm{T}_C(x)}(f(x)) \in \mathrm{T}^B_S(x)$,

where $\mathrm{T}_C(x)$ is the tangent cone in the sense of convex analysis. These conditions have a clear geometrical meaning (see the figure below)

$$\dot{x}(t) \in x(t) - \mathrm{N}_{B(\theta,r)}(x(t)) \text{ a.e. } t \geq 0$$

S is invariant set
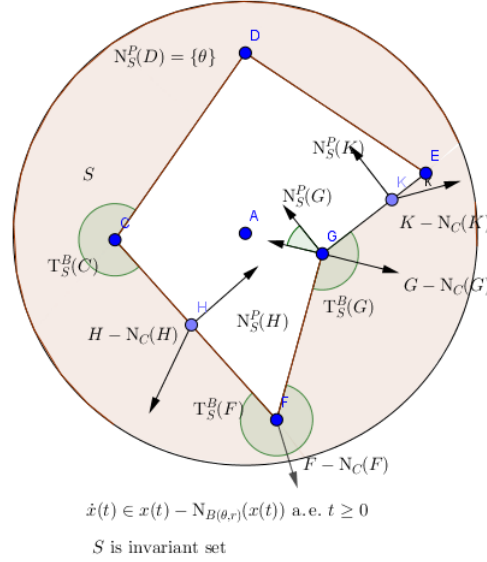
(ii) For every $x \in S \cap \mathrm{dom}A$

$$\sup_{\xi \in \mathrm{N}_S(x)} \langle \xi, f(x) - \Pi_{A(x)}(f(x)) \rangle \leq 0,$$

where N_S stands for either the proximal normal cone N_S^P or the Fréchet normal cone N_S^F. It is worth observing that when S is a closed convex set, this last condition reads $f(x) - \Pi_{A(x)}(f(x)) \in (\mathrm{N}_S(x))^\circ = \mathrm{T}_S(x)$, and one goes back to condition (i) above. However, this argument cannot be extended outside convex sets since that the relation $(\mathrm{N}_S(x))^\circ = \mathrm{T}_S(x)$ is not true in general for closed sets which are not convex. This is to say that the last condition is meaningful and may have an interpretation which differs from the one in (i); however, both are equivalent.

(iii) For every $x \in S \cap \mathrm{dom}A$

$$\sup_{\xi \in \mathrm{N}_S(x)} \inf_{x^* \in f(x) - A(x)} \langle \xi, x^* \rangle \leq 0.$$

This condition is very practical since it only appeals to the values of the data, which are the mapping f and the operator A, and thus no projection is needed.

The main feature of criteria (i)-(ii)-(iii) above is that they only involve the position of the set S regarding the values of A and f, as the figure above shows.

The generality of our setting, dealing with general lower semi-continuous extended-real-valued functions, allows us to make an exact correspondence between

a-Lyapunov pairs and invariant sets. For instance, it is not difficult to verify that a lower semi-continuous function $V : H \rightarrow \mathbb{R} \cup \{+\infty\}$ is Lyapunov for (1.5) if and only if the epigraph of V is invariant with respect to the following augmented differential inclusion given in $H \times \mathbb{R}$,

$$(\dot{x}(t; x_0), \dot{\alpha}(t; \alpha_0)) \in (f(x(t; x_0)) - A(x(t; x_0)), 0), \; \; t \in [0, T].$$

Hence, then invariance criterion above is naturally rewritten into a criterion for a-Lyapunov pairs in the following form: a pair (V, W) of two proper lower semi-continuous functions forms an a-Lyapunov pair for differential inclusion (1.5) whenever for every $x \in \text{dom}V \cap \text{dom}A$ there exists a large enough $m > 0$ such that for all y closed to x we have that

$$\sup_{\xi \in \partial_P (V + \text{I}_{A_m})(y)} \inf_{y^* \in A(y) \cap \text{B}(\theta, m)} \langle \xi, f(y) - y^* \rangle + aV(x) + W(x) \leq 0,$$

where the set A_m plays a similar role as the set S_m above. Similarly, when A is locally bounded, we prove that (V, W) forms an a-Lyapunov pair with respect to system (1.5) if and only if one of the following conditions is satisfied:

(i') For any $x \in \text{dom}V \cap \text{dom}A$

$$\sup_{\xi \in \partial V(x)} \langle \xi, (f(x) - A(x))^\circ \rangle + aV(x) + W(x) \leq 0,$$

where ∂ stands for either the proximal subdifferential ∂_P or the Fréchet subdifferential ∂_F.

(ii') For any $x \in \text{dom}V \cap \text{dom}A$

$$\sup_{\xi \in \partial V(x)} \inf_{x^* \in A(x)} \langle \xi, f(x) - x^* \rangle + aV(x) + W(x) \leq 0.$$

(iii') For any $x \in \text{dom}V \cap \text{dom}A$

$$\inf_{v \in A(x)} V'(x; f(x) - v) + aV(x) + W(x) \leq 0.$$

Let us observe that, according to theses equivalences, when looking for complete characterizations of Lyapunov pairs it doesn't matter to consider either the proximal or the Fréchet subdifferentials. However, when verifying the validity of conditions (i') or (ii'), it is more natural in practice to check the inequalities in (i') and (ii') only for the proximal subdifferential, which is in general smaller than

the Fréchet subdifferential. For instance, we know that for differentiable functions in finite dimensions we always have that $\partial_F V = \nabla V$, while $\partial_P V$ may be empty at some differentiability points of V even for C^1 functions. We also observe that for differentiable function V, each one of the relations above is equivalent to

$$\inf_{x^* \in A(x)} \langle \nabla V(x), f(x) - x^* \rangle + aV(x) + W(x) \leq 0.$$

However, as we have just commented above, it is enough to verify this last inequality only for points x where $\partial_P V(x)$ is not empty. It is then clear that the main advantage in using the proximal mapping comes from being the smallest one among well-known subdifferentials.

Condition (ii') is a kind of Hamilton-Jacobi inequality. Let us denote

$$h(x, p) := \inf_{x^* \in A(x)} \langle p, f(x) - x^* \rangle,$$

so that condition (ii') is written as

$$h(x, \partial V(x)) := \sup_{p \in \partial V(x)} h(x, p) \leq -aV(x) - W(x),$$

and V is seen as a lower solution of the Hamilton-Jacobi inequality ([30])

$$h(x, \partial V(x)) \leq -aV(x) - W(x).$$

As with the invariance criteria above, only the data of system (1.5) are used. When the minimal norm mapping A is not necessarily locally bounded, one needs to consider in the criteria above the singular subddifferential of V, $\partial_\infty V$ (see [7, 8]).

The results above extend the ones in [7, 8] by removing the assumption of the weak closedness of the candidate invariant sets. Only the data of the system represented by A and f are appealed to within the presented criteria and, so, no need to solve explicitly the differential inclusion (1.1). The invariant results of this work are then rewritten as criteria for lower semi-continuous a-Lyapunov pairs which are non-necessarily weakly lower semi-continuous functions. Because the sets we consider are not necessarily convex or smooth, and the candidate Lyapunov functions are not necessarily sufficiently regular, we use techniques from nonsmooth analysis (e.g. [30, 64, 76]), including general subdifferentials. The main invariance criterion is given by means of the normal cone to the nominal set. Other invariance results are given by means of primal and dual conditions. These results are next

applied to obtain criteria for a-Lyapunov pairs associated to differential inclusion (1.5), including primal conditions using the directional derivatives of the Lyapunov candidate functions, and dual ones using general subdifferentials of such functions like the Proximal, the Fréchet, the singular, and the limiting subdifferentials. The result of this part are presented in Chapter 3.

B. Lyapunov stability of differential inclusions with prox-regular sets

We study the case when $F = f - N_C$, so that differential inclusion (1.1) takes the form

$$\dot{x}(t; x_0) \in f(x(t; x_0)) - N_C(x(t; x_0)) \text{ a.e. } t \geq 0, \ x(0; x_0) = x_0 \in C, \qquad (1.7)$$

where C is a uniformly prox-regular set and $f : H \to H$ is a Lipschitz mapping. Here, N_C refers to the proximal normal cone to the set C. In general, the set C may depend on the time parameter, in which case the underlying system is refered to as a sweeping process, a name coined in the sixties by Moreau who studied the convex case ($C(t)$ convex) and applied it in mathematical models of elastoplacity mechanical problems. Differential inclusion (1.7) appears in the modeling of many concrete problems in economics, unilateral mechanics, electrical engineering as well as optimal control (see, eg., [1, 33, 62, 82] and references therein). The model above is also used as a companion system for differential equations of the form $\dot{x} = f(x)$, for which C is not necessarily invariant. In this case, system (1.7) above is a reasonable approximation of this differential equation, since the corresponding solution naturally remains in the set C ([35]). The family of uniformly prox-regular sets contains and is larger than the family of convex set; for example, the union of two disjoint convex sets is uniformly prox-regular, but obviously is not necessarily convex. Also, the graph of C^2-functions are prox-regular sets ([19]).

In the current thesis, we restrict ourselves to time-independent constraint sets, in order to provide a new and natural approach to prove existence of solutions. We also establish new criteria for the associated a-Lyapunov pairs. This model inherits the main difficulties of differential inclusion (1.7), namely, the right-hand side is naturally unbounded, and may even be empty (at points outside the set C). As well, the multimapping $-N_C$ is not upper semi-continuous in general.

Existence of solutions of (1.7) are known to occur for general uniformly prox-regular time-depending sets; indeed, in finite-dimension, (1.7) has solutions without any regularity assumption on C ([17, 18]). However, the methods used in the literature for this prox-regular setting are very similar to the convex one,

originated by Moreau for convex sweeping processes. In our case, we shall follow a different and direct approach, which could use the results already known in the convex case. For this aim, we transform (1.7) into a differential inclusion of the form (1.5), so that we can use and apply invariance and a-Lyapunov criteria established for (1.5) to investigate the existence and properties of solutions, and to give explicit criteria for the invariance and a-Lyapunov pairs associated to (1.7). Then we prove in this case that a closed subset S of C is invariant with respect to system (1.7) if and only if one of the following conditions is satisfied:

(i) For every $x \in S$

$$(f(x) - \mathrm{N}_C(x))^\circ \in \mathrm{T}_S^B(x).$$

(ii) For every $x \in S$

$$[f(x) - \mathrm{N}_C(x)] \cap \mathrm{T}_S^B(x) \cap \mathrm{B}(\theta, \|f(x)\|) \neq \emptyset.$$

(iii) For every $x \in S$

$$\inf_{x^* \in [f(x) - \mathrm{N}_C(x)] \cap \mathrm{B}(\theta, \|f(x)\|)} \langle \xi, x^* \rangle \leq 0.$$

Similarly, we also have the following primal and dual characterizations for a-Lyapunov pairs of proper lower semi-continuous functions real-extended-valued functions V, W, associated to differential inclusion (1.7):

(i) For every $x \in \mathrm{dom}V$ and $\xi \in \partial V(x)$

$$\langle \xi, (f(x) - \mathrm{N}_C(x))^\circ \rangle + aV(x) + W(x) \leq 0.$$

(ii) For every $x \in \mathrm{dom}V$ and $\xi \in \partial V(x)$

$$\min_{x^* \in \mathrm{N}_C(x) \cap \mathrm{B}(\theta, \|f(x)\|)} \langle \xi, f(x) - x^* \rangle + aV(x) + W(x) \leq 0.$$

(iii) For every $x \in \mathrm{dom}V$

$$\inf_{x^* \in \mathrm{N}_C(x) \cap \mathrm{B}(\theta, \|f(x)\|)} V'(x; f(x) - x^*) + aV(x) + W(x) \leq 0.$$

These results are next applied to study the stability and observers design of Lur'e systems involving non-monotone set-valued nonlinearities.

C. Lyapunov stability of differential inclusions with Lipschitz Cusco perturbations of maximal monotone operators

In this part, we study differential inclusion (1.1) when

$$F = f - A,$$

with $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ being a maximal monotone operator defined on $\mathbb{R}^n$, and $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a Lipschitz multifunction. The study of the Lyapunov stability of this problem by means of criteria in the form given in paragraphs above **A** and **B**, has not been adressed before in the current generality. This is why we restrict ourselves in this work to the finite-dimensional setting. Compared to paragraphs **A** and **B** above, we face here the problem of having multiple solutions, and this leads us to consider weak and strong invariant sets, as well as weak and strong a-Lyapunov pairs for our differential inclusion (1.1), which takes the form

$$\dot{x}(t; x_0) \in f(x(t; x_0)) - A(x(t; x_0)) \text{ a.e. } t \geq 0, \, x(0; x_0) = x_0 \in \overline{\mathrm{dom}A}. \quad (1.8)$$

In this case, we show that a set $S \subset \mathrm{dom}A$ is strong invariant for this differential inclusion if and only if one of the following conditions holds:

(i) For every $x \in S \cap \mathrm{dom}A$

$$v - \Pi_{A(x)}(v) \in \mathrm{T}_S^B(x) \; \forall v \in F(x).$$

(ii) For every $x \in S \cap \mathrm{dom}A$

$$[v - A(x)] \cap \mathrm{T}_S^B(x) \neq \emptyset \; \forall v \in F(x).$$

(iii) For every $x \in S \cap \mathrm{dom}A$

$$\sup_{\xi \in \mathrm{N}_S(x)} \sup_{v \in F(x)} \langle \xi, v - \Pi_{A(x)}(v) \rangle \leq 0.$$

(iv) For every $x \in S \cap \mathrm{dom}A$

$$\sup_{\xi \in \mathrm{N}_S(x)} \sup_{v \in F(x)} \inf_{x^* \in A(x)} \langle \xi, v - x^* \rangle \leq 0.$$

It is worth obersving that we do not assume here that the minimal section A° is locally bounded. On the other hand, a closed $S \subset \mathrm{dom}A$ such that A° is locally bounded on S, is weak invariant for differential inclusion (1.8) if and only if one of the following conditions holds:

(i) For every $x \in S$, there exist $v \in F(x), x^* \in A(x) \cap \mathrm{B}(\theta, \|F(x)\| + \mathfrak{m}(x))$

such that

$$v - x^* \in \mathrm{T}_S^B(x).$$

(*ii*) For every $x \in S$

$$\sup_{\xi \in \mathrm{N}_S(x)} \; \inf_{v \in F(x)} \; \inf_{x^* \in A(x) \cap \mathrm{B}(\theta, \|F(x)\| + \mathfrak{m}(x))} \langle \xi, v - x^* \rangle \leq 0.$$

where $\mathfrak{m}(x) := \limsup_{y \to x, y \in S} \|A^\circ(y)\|$.

Concerning Lyapunov pairs, we obtain that a pair (V, W) of two proper lower semi-continuous extended-real-valued functions such that $\mathrm{dom}V \subset \mathrm{dom}A$, forms a strong a-Lyapunov pair for differential inclusion (1.8) if and only if for every $x \in \mathrm{dom}V$

$$\sup_{\xi \in \partial_P V(x)} \; \sup_{v \in F(x)} \; \inf x^* \in A(x) \langle \xi, v - x^* \rangle + aV(x) + W(x) \leq 0$$

and

$$\sup_{\xi \in \partial_{P,\infty} V(x)} \; \sup_{v \in F(x)} \; \inf_{x^* \in A(x)} \langle \xi, v - x^* \rangle \leq 0.$$

In this case, we need to consider the singular proximal subdifferential of function V, $\partial_{P,\infty} V$. However, if A° is locally bounded on $\mathrm{dom}V$, then (V, W) forms a weak a-Lyapunov pair for differential inclusion (1.8) if and only if one of following hold

(*i*) For any $x \in \mathrm{dom}V$

$$\sup_{\xi \in \partial V(x)} \; \inf_{v \in F(x)} \; \inf_{x^* \in A(x) \cap \mathrm{B}(\theta, \mathfrak{m}(x))} \langle \xi, v - x^* \rangle + aV(x) + W(x) \leq 0,$$

where ∂ stands for either ∂_P, ∂_F, or ∂_L.

(*ii*) For any $x \in \mathrm{dom}V$

$$\inf_{v \in F(x)} \; \inf_{x^* \in A(x) \cap \mathrm{B}(\theta, \|F(x)\| + \mathfrak{m}(x))} V'(x; v - x^*) + aV(x) + W(x) \leq 0.$$

This result is also applied to systems involving uniformly prox-regular sets, given in the form

$$\dot{x}(t; x_0) \in f(x(t; x_0)) - \mathrm{N}_C(x(t; x_0)) \text{ a.e. } t \geq 0, \; x(0; x_0) = x_0 \in C, \qquad (1.9)$$

with f being a Lipschitz Cusco multifunction. We show that for $V, W : H \to \overline{\overline{\mathbb{R}}}$ as above, such that $\mathrm{dom}V \subset C$, form a strong a-Lyapunov pair $(a \geq 0)$ for differential inclusion (1.9) if and only if one of the following conditions holds:

(i) for every $x \in \mathrm{dom}V$

$$\sup_{\xi \in \partial V(x)} \sup_{v \in F(x)} \langle \xi, v - \Pi_{\mathrm{N}_C(x)}(v) \rangle + aV(x) + W(x) \le 0.$$

(ii) for every $x \in \mathrm{dom}V$

$$\sup_{\xi \in \partial V(x)} \sup_{v \in F(x)} \inf_{x^* \in \mathrm{N}_C(x) \cap \mathrm{B}(\theta, \|F(x)\|)} \langle \xi, v - x^* \rangle + aV(x) + W(x) \le 0;$$

(iii) for every $x \in \mathrm{dom}V$

$$\sup_{v \in F(x)} V'(x; v - \Pi_{\mathrm{N}_C(x)}(v)) + aV(x) + W(x) \le 0;$$

(iv) for every $x \in \mathrm{dom}V$

$$\sup_{v \in F(x)} \inf_{x^* \in \mathrm{N}_C(x) \cap \mathrm{B}(\theta, \|F(x)\|)} V'(x; v - x^*) + aV(x) + W(x) \le 0.$$

D. Application to the geometry of maximal monotone operators: In this part, we characterize the boundary of the values of maximal monotone operators defined in Hilbert spaces, by means only of the values at nearby points, which are closed enough to the reference point but distinct of it. This allows to write the values of such operators using finite convex (2-)combinations of the values at such nearby points. We also provide similar characterizations for the normal cone to prox-regular sets.

For instance, given a maximal monotone operator $A : H \rightrightarrows H$, defined on a Hilbert space H, for every $x \in H$ we have

$$\mathrm{bd}(A(x)) = \mathrm{Limsup}_{y \to_{\neq} x} \mathrm{bd}(A(y)) = \mathrm{Limsup}_{y \to_{\neq} x} A(y),$$

and, consequently, for every $x \in \mathrm{dom}A$ such that $\mathrm{bd}(A(x)) \neq \emptyset$ we obtain

$$A(x) = \mathrm{N}_{\mathrm{cl}(\mathrm{dom}A)}(x) + \mathrm{co}_2 \left\{ \mathrm{Limsup}_{y \to_{\neq} x} A(y) \right\}.$$

1.2   Previous results from the literature

First results dealing with Lyapunov pairs and functions associated to (1.5) have been established by Pazy in [68, 70] in the case of homogeneous systems governed

by maximal monotone operators; that is, $F \equiv -A$. The motivations of Pazy came from the investigation of some regularity properties of partial differential equations. Pazy's criteria for Lyapunov ($a = 0$) pairs are given by means of directional-like derivatives of the candidate functions, using the Moreau-Yoshida approximation of operator A. Namely, (V, W) is a Lyapunov pair for (1.1) if the following relation holds uniformly on bounded sets of dom$A$,

$$\limsup_{\lambda \downarrow 0} \frac{V(J_\lambda(x)) - V(x)}{\lambda} + W(x) \leq 0,$$

where $J_\lambda := (I + \lambda A)^{-1}$ is the resolvent operator A.

Pazy's results have been extended by Kocan and Soravia [54] (see, also, [26]) to the non-homogeneous case; that is, when (1.1) is written as

$$\dot{x}(t; x_0) \in f(x(t; x_0)) - A(x(t; x_0)) \ \ t \in [0, T], \ x(0; x_0) = x_0 \in H, \qquad (1.10)$$

with f being a Lipschitz mapping. The approach of [26, 54] uses implicit criteria depending heavily on the semi-group generated by the maximal monotone operator A. For instance, Kocan and Soravia show that (V, W) is a Lyapunov pair for differential inclusion (1.10) if and only if (V, W) is a solution in the viscosity sense of the following differential inequality

$$\langle A(x), DV(x) \rangle - \langle f(x), DV(x) \rangle \geq W(x).$$

Recently, always dealing with Lipschitz perturbations of maximal monotone operators, different criteria for weak lower semi-continuous a-Lyapunov pairs have been investigated in [7, 8], using the condition that for any x in the domain of V (which is a subset of the closure of the domain of A, dom$A$) and $\delta > 0$ it holds

$$\sup_{\xi \in \partial_P V(x)} \liminf_{y \stackrel{\mathrm{dom}A}{\longrightarrow} x} \inf_{y^* \in A(y)} \langle \xi + \delta(y - x), f(y) - y^* \rangle + aV(x) + W(x) \leq 0,$$

$$\sup_{\xi \in \partial_{P,\infty} V(x)} \liminf_{y \stackrel{\mathrm{dom}A}{\longrightarrow} x} \inf_{y^* \in A(y)} \langle \xi + y - x, f(y) - y^* \rangle \leq 0,$$

where ∂_P and $\partial_{P,\infty}$ are the proximal and the proximal singular subdifferential operators, respectively.

More early at the beginning of the twentieth century, in his the pioneering work [47, 58], Lyapunov studied stability properties of linear systems that he extended to nonlinear differential equations. These results are known as the first and the second

methods of Lyapunov. Since then, this approach continues to be fundamental in the study of dynamical systems from both the theoritecal and applicable points of view. There are other famous results like the ones of Zubov and Krasovskii on the stability of differential equations; i.e., $F = f$, and which use Lyapunov functions. Lyapunov himself also investigated instability of differential equations (see [47, 56, 58, 60]).

We may distinguish in this study two main important types of differential inclusion (1.1), the one involving maximal monotone operators, and the other one governed by upper semi-continuous multifunctions F with nonempty, bounded (or compact), and closed values, which are refereed to as Cusco mappings. We shall also consider in some cases, specially in finite-dimensions, coupled systems covering both situations.

In the case of Cusco multifunctions F defined on $\mathbb{R}^n$, considered as the natural extension of classical differential equations, under some standard linear growth hypothesis, differential inclusion (5.7) has solutions (may be not unique) [11, 42]. In this case, (strong and weak) invariant sets and Lyapunov pairs associated to (5.7) have been studied in Clarke *et all* [30] (and references therein), using Euler-like approximations that in the finite-dimensional setting lead to the required solution. It is worth observing that strong invariant sets and strong Lyapunov pairs require in [30] Lipschitz assumptions on multifunction F. Donchev, Ríos, Wolenski [39] extended the strong invariant results to the class of one-side Lipschitz time-dependent multifunctions, a family which is less restrictive than the class of Lipschitz multifunctions.
Colombo, Palladino [33] provided similar results to [30] for classes of time-depend multifunctions of the form $F(t, x) = G(t, x) - \mathrm{N}_{C(t)}(x)$, where G is Lipschitz with respect to the second variable x, and $C(t)$ is a uniformly prox-regular set with Lipschitz dependence on t. This class of problems is called Sweeping process. This problem has been introduced and studied by Moreau [65]. More precisely, Moreau studied the existence of solution of the differential inclusion which has the form

$$\dot{x}(t) \in -\mathrm{N}_{C(t)}(x(t)) \ \ t \geq 0, \ x(0) = x_0 \in C(0), \tag{1.11}$$

where $C(t)$ is closed convex set in a Hilbert space. This problem has been developed by Castaing and his collaborators to study the following differential inclusion

$$\dot{x}(t) \in -\mathrm{N}_{C(t)}(x(t)) + F(t, x(t)) \ \ t \geq 0, \ x(0) = x_0 \in C(0), \tag{1.12}$$

where all the sets $C(t)$ are either convex or complements of open convex sets (in finite-dimensional settings). Differential inclusion (1.12) was studied by Mazade, Thibault [62, 63] for the case when $C(t) = C$ is a uniformly prox-regular subset of a Hilbert space, while in Adly, Haddad and Thibault [3], the authors consider the case when $C(t)$ are closed convex sets. These last papers only studied the existence of solutions, but recently, Mazade and Hantoute [61] have given characterizations for Lyapunov pairs (V, W) (with V being weak lower semi-continuous) with respect to (1.12), for the cases $F = -\mathrm{N}_C + f$, where C is uniformly prox-regular subset of a Hilbert space.

Chapter 2

Notation and main concepts

Throughout the thesis, we frequently work in a real Hilbert space H which is endowed with the inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|$, and identified to its dual space H'.

We use the notations $\to, \rightharpoonup$ to the denote strong and weak convergence, respectively, and write $\overset{S}{\to}, \overset{S}{\rightharpoonup}$ if the convergence is restricted to a set $S \subset H$. We denote by $\mathrm{B}(x, r)$ the closed ball in H with center x and radius r, in particular, we denote $\mathbb{B} = \mathrm{B}(\theta, 1)$ and by B_r the ball $\mathrm{B}(\theta, r)$.

Given a subset S of H, we denote by

$$\overline{S} \,(\text{ or } \mathrm{cl}(S)),\ \mathrm{co}(S),\ \overline{\mathrm{co}}(S),\ \mathrm{cone}(S),\ \mathrm{int}(S),\ \mathrm{bd}(S),$$

the closure, the convex hull, the closed convex hull, the conic hull, the interior, and the boundary of the set S, respectively. S° denotes the set of points of *minimal norm* in S, i.e.,

$$S^\circ := \{x \in S \mid \|x\| \leq \|s\|, \text{ for all } s \in S\}.$$

Suppose that K is a convex, closed subset of H. We denote

$$(S - K)^\circ := \{(s - K)^\circ \mid s \in S\}.$$

The dual cone set of S is the set

$$S^* := \{x^* \in H \mid \langle x^*, x \rangle \leq 0 \text{ for all } x \in S\}.$$

The *indicator function* and the *distance function* are respectively given by

$$\mathrm{I}_S(x) := 0 \text{ if } x \in S;\ +\infty \text{ otherwise, and } d_S(x) := \inf\{\|x - y\| \mid y \in S\}.$$

The *support function* of a non-empty set S is given by

$$\sigma_S(x) := \sup\{\langle x, s \rangle \mid s \in S\}.$$

For $\delta \geq 0$, we denote Π_S^δ the (*orthogonal*) *δ-projection mapping* onto S defined as

$$\Pi_S^\delta(x) := \{y \in S : \|x - y\|^2 \leq d_S^2(x) + \delta^2\}.$$

When $\delta = 0$, we simply write $\Pi_S^0(x) =: \Pi_S(x)$. It is known that Π_S is nonempty-valued on a dense subset of $H \setminus S$ (see [30]). We have the following theorem:

Theorem 2.1. *[32, 74] Suppose that S is closed. Then, for any $x \in H$ and any $s \in \Pi_S^\delta(x)$, with $\delta > 0$, there exist $s_\delta \in S$ and $y \in H$ such that*

$$\begin{cases} y - s_\delta \in \mathrm{N}_S^P(s_\delta), \\ \|y - s_\delta - (x - s)\| \leq 2\delta, \\ \|s - s_\delta\| \leq \delta, \ \|x - y\| \leq \delta. \end{cases} \tag{2.1}$$

In addition, if $x \in \mathrm{B}(x_0, \sigma)$ for some $x_0 \in S$ and $\sigma > 0$, then s_δ satisfies

$$\|s_\delta - x_0\|^2 \leq 6\sigma^2 + 8\delta^2. \tag{2.2}$$

Proof. We can find the proof of the first part of the theorem in [32, 74]. We now suppose that $x \in \mathrm{B}(x_0, \sigma)$. One has

$$\begin{aligned} \|s_\delta - x_0\|^2 &\leq 2\|s_\delta - x\|^2 + 2\|x - x_0\|^2 \\ &\leq 4\|s_\delta - s\|^2 + 4\|x - s\|^2 + 2\sigma^2 \\ &\leq 4\delta^2 + 4(d_S^2(x) + \delta^2) + 2\sigma^2 \\ &\leq 8\delta^2 + \delta^2 + 4\|x - x_0\|^2 + 2\sigma^2 \\ &\leq 8\delta^2 + 6\sigma^2, \end{aligned}$$

which completes the proof of the theorem. $\square$

Finally, let a function $\varphi : H \to \overline{\overline{\mathbb{R}}} := \mathbb{R} \cup \{+\infty\}$ be given. The *domain* of φ is

$$\mathrm{dom}\varphi := \{x \in H \mid \varphi(x) < +\infty\},$$

and the *epigraph* of φ is

$$\mathrm{epi}\varphi := \{(x, \alpha) \in H \times \mathbb{R} \mid \varphi(x) \leq \alpha\}.$$

The function φ is *proper* if $\text{dom}\varphi \neq \emptyset$.

Definition 2.2. Let $\varphi : H \to \overline{\mathbb{R}}$ and let $x \in \text{dom}\varphi$. The function φ is said to be lower semi-continuous at x if

$$\varphi(x) \leq \liminf_{y \to x} \varphi(y).$$

We say that φ is lower semi-continuous on H when it is lower semi-continuous at every point in H. We denote by $\mathcal{F}(H)$ the set of lower semi-continuous function on H.

The function φ is lower semi-continuous iff $\text{epi}\varphi$ is closed in $H \times \mathbb{R}$.

Example 2.3. The function $\text{I}_S(\cdot)$ is lower semi-continuous if and only if S is a closed set.

2.1 Basic definitions and some properties of nonsmooth analysis

In this section, we remind some basic concepts of nonsmooth analysis theory and the theory of solutions of differential inclusions which play an important role in our work.

First, let us remind some concepts and properties of convex analysis.

Definition 2.4.   (i) A subset S is called convex if for any two points $x, y \in S$ and any $\alpha \in [0, 1]$, one has $\alpha x + (1 - \alpha)y \in S$.

(ii) A function $\varphi : \text{H} \to \overline{\mathbb{R}}$ is called convex if for any $x, y \in \text{dom}\varphi$ and any $\alpha \in [0, 1]$, one has

$$\varphi(\alpha x + (1 - \alpha)y) \leq \alpha\varphi(x) + (1 - \alpha)\varphi(y)$$

(with $0.\infty = \infty$).

Example 2.5.   (i) The closed balls $\text{B}(x_0, r), r \geq 0$ are convex.

(ii) The function $V(x) := \|x\|^p$ is convex whenever $p \geq 1$.

(iii) Suppose that $\varphi \in \mathcal{F}(H)$. Then the conjugate function φ^* of φ which is defined by

$$\varphi^*(x) = \sup_{y \in H}\{\langle x, y \rangle - \varphi(y)\}$$

is convex.

We have

$$\mathrm{I}_S^* = \sigma_S, \ \sigma_S^* = \mathrm{I}_{\overline{\mathrm{co}}(S)}, \ (\frac{1}{2}\|x\|^2)^* = \frac{1}{2}\|x\|^2.$$

(iv) A subset S is convex if and only if the function $\mathrm{I}_S(\cdot)$ is convex.

Next, we remind some basic concepts in nonsmooth analysis

Definition 2.6. Let $\varphi \in \mathcal{F}(H)$ and let $x \in \mathrm{dom}\varphi$. The *contingent directional derivative* of φ at $x \in \mathrm{dom}\varphi$ in the direction $v \in H$ is

$$\varphi'(x; v) := \liminf_{t \to 0^+, w \to v} \frac{\varphi(x + tw) - \varphi(x)}{t}. \tag{2.3}$$

Definition 2.7. Let $\varphi \in \mathcal{F}(H)$ and let $x \in \mathrm{dom}\varphi$.

(i) A vector $\xi \in H$ is called a *proximal subgradient* of φ at x, written $\xi \in \partial_P \varphi(x)$, if there exist $\rho > 0$ and $\sigma \geq 0$ such that

$$\varphi(y) \geq \varphi(x) + \langle \xi, y - x \rangle - \sigma \|x - y\|^2 \ \forall y \in \mathrm{B}(x, \rho).$$

(ii) A vector $\xi \in H$ is called a *Fréchet subgradient* of φ at x, written $\xi \in \partial_F \varphi(x)$, if the following inequality holds

$$\varphi(y) \geq \varphi(x) + \langle \xi, y - x \rangle + o(\|y - x\|) \ \forall y \in H.$$

(iii) A vector $\xi \in H$ is called a *basic/limiting subgradient* of φ at x, written $\xi \in \partial_L \varphi(x)$, if there are sequences $x_i \xrightarrow{\varphi} x$, (i.e., $x_i \to x$, $\varphi(x_i) \to \varphi(x)$) and $\xi_i \rightharpoonup \xi$ such that $\xi_i \in \partial_P \varphi(x_i)$ for all $i \in \mathbb{N}$.

(iv) A vector $\xi \in H$ is called a *singular subdifferential* of φ at x, written $\xi \in \partial_\infty \varphi(x)$, if there exist sequences $(\alpha_i)_i \subset \mathbb{R}_+$ and $(x_i)_i, (\xi_i)_i \subset H$ such that

$$\alpha_i \downarrow 0, \ x_i \xrightarrow{\varphi} x, \ \xi_k \in \partial_P \varphi(x_k), \ \alpha_i \xi_k \rightharpoonup \xi.$$

(v) A vector $\xi \in H$ is called a *Clarke subdifferential* of φ at x, written $\xi \in \partial_C \varphi(x)$, if ξ belongs to the set

$$\partial_C \varphi(x) := \overline{\mathrm{co}}(\partial_L \varphi(x) + \partial_\infty \varphi(x)).$$

In the case $x \notin \mathrm{dom}\varphi$, we put $\partial_P \varphi(x) = \partial_F \varphi(x) = \partial_L \varphi(x) = \partial_C \varphi(x) = \emptyset$.

From the definitions above, it is clear that

$$\partial_P \varphi(x) \subset \partial_F \varphi(x) \subset \partial_L \varphi(x) \subset \partial_C \varphi(x) \ \forall x \in H, \tag{2.4}$$

and for any $x \in \mathrm{dom} V$, one has

$$\sigma_{\partial_P \varphi(x)}(\cdot) \leq \sigma_{\partial_F \varphi(x)}(\cdot) \leq \varphi'(x; \cdot).$$

When $\varphi \in \mathcal{F}(H)$ and is convex, then for every x, one has

$$\partial_P \varphi(x) = \partial_F \varphi(x) = \partial_L \varphi(x) = \partial_C \varphi(x) = \partial \varphi(x),$$

where $\partial \varphi(x)$ is Fenchel subdifferential of φ at x which is defined by

$$\partial \varphi(x) := \{\xi \in H \mid \varphi(y) \geq \varphi(x) + \langle \xi, y - x \rangle \ \forall y \in H\}.$$

In general, the subdifferential of φ at $x \in \mathrm{dom}\varphi$ may be empty. However, the set $\mathrm{dom}(\partial_P \varphi)$ of points where $\partial_P \varphi$ is nonempty dense in $\mathrm{dom}\varphi$.

Theorem 2.8. *[30]* Let $\varphi \in \mathcal{F}(H), x_0 \in \mathrm{dom}\varphi$ and let $\varepsilon > 0$ be given. Then there exists a point $y \in \mathrm{B}(x_0, \varepsilon)$ such that

$$\partial_P \varphi(y) \neq \emptyset \text{ and } \varphi(x_0) - \varepsilon \leq \varphi(y) \leq \varphi(x_0).$$

In particular, $\mathrm{dom}\,\partial_P \varphi$ is dense in $\mathrm{dom}\,\varphi$.

From this theorem and the inclusions in (2.4), the domains of all the subdifferentials defined above are dense in $\mathrm{dom}\varphi$.

Given a closed set S and $x \in S$, we define the *proximal normal cone, Fréchet normal cone, limiting normal cone, Clarke normal cone*, respectively, by

$$\mathrm{N}_S^P(x) := \partial_P \mathrm{I}_S(x), \ \mathrm{N}_S^F(x) := \partial_F \mathrm{I}_S(x), \ \mathrm{N}_S^L(x) := \partial_L \mathrm{I}_S(x), \mathrm{N}_S^C(x) := \partial_C \mathrm{I}_S(x).$$

From inclusions in (2.4), we derive that (refer [64])

$$\mathrm{N}_S^P(x) \subset \mathrm{N}_S^F(x) \subset \mathrm{N}_S^L(x) \subset \overline{\mathrm{co}}(\mathrm{N}_S^L(x)) = \mathrm{N}_S^C(x) \ \forall x \in S. \tag{2.5}$$

We have the relationships between subdifferentials and geometric characterizations

of normal cones:

$$\xi \in \partial_\infty \varphi(x) \Leftrightarrow (\xi, 0) \in \mathrm{N}^L_{\mathrm{epi}\varphi}(x, \varphi);$$
$$\xi \in \partial_\dagger \varphi(x) \Leftrightarrow (\xi, -1) \in \mathrm{N}^\dagger_{\mathrm{epi}\varphi}(x, \varphi(x)),$$

(2.6)

where " $\dagger$ " stands for P, F, L, C respectively.

In the thesis, we also define the *singular proximal subdifferential* of φ at x as follows:

$$\partial_{P,\infty} \varphi(x) := \{\xi \mid (\xi, 0) \in \mathrm{N}^P_{\mathrm{epi}\varphi}(x; \varphi(x))\}.$$

According to [64], if $\xi \in \partial_{P,\infty} \varphi(x)$, then there exist sequences $(x_n) \subset \mathrm{dom}\varphi, (\xi_n), (\alpha_n)$ such that

$$x_n \xrightarrow{\varphi} x, \ \xi_n \in \partial_P \varphi(x_n), \ \alpha_n \downarrow 0, \ \alpha_n \xi_n \to \xi \text{ as } n \to \infty.$$

Proposition 2.9. *[30] Let $\varphi \in \mathcal{F}(H)$ and let $(x, \alpha) \in \mathrm{epi}\varphi$. We have that*

$$\mathrm{N}^P_{\mathrm{epi}\varphi}(x, \alpha) \subset \mathrm{N}^P_{\mathrm{epi}\varphi}(x, \varphi(x)). \tag{2.7}$$

Moreover, if $\alpha > \varphi(x)$ and $(\xi, -\kappa) \in \mathrm{N}^P_{\mathrm{epi}\varphi}(x, \alpha), \kappa \geq 0$ then $\kappa = 0$.

Proof. If $\alpha = \varphi(x)$ then (2.7) holds. We now suppose that $\alpha > \varphi(x)$ and $(\xi, -\kappa) \in \mathrm{N}^P_{\mathrm{epi}\varphi}(x, \alpha)$. There exist $\eta > 0, \delta \geq 0$ such that

$$\langle (\xi, -\kappa), (y, \beta) - (x, \alpha) \rangle \leq \delta(\|y - x\|^2 + (\beta - \alpha)^2) \quad \forall (y, \beta) \in \mathrm{B}((x, \alpha), \eta) \cap \mathrm{epi}\varphi.$$

(2.8)

Since $\alpha > \varphi(x)$, then exists $\eta' \in (0, \eta)$ such that $(x, \beta) \in \mathrm{epi}\varphi$ whenever $|\beta - \alpha| \leq \eta'$. Hence, from the last inequality, we obtain that

$$-\kappa(\beta - \alpha) \leq (\beta - \alpha)^2 \quad \forall \beta \in (\alpha - \eta', \alpha + \eta'),$$

which implies that $\kappa = 0$.

We now suppose that $(y, \beta) \in \mathrm{B}((x, \varphi(x)), \eta) \cap \mathrm{epi}\varphi$. Then it is clear that

$$(y, \beta + \alpha - \varphi(x)) \in \mathrm{B}((x, \alpha), \eta) \cap \mathrm{epi}\varphi.$$

Hence, from inequality (2.8), one has

$$\langle (\xi, 0), (y, \beta) - (x, \varphi(x)) \rangle = \langle (\xi, 0), (y, \beta + \alpha - \varphi(x)) - (x, \alpha) \rangle$$
$$\leq \delta(\|y - x\|^2 + (\beta - \varphi(x))^2)$$

which implies that $(\xi, -\kappa) \in \mathrm{N}^P_{\mathrm{epi}\varphi}(x, \varphi)$, and we complete the proof of the proposition. $\qquad\square$

We will continue with the concept *tangent* cones

Definition 2.10. Let $S \subset H$ and let $x \in S$ be given.

(i) The *Bouligand tangent cone* to S at x is defined by

$$\mathrm{T}^B_S(x) := \{v \in H \mid \exists\, x_i \xrightarrow{S} x, \exists\, t_i \downarrow 0 \text{ such that } \frac{x_i - x}{t_i} \to v \text{ as } i \to \infty\}.$$

(ii) The *Clarke tangent cone* to S at x is defined by

$$\mathrm{T}^C_S(x) := \{v \in H \mid \forall\, x_i \xrightarrow{S} x, \forall\, t_i \downarrow 0, \exists\, v_i \to v \text{ such that } x_i + t_i v_i \in S \text{ for all } i \in \mathbb{N}\}.$$

From this definition, it is clear that $\mathrm{T}^C_S(x)$ is always closed, convex and

$$\mathrm{T}^C_S(x) \subset \mathrm{T}^B_S(x) \ \forall x \in S. \tag{2.9}$$

Furthermore, we have

$$\mathrm{N}^C_S(x) = (\mathrm{T}^C_S(x))^*, \mathrm{T}^C_S(x) = (\mathrm{N}^C_S(x))^*.$$

When S is convex, we have that

$$\mathrm{N}^P_S(x) = \mathrm{N}^C_S(x) = \{x^* \mid \langle x^*, y - x \rangle \le 0 \ \forall y \in S\}, \mathrm{T}^B_S(x) = \mathrm{T}^C_S(x) = \overline{\mathbb{R}_+(S - x)}. \tag{2.10}$$

Next, we remind the definition of prox-regular sets and some of their properties.

Definition 2.11. [63, 73] For positive numbers r and α, a closed set S is said to be (r, α)-prox-regular at $\overline{x} \in S$ provided that one has $x = \Pi_S(x + v)$, for all $x \in S \cap \mathrm{B}(\overline{x}, \alpha)$ and all $v \in \mathrm{N}^P_S(x)$ such that $\|v\| < r$.
The set S is r-prox-regular (resp., prox-regular) at $\overline{x}$ when it is (r, α)-prox-regular at, $\overline{x}$ for some real $\alpha > 0$ (resp., for some numbers $r, \alpha > 0$). The set S is said to be r-uniformly prox-regular when $\alpha = +\infty$.

From the definition above, it is clear that if S is r-uniformly prox-regular then S is also r'-uniformly prox-regular for every $r' \le r$.

Example 2.12. (i) Any closed convex set of H is r-uniformly prox-regular for any $r \ge 0$.

(ii) The unit sphere S of H is not convex, but it is r-uniformly prox-regular for any $r \leq 1$.

When S is r-uniformly prox-regular, then all of normal cones that mention above coincide, i.e.,

$$\mathrm{N}_S^P(x) = \mathrm{N}_S^F(x) = \mathrm{N}_S^L(x) = \mathrm{N}_S^C(x) \,\forall x \in S$$

and

$$\mathrm{T}_S^B(x) = \mathrm{T}_S^C(x) \,\forall x \in S;$$

indeed, we have that

$$\mathrm{T}_S^B(x) \subset (\mathrm{N}_S^P(x))^* = (\mathrm{N}_S^C(x))^* = \mathrm{T}_S^C(x) \subset \mathrm{T}_S^B(x).$$

Hence, we denote $\mathrm{T}_S(x)$ for these tangent cones. According to Poliquin, Rockafellar and Thibault [73], we have the following property of r-uniformly prox-regular sets:

Proposition 2.13. *Let S be a closed subset of H. If S is r-uniformly prox-regular, then the set-valued mapping defined by $x \mapsto \mathrm{N}_S^P(x) \cap \mathbb{B}$ is $\frac{1}{r}$-hypomonotone, i.e., for any $x, y \in C, x^* \in \mathrm{N}_S^P(x) \cap \mathbb{B}, y^* \in \mathrm{N}_S^P(y) \cap \mathbb{B}$, one has*

$$\langle x^* - y^* + \frac{1}{r}(x - y), x - y \rangle \geq 0.$$

Lemma 2.14. *Let C be a r-uniformly-prox-regular set of $\mathbb{R}^n$ and let $\kappa > 0$. Suppose that A_C is any maximal monotone extension of the mapping $x \mapsto \mathrm{N}_C(x) \cap \mathrm{B}(\theta, \kappa) + \frac{\kappa}{r}x$, then for every $x \in C$, one has*

$$\mathrm{N}_C(x) \cap \mathrm{B}(\theta, \kappa) + \frac{\kappa}{r}x \subset A_C(x) \subset \mathrm{N}_C(x) + \frac{\kappa}{r}x \qquad (2.11)$$

and for any v such that $\|v\| \leq \kappa$,

$$(v - \mathrm{N}_C(x))^\circ = (v + \frac{\kappa}{r}x - A_C(x))^\circ. \qquad (2.12)$$

Proof. We refer [5] for the first part of the lemma. We now justify the second part of the lemma. By the first part of the lemma, we have that

$$v - \mathrm{N}_C(x) \cap \mathrm{B}(\theta, \kappa) \subset v + \frac{\kappa}{r}x - A_C(x) \subset v - \mathrm{N}_C(x).$$

Since $\|v\| \leq \kappa$, it is clear that $(v - \mathrm{N}_C(x) \cap \mathrm{B}(\theta, \kappa))^\circ = (v - \mathrm{N}_C(x))^\circ$. Hence, one

has

$$(v - \mathrm{N}_C(x) \cap \mathrm{B}(\theta, \kappa))^\circ = (v + \frac{\kappa}{r}x - A_C(x))^\circ = (v - \mathrm{N}_C(x))^\circ.$$

$\square$

2.2 Maximal monotone operators

In this section, we remind the concept of maximal monotone operators.

Let $A : H \rightrightarrows H$ be a multivalued operator. We define the *domain* and the *graph* of operator A, respectively, as

$$\mathrm{dom}A := \{x \in H \mid A(x) \neq \emptyset\}, \ \mathrm{Gr}(A) := \{(x, x^*) \in H \times H \mid x^* \in A(x)\}.$$

To simplify, we identify A to its graph.

The *inverse operator* $A^{-1} : H \rightrightarrows H$ of A defined by

$$A^{-1}(y) := \{x \in H \mid y \in A(x)\}.$$

Definition 2.15. Let $A : H \rightrightarrows H$ be a operator.

(i) The operator A is called monotone if for any two points $(x, x^*), (y, y^*) \in A$, one has

$$\langle x^* - y^*, x - y \rangle \geq 0.$$

(ii) The operator A is called maximal monotone if A is monotone and there exists no monotone operator that contains it strictly.

Example 2.16. Let $\varphi \in \mathcal{F}(H)$ be a convex function. Then the subdifferential $\partial\varphi$ is maximal monotone. In particular, the normal cone $\mathrm{N}_S(\cdot)$ is maximal monotone whenever S is a closed convex set.

Proposition 2.17. *[16] Let $A : H \rightrightarrows H$ be a maximal monotone operator. The following assertions hold:*

(i) $\overline{\mathrm{dom}A}$ *is convex.*

(ii) $A(x)$ *is closed and convex for every $x \in H$ and*

$$A(x) = A(x) + \mathrm{N}_{\overline{\mathrm{dom}A}}(x). \tag{2.13}$$

(iii) Suppose that $x_n^* \in A(x_n)$ for all $n \in \mathbb{N}$ and $x_n \to x, x_n^* \rightharpoonup x^*$ as $n \to \infty$, then $x \in \mathrm{dom}A$ and $x^* \in A(x)$.

Definition 2.18. Let $A : H \rightrightarrows H$ be a maximal monotone and let S be a subset of H.

(i) A is called locally bounded at x if there exist $r, m > 0$ such that

$$\|y^*\| \le m \ \ \forall y^* \in A(y), \ \forall y \in \mathrm{B}(x, r).$$

(ii) A is called locally bounded at x respect to S if there exist $r, m > 0$ such that

$$\|y^*\| \le m \ \ \forall y^* \in A(y), \ \forall y \in \mathrm{B}(x, r) \cap \mathrm{dom} A \cap S.$$

From (2.13), we obtain the following proposition

Proposition 2.19. *[71, 72] Let $A : H \rightrightarrows H$ be a maximal monotone operator. A is locally bounded at x if and only if $x \in \mathrm{int}(\mathrm{dom} A)$.*

We now remind Minty's Theorem

Theorem 2.20. *Let $A : H \rightrightarrows H$ be monotone. Then A is a maximal monotone operator if and only if*

$$\mathrm{rank}(\mathrm{I} + A) = H.$$

Let A be a maximal monotone and let $\lambda > 0$. We define the *resolvent* and the *Yoshida approximation* of A, respectively, by

$$J_\lambda := (\mathrm{I} + \lambda A)^{-1}, \ \ A_\lambda := \frac{\mathrm{I} - J_\lambda}{\lambda}. \tag{2.14}$$

According to Bauschke and Combettes [16], Brézis [21], Phelps [71, 72], etc, for any $\lambda > 0$, J_λ and A_λ are singular and maximal monotone with $\mathrm{dom} J_\lambda = \mathrm{dom} A_\lambda = H$. Moreover, J_λ is $1-$Lipschitz, A_λ is $\frac{1}{\lambda}-$Lipschitz and

$$A_\lambda(x) \in A(J_\lambda(x)) \ \forall x \in H.$$

Theorem 2.21. *[16, 21, 71, 72] Let $A : H \rightrightarrows H$ be a maximal monotone operator. The following assertions hold:*

(i) For every $x \in H$, one has

$$J_\lambda(x) \to \Pi_{\overline{\mathrm{dom} A}}(x) \ \ \ as \ \lambda \downarrow 0.$$

In particular, for any $x \in \overline{\mathrm{dom} A}$, then $J_\lambda(x) \to x$ as $\lambda \downarrow 0$.

(ii) For any $\lambda, \mu > 0$, one has $(A_\lambda)_\mu = A_{\lambda+\mu}$.

(iii) For every $x \in \mathrm{dom}A$, then $\|A_\lambda(x)\| \to \|A^\circ(x)\|$ and $A_\lambda(x) \to A^\circ(x)$ as $\lambda \downarrow 0$. More generally, we have that

$$\|A_\lambda(x) - A^\circ(x)\|^2 \leq \|A^\circ(x)\|^2 - \|A_\lambda(x)\|^2,$$

where $A^\circ(x) := \Pi_{A(x)}(\theta)$.

(iv) If $x \notin \mathrm{dom}A$ then $\|A_\lambda(x)\| \uparrow +\infty$ as $\lambda \downarrow 0$.

2.3 Differential inclusions with maximal monotone perturbations

In this section, we provide some basic knowledges of the following differential inclusion

$$\begin{cases} \dot{x}(t) \in f(x(t)) - A(x(t)) \ \ t \geq 0, \\ x(0) = x_0 \in \overline{\mathrm{dom}A}, \end{cases} \tag{2.15}$$

where $f : H \to H$ is an l-Lipschitz mapping and $A : H \rightrightarrows H$ is a maximal monotone operator. We refer to [11, 12, 21] for all of results of this section.

Definition 2.22. A function $x : [a, b] \to H$ is said to be absolutely continuous if it can be expressed in the form

$$x(t) = x(a) + \int_0^t v(\tau)d\tau \ \ \forall t \in [a, b],$$

for some integrable function v. In this case, we have

$$\dot{x}(t) = v(t) \text{ a.e. } t \in [a, b].$$

Theorem 2.23. *[14, 21] We consider the differential inclusion (2.15). For any $x_0 \in \mathrm{dom}A$, there exists a unique function $x : [0, \infty) \to H$ such that*

(i) $x(0) = x_0, x(t) \in \mathrm{dom}A \ \ \forall t \geq 0$,

(ii) $x(\cdot)$ is differentiable almost everywhere and the right derivative $\frac{d^+x(t)}{dt} := \lim_{h \downarrow 0} \frac{x(t+h)-x(t)}{h}$ exists and

$$\frac{d^+x(t)}{dt} = f(x(t)) - \Pi_{A(x(t))}(f(x(t)) \ \ \forall t \geq 0.$$

Furthermore, the function $t \mapsto \frac{d^+ x(t)}{dt}$ is right continuous and satisfies

$$\left\| \frac{d^+ x(t)}{dt} \right\| \leq e^{lt} \|(f(x_0) - A(x_0))^\circ\| \quad \forall t \geq 0.$$

In particular, we have $\frac{d^+ x(0)}{dt} = (f(x_0) - A(x_0))^\circ$.

(iv) *Suppose that $y_0 \in \mathrm{dom}A$ and $y(\cdot)$ also satisfies $(i), (ii), (iii)$ with respect to the initial value y_0. Then we have that*

$$\|x(t) - y(t)\| \leq e^{lt} \|x_0 - y_0\| \quad \forall t \geq 0. \tag{2.16}$$

Definition 2.24. We consider differential inclusion (2.15) on $[0, T]$.

(i) A strong solution of (2.15) is a continuous function $x(\cdot; x_0) : [0, T] \to H$, absolutely continuous on every interval $[a, b] \subset (0, T]$, and for a.e. $t \in [0, T]$, one has

$$\dot{x}(t; x_0) \in f(x(t; x_0)) - A(x(t; x_0)).$$

(ii) A weak solution of (2.15) is a continuous function $x(\cdot; x_0) : [0, T] \to H$ which is the limit of any sequence of solutions $(x(\cdot; x_n))_{n \in \mathbb{N}^*}$, where $x_n \in \mathrm{dom}A$ and $x_n \to x_0$ as $n \to \infty$.

Any strong solution of (2.15) is a weak solution. The following theorem follows directly from Theorem 2.23 and Definition 2.24.

Theorem 2.25. *[14, 21] We consider differential inclusion (2.15) on $[0, T]$. The following assertions hold:*

(i) *If $x_0 \in \mathrm{dom}A$, then system (2.15) has a unique strong solution.*

(ii) *For any $x_0 \in \overline{\mathrm{dom}A}$, system (2.15) always has a unique weak solution $x(\cdot; x_0)$ which satisfies*

$$x(t; x(s; x_0)) = x(t + s; x_0) \quad \forall t \geq 0, \forall s \geq 0.$$

(iii) *Suppose that $x(\cdot; x_0)$ and $x(\cdot; y_0)$ are two weak solutions of (2.15) with respect to the two initial values $x_0, y_0 \in \overline{\mathrm{dom}A}$, we have inequality*

$$\|x(t; x_0) - x(t; y_0)\| \leq e^{lt} \|x_0 - y_0\| \quad \forall t \geq 0.$$

For each $\lambda > 0$, A_λ is a Lipschitz mapping. Hence, the following differential inclusion

$$\begin{cases} \dot{x}(t) = f(x(t)) - A_\lambda(x(t)) \text{ a.e. } t \geq 0, \\ x(0) = x_0 \in H \end{cases}$$

always has a unique strong solution. The proposition below proves that the solution of (2.15) is the limit of the solutions of the differential equations above.

Proposition 2.26. *For any $x_0 \in \overline{\mathrm{dom}A}$ and $T > 0$, differential inclusion (2.15) has a unique continuous solution, which is the uniform limit on $[0,T]$ of $x_\lambda(\cdot;x_0)$ (as $\lambda \downarrow 0$), where $x_\lambda(\cdot;x_0)$ is the solution of the following differential equation*

$$\dot{x}(t) = f(x(t)) - A_\lambda(x(t)) \ a.e. \ t \in [0,T], \ x(0) = x_0.$$

Proof. We can refer to Brézis [21] and Barbu [14] for the case $x_0 \in \mathrm{dom}A$. Now we consider the case $x_0 \notin \mathrm{dom}A$. Let us fix $\varepsilon > 0$ and any sequence $\lambda \downarrow 0$ and any $z \in \mathrm{dom}A$ such that $\|z - x_0\| \leq e^{-lT}\varepsilon$. According to Theorem 2.23 then

$$\max\{\|x_\lambda(t;x_0) - x_\lambda(t;z)\|, \|x(t;x_0) - x(t;z)\|\} \leq e^{lt}\|x_0 - z\| \leq \varepsilon \ \forall t \in [0,T].$$

Since the proposition holds whenever the initial-value belongs to $\mathrm{dom}A$, there exists $\lambda_0 > 0$ such that for every $\lambda < \lambda_0$, one has

$$\|x(t;z) - x_\lambda(t;z)\| \leq \varepsilon \ \forall t \in [0,T].$$

Combining the above results, we obtain that for any $\lambda \leq \lambda_0$, one has

$$\|x(t;x_0) - x_\lambda(t;x_0)\| \leq 3\varepsilon \ \forall t \in [0,T],$$

we complete the proof of the proposition. $\square$

Example 2.27. (i) If $\mathrm{int}(\mathrm{dom}A) \neq \emptyset$, then every weak solution of differential inclusion (2.15) is a strong solution. In particular, the conclusion still hold when $\dim H < +\infty$.

(ii) The weak solution of the following differential inclusion

$$\begin{cases} \dot{x}(t) \in -\partial\varphi(x(t)) \text{ a.e. } t \geq 0, \\ x(0) = x_0 \in \overline{\mathrm{dom}\varphi}, \end{cases}$$

where $\varphi \in \mathcal{F}(H)$ and is convex, is a strong solution (see [11, 21]).

Let $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a maximal monotone operator, $g \in L^1(0, T; \mathbb{R}^n)$ (space of integrable functions from $[0, T]$ to $\mathbb{R}^n$), and consider the following differential inclusion

$$\begin{cases} \dot{x}(t) \in g(t) - A(x(t)) \ \text{ a.e. } t \in [0, T], \\ x(0) = x_0 \in \overline{\text{dom}A}. \end{cases} \tag{2.17}$$

Because of the finite-dimension, we can suppose that $\text{int}(\text{dom}A) \neq \emptyset$ (see [21]). Hence, the differential inclusion above always has a unique continuous solution satisfying (2.17). The following theorem provides the exact valued of the right-derivative of the solution (Brézis [21] and Barbu [14]).

Theorem 2.28. *Differential inclusion (2.17) has a unique solution denoted by* $x(\cdot; x_0)$. *Moreover, we have that*

$$\dot{x}(t; x_0) = g(t) - \Pi_{A(x(t))}(g(t)) \ \text{ a.e. } t \in [0, T].$$

Finally, to end this section, let us remind one version of Gronwall's Lemma.

Lemma 2.29. *(Gronwall's Lemma, [2]) Let* $T > 0$ *and* $a, b \in L^1(t_0, t_0 + T; \mathbb{R})$ *such that* $b(t) \geq 0$ *a.e.* $t \in [t_0, t_0 + T]$. *If an absolutely continuous function* $w : [t_0, t_0 + T] \to \mathbb{R}_+$ *satisfies, for* $0 \leq \alpha < 1$,

$$(1 - \alpha)w'(t) \leq a(t)w(t) + b(t)w^\alpha(t) \ \text{ a.e. } t \in [t_0, t_0 + T],$$

then

$$w^{1-\alpha}(t) \leq w^{1-\alpha}(t_0)e^{\int_{t_0}^t a(\tau)d\tau} + \int_{t_0}^t e^{\int_s^t a(\tau)d\tau}b(s)ds \ \ \forall t \in [t_0, t_0 + T].$$

Chapter 3

Invariant sets and Lyapunov pairs for differential inclusions with maximal monotone operators

We give different conditions for the invariance of closed sets with respect to differential inclusions governed by a maximal monotone operator defined on Hilbert spaces, which is subject to a Lipschitz continuous perturbation depending on the state. These sets are not necessarily weakly closed as in [7, 8], while the invariance criteria are still written by using only the data of the system. So, no need to the explicit knowledge of neither the solution of this differential inclusion, nor the semi-group generated by the maximal monotone operator. These invariant/viability results are next applied to derive explicit criteria for a-Lyapunov pairs of lower semi-continuous (not necessarily weakly lower semi-continuous) functions associated to these differential inclusions. The lack of differentiability of the candidate Lyapunov functions and the consideration of general invariant sets (possibly not convex or smooth) are carried out by using techniques from nonsmooth analysis.

3.1   Introduction

We provide sufficient and, in many different interesting situations, necessary criteria for the invariance property of closed subsets with respect to the following differential inclusion, given in a Hilbert space H,

$$\dot{x}(t) \in f(x(t)) - A(x(t)), \ \ x(0) = x_0 \in \overline{\mathrm{dom}A}, \ \mathrm{a.e.} \ \ t \geq 0 \qquad (3.1)$$

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where A is a maximal monotone operator which is subject to a Lipschitzian perturbation f. Equivalently, we establish many primal and dual explicit criteria for a-Lyapunov pairs and functions associated to the differential inclusion above. The current work extends and improves some of the results given in [7, 8] on weakly closed invariant sets and weakly lower semi-continuous a-Lyapunov pairs.

The domain of A does not need to be closed, nor the values of A are supposed to be bounded or even nonempty. Thus, the scope of the equation above goes beyond the differential inclusions treated in [11, 12, 30, 31, 43], where the right-hand side is generally represented by a cusco set-valued mapping (in particular, with nonempty and weak*-compact multi-valued operator). It is the monotonicity of A which compensates the lack of compacity in our differential inclusion, while the maximality of this operator guaranties, among other properties, the existence and the regularity of solutions. These two facts are also essential when checking the invariance of closed sets.

In front of the lack to a direct access to the explicit calculus of either the solution of the inclusion above or to the semi-group generated by A, the current work aims at finding weaker conditions for the invariance of closed sets, which only appeal to the fresh input data, namely the maximal monotone operator and the Lipschitz mapping. These conditions are applicable to a large variety of closed sets which do not need to be convex or smooth. Our approach fits the general scope and the main ideas behind Lyapunov's stability, which consists of looking for an adjacent function to the system described by the inclusion above; namely, an energy-like function which decreases along the trajectories and, so, under some extra usual conditions, forces the system to converge towards its equilibrium state and to remain there. Since our analysis allows to deal with extended-real valued functions, the invariance of a set occurs as long as the associated indicator function is a Lyapunov's function. However, our approach is more geometric since we first establish criteria for the invariance property and next deduce the adequate conditions for Lyapunov pairs and functions.

Invariant sets associated to general differential inclusions/equations have been the subject of extensive research during the last decades; namely, in relation with differential inclusions involving cusco mappings in their right-hand side (see, e.g., [11]). First results dealing with Lyapunov pairs and functions associated to the differential inclusions above have been first established in [69, 70] in the case of homogeneous systems; that is, $f \equiv 0$. Pazy's criteria for a-Lyapunov pairs are given by means of directional-like derivative using the Moreau-Yoshida approximation

of the operator A. This result has been extended to the general inclusion above in [26, 54], with the use of implicit criteria depending heavily on the semi-group generated by the maximal monotone operator A. Recently, different criteria for weakly lower semi-continuous a-Lyapunov pairs have been investigated in [7, 8].

The need of more explicit conditions, not depending on the semi-group generated by A, is of utmost importance for many reasons, one of which is that the inclusion above is sometimes evoked as a companion tool to analyze other differential inclusions. In that case, the operator A may not be known explicitly, and this facts makes the access to its semi-group more complicated. For instance, in our work [5] we have investigated the existence of solutions to a differential inclusion governed by the normal cone to a prox-regular set ([73]), by rewriting it in the form of (3.1) with A being some intrinsic maximal monotone operator to this prox-regular set. Such an operator A is not known explicitly but it processes enough information in order to check the invariance of the involved prox-regular set with respect to (3.1). This was sufficient to get the desired existence results; for more details, we refer the reader to [5].

Invariant sets are also referred to in the wide literature as viable sets [11–13], and are of crucial use in many domains, as in economic, renewable resources, biology, diseases propagation, control processes of species and so on. It is manifest, in recent papers [85], that the investigation of certain algebraic varieties is sufficient to characterize invariant sets forced by symmetries. Lyapunov pairs and functions are used extensively in dynamic systems and control theory, among many other applications; see, e.g., [1, 22].

In this work, we provide different criteria to characterize those sets which are invariant with respect to the differential inclusion (3.1). Only the data, A and f, will be appealed to and no need to solve explicitly the equation. These invariant results are then rewritten as criteria for a-Lyapunov pairs, which are crucial for Lyapunov stability of (3.1). Because the sets we consider are not necessary convex or smooth, and the candidate Lyapunov functions are not necessarily sufficiently regular, we use techniques of nonsmooth analysis (e.g.. [30, 64, 76]).

The organization of the paper is as follows. After an introductory section to present the main notations and tools which are used through this work, we give in Section 3.3 the main invariance criterion in Theorem 3.6, using the normal cone to the nominal set. Other corollaries follow in order to simplify this invariance criterion and provide equivalent primal and dual conditions. In Section 3.4, we

apply the previous invariance result to investigate a-Lyapunov pairs associated to differential inclusion (3.1).

3.2 Notation and preliminary results

Let $(H, \langle \cdot, \cdot \rangle, \|\cdot\|)$ be a Hilbert space, with origin θ. Given a set $S \subset H$, by $\overline{S}$ and S^* we denote the *closure* of S and the *polar* of S, respectively, where

$$S^* := \{x^* \in H \mid \langle x^*, x \rangle \leq 0, \text{ for all } x \in S\}.$$

The *indicator* and the *distance functions* are respectively given by

$$\mathrm{I}_S(x) := 0 \text{ if } x \in S; \ +\infty \text{ otherwise, and } d_S(x) := \inf\{\|x - y\| : \ y \in S\}$$

(in the sequel we shall adopt the convention $\inf_\emptyset = +\infty$). For $\delta \geq 0$, we denote Π_S^δ the *(orthogonal) δ-projection mapping* onto S defined as

$$\Pi_S^\delta(x) := \{y \in S : \ \|x - y\|^2 \leq d_S^2(x) + \delta^2\};$$

for $\delta = 0$, we simply write $\Pi_S(x) := \Pi_S^0(x)$. It is known that Π_S is nonempty-valued on a dense subset of $H \setminus S$ ([30]). For an extended-real valued function $\varphi : H \to \overline{\mathbb{R}} := (-\infty, +\infty]$, we denote $\mathrm{dom}\varphi := \{x \in H \mid \varphi(x) < +\infty\}$ and $\mathrm{epi}\varphi := \{(x, \alpha) \in H \times \mathbb{R} \mid \varphi(x) \leq \alpha\}$. Function φ is lower semi-continuous if $\mathrm{epi}\varphi$ is closed. The *contingent directional derivative* of φ at $x \in \mathrm{dom}\varphi$ in the direction $v \in H$ is

$$\varphi'(x; v) := \liminf_{t \to 0^+, w \to v} \frac{\varphi(x + tw) - \varphi(x)}{t}.$$

A vector $\xi \in H$ is called a *proximal subgradient* of φ at $x \in H$, written $\xi \in \partial_P \varphi(x)$, if there are $\rho > 0$ and $\sigma \geq 0$ such that

$$\varphi(y) \geq \varphi(x) + \langle \xi, y - x \rangle - \sigma \|y - x\|^2, \ \forall y \in B_\rho(x),$$

where $\mathrm{B}_\rho(x) \ (=: \mathrm{B}(x, \rho))$ is the closed ball centered at $x \in H$ of radius $\rho > 0$. The vector ξ is called a *Fréchet subgradient* of φ at x, written $\xi \in \partial_F \varphi(x)$, if

$$\varphi(y) \geq \varphi(x) + \langle \xi, y - x \rangle + o(\|y - x\|) \ \forall y \in H;$$

and a *basic (or Limiting) subgradient* of φ at x, written $\xi \in \partial_L\varphi(x)$, if there exist sequences $(x_k)_k$ and $(\xi_k)_k$ such that

$$x_k \overset{\varphi}{\to} x, \ \xi_k \in \partial_P\varphi(x_k) \ \xi_k \rightharpoonup \xi,$$

where $\rightharpoonup$ refers to the weak convergence in H, and $x_k \overset{\varphi}{\to} x$ means that $x_k \to x$ together with $\varphi(x_k) \to \varphi(x)$.

If $x \notin \text{dom}\varphi$, we write $\partial_P\varphi(x) = \partial_F\varphi(x) = \partial_L\varphi(x) = \emptyset$. If S is a closed set and $s \in S$, we define the *proximal normal cone* to S at s as $\text{N}_S^P(s) = \partial_P\text{I}_S(s)$, the *Fréchet normal* cone to S at s as $\text{N}_S^F(s) = \partial_F\text{I}_S(s)$, the *limiting normal cone* to S at s as $\text{N}_S^L(s) = \partial_L\text{I}_S(s)$, and *the Clarke normal cone* to S at s as $\text{N}_S^C(s) = \overline{\text{co}}(\text{N}_S^L(s))$. Equivalently, we have that $\text{N}_S^P(s) = \text{cone}(\Pi_S^{-1}(s) - s)$, where $\Pi_S^{-1}(s) := \{x \in H \mid s \in \Pi_S(x)\}$. The *Bouligand tangent cone* to S at x is defined as

$$\text{T}_S^B(x) := \left\{v \in H \mid \exists \, x_k \in S, \exists \, t_k \to 0, \ \text{st.} \ t_k^{-1}(x_k - x) \to v \ \text{as} \ k \to +\infty\right\}.$$

We also define the *Clarke subgradients* of φ at x as the vectors $\xi \in H$ such that $(\xi, -1) \in \text{N}_{\text{epi}\varphi}^C(x, \varphi(x))$, and denote $\partial_C\varphi(x)$ the Clarke subdifferential of φ at x. The *singular subdifferential* of φ at x, written $\partial_\infty\varphi(x)$, is the set of vectors $\xi \in H$ for which there are sequences $x_k \overset{\varphi}{\to} x$, $\xi_k \in \partial_P\varphi(x_k)$ and $\lambda_k \to 0^+$ such that $\lambda_k\xi_k \rightharpoonup \xi$; equivalently, $\xi \in \partial_\infty\varphi(x)$ iff $(\xi, 0) \in \text{N}_{\text{epi}\varphi}^L(x, \varphi(x))$ (see [64, Theorem 2.38]). It is known that every $\xi \in H$ such that $(\xi, 0) \in \text{N}_{\text{epi}\varphi}^P(x, \varphi(x))$ belongs to $\partial_\infty\varphi(x)$ and, moreover, there exist sequences as in the definition before but with $\lambda_k\xi_k \to \xi$ instead of $\lambda_k\xi_k \rightharpoonup \xi$ (see [64, Lemma 2.37]). Observe that $\partial_P\varphi(x) \subset \partial_F\varphi(x) \subset \partial_L\varphi(x) \subset \partial_C\varphi(x)$. For all these concepts and properties we refer to [64, 76].

We shall use the following version of Gronwall's Lemma:

Lemma 3.1. (Gronwall's Lemma [2]) *Let $T > 0$ and $a, b \in L^1(t_0, t_0 + T; \mathbb{R})$ such that $b(t) \geq 0$ a.e. $t \in [t_0, t_0 + T]$. If an absolutely continuous function $w : [t_0, t_0 + T] \to \mathbb{R}_+$ satisfies, for $0 \leq \alpha < 1$,*

$$(1 - \alpha)w'(t) \leq a(t)w(t) + b(t)w^\alpha(t) \quad a.e. \ t \in [t_0, t_0 + T],$$

then

$$w^{1-\alpha}(t) \leq w^{1-\alpha}(t_0)e^{\int_{t_0}^t a(\tau)d\tau} + \int_{t_0}^t e^{\int_s^t a(\tau)d\tau}b(s)ds, \ \forall t \in [t_0, t_0 + T].$$

Next, we review some facts about monotone and maximal monotone operators. Given a set-valued operator $A : H \rightrightarrows H$, which we identify with its graph, we denote its *domain* by $\mathrm{dom}A := \{x \in H \mid A(x) \neq \emptyset\}$. Operator A is *monotone* if

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0 \quad \text{for all} \quad (x_1, y_1), (x_2, y_2) \in A.$$

We say that A is *maximal monotone* if A is monotone and coincides with every monotone operator containing its graph. In such a case, it is known that $A(x)$ is convex and closed for every $x \in H$; moreover, for every $\lambda > 0$ there exists a unique vector $J_\lambda(x) \in (\mathrm{id} + \lambda A)^{-1}(x)$, which is the *resolvent* of the (maximal monotone) operator A, while $A_\lambda(x) := \frac{x - J_\lambda x}{\lambda}$ is the *Moreau-Yoshida approximation* of A. If $S \subset H$ is a closed convex set, we denote $S^\circ := \{y \in S \mid \|y\| = \min\limits_{z \in S} \|z\|\}$; in particular, we write $A^\circ(x) := (A(x))^\circ$, $x \in \mathrm{dom}A$.

Associated with a maximal monotone operator $A : H \rightrightarrows H$ we consider the differential inclusion given in (3.1) :

$$\dot{x}(t) \in f(x(t)) - A(x(t)), \quad \text{a.e. } t \geq 0, \ x(0) = x_0 \in \overline{\mathrm{dom}A},$$

where $f : H \to H$ is a given (l-)Lipschitz continuous mapping. Every solution of differential inclusion (3.1) will be denoted by $x(\cdot; x_0)$.

We introduce the concept of invariant sets (see, e.g., [11, 30, 32]):

Definition 3.2. A set $S \subset \overline{\mathrm{dom}A}$ is said to be invariant for (3.1) provided that $x(t; x_0) \in S$ for every $x_0 \in S$ and every $t \geq 0$.

We also recall the following result on the existence of solutions of (3.1); for more details, we refer to [21].

Proposition 3.3. *For any $x_0 \in \overline{\mathrm{dom}A}$ and $T > 0$, system (3.1) has a unique continuous solution, which is the uniform limit on $[0, T]$ of $x_\lambda(\cdot; x_0)$ (as $\lambda \downarrow 0$), where $x_\lambda(\cdot; x_0)$ is the solution of the differential equation*

$$\dot{x}_\lambda(t) = f(x_\lambda(t)) - A_\lambda(x_\lambda(t)), \ x_\lambda(0) = x_0.$$

Moreover, the following holds:
(i) *For all $s, t \geq 0$ and all $y_0 \in \overline{\mathrm{dom}A}$ we have that*

$$x(s; x(t; x_0)) = x(t + s; x_0), \quad \|x(t; x_0) - x(t; y_0)\| \leq e^{lt} \|x_0 - y_0\|.$$

(ii) *If $x(t_0, x_0) \in \mathrm{dom}A$ for some $t_0 \geq 0$, then*

$$\frac{d^+ x(t_0; x_0)}{dt} = (f(x(t_0; x_0)) - A(x(t_0; x_0)))^\circ.$$

(iii) *The function $t \to \frac{d^+ x(t; x_0)}{dt}$ is right-continuous at every $t \geq t_0$, where $t_0 \geq 0$ is such that $x(t_0; x_0) \in \mathrm{dom}A$, and we have*

$$\left\| \frac{d^+ x(t; x_0)}{dt} \right\| \leq e^{l(t-t_0)} \left\| \frac{d^+ x(t_0; x_0)}{dt} \right\|.$$

3.3   Invariant sets

In this section, we achieve our first goal to characterize those closed sets in the Hilbert space H, which are invariant with respect to differential inclusion (3.1):

$$\dot{x}(t) \in f(x(t)) - A(x(t)), \quad t \in [0, \infty), \ x(0) = x_0 \in \overline{\mathrm{dom}A};$$

the unique solution of this inclusion is written $x(\cdot; x_0)$.

It is worth observing that whenever differential inclusion (3.1) possesses a strong solution starting from S ($x_0 \in S$), which is an absolutely continuous function such that $x(t; x_0) \in \mathrm{dom}A$ for all $t > 0$, each invariant closed set $S \subset \overline{\mathrm{dom}A}$ satisfies the condition

$$S = \overline{\mathrm{dom}A \cap S}. \tag{3.2}$$

However, this condition may not be true when only weak solutions exist. This is why we shall assume in what follows that our invariance candidate sets satisfy this "almost necessary" condition.

Remark 3.4. Theorem 3.6 below gives the main invariance criterion, given in (3.3), for closed sets with respect to differential inclusion (3.1), using only the data in (3.1) which are the operator A and the mapping f. Hence, explicit calculus of either the solution or the semigroup generated by A are not required. Criterion (3.3) extends and adapts some of the results given in [7, 8] on weakly closed invariant sets. Its geometric meaning is very similar to the classical ones established in [30, 31] for differential inclusions of the form

$$\dot{x}(t) \in F(x(t)),$$

with a w^*-compact, nonempty and convex multifunction F. In our case, condition (3.3) takes into account that the right-hand side in (3.1), which is governed by a general maximal monotone operator, may have empty or unbounded values. As well, another crucial difference between (3.1) and the last inclusion above is that our analysis also allows the initial condition in (3.1) to start from the larger set $\overline{\mathrm{dom}A}$. Thus, the scope of our analysis goes beyond the differential inclusions treated in [11, 12, 30, 31, 43]. First invariance criteria for differential inclusions involving maximal monotone operators have been given in [70] (see, also, [21]) without considering the Lipschitzian perturbation. Such results have been extended in [26, 54] to maximal monotone operators which are subject to Lipschitz perturbations, using criteria which depend on the semi-group of contractions generated by $-A$. Compared to [26, 54] (see, also, references therein), condition (3.3) relies exclusively on the geometry of C as in [30, 31].

Before we state the main theorem of this section, Theorem 3.6 below, we give the following lemma.

Lemma 3.5. *Given a closed set $S \subset H$ and an $m \geq 0$, we denote*

$$S_m := \{x \in S \cap \mathrm{dom}A \mid \|(f(x) - A(x))^\circ\| \leq m\}.$$

Then the set S_m is closed.

Proof. Take a sequence $(x_k)_k \subset S_m$ such that $x_k \to x \ (\in S)$. Without loss of generality, and taking into account the norm-weak upper semi-continuity of the maximal monotone operator A, we conclude that the sequence $(\Pi_{A(x_k)}(f(x_k)))_k$ weakly converges to some $z \in A(x)$. Then

$$
\begin{aligned}
\|(f(x) - A(x))^\circ\| &\leq \|f(x) - z\| \\
&\leq \liminf_{k\to\infty} \|f(x_k) - \Pi_{A(x_k)}(f(x_k))\| \\
&= \liminf_{k\to\infty} \|(f(x_k) - A(x_k))^\circ\| \leq m,
\end{aligned}
$$

so that $x \in S_m$. $\qquad\square$

Theorem 3.6. *Given a closed set $S \subset \overline{\mathrm{dom}A \cap S}$, we assume that for every $x \in S \cap \mathrm{dom}A$ there exist $m, r > 0$ such that $\left\|\Pi_{A(x)}(f(x))\right\| \leq m$ and*

$$\sup_{\xi \in \mathrm{N}^P_{S_m}(y)} \min_{y^* \in A(y) \cap \mathrm{B}(\theta, m)} \langle \xi, f(y) - y^* \rangle \leq 0 \quad \text{for all } y \in \mathrm{B}(x, r). \qquad (3.3)$$

Then S is invariant for (3.1).

Proof. We fix $x_0 \in S \cap \mathrm{dom} A$ and $\varepsilon > 0$. Let $m, r > 0$ be as in the current assumption (with $x = x_0$), and choose an $M > 0$ such that

$$f(y) - A(y) \cap \mathrm{B}(\theta, m) \subset \mathrm{B}(\theta, M) \ \text{ for all } y \in K := S_m \cap \mathrm{B}(x_0, r). \tag{3.4}$$

We also choose sufficiently small numbers $\bar{t}, \delta > 0$ and a sufficiently large integer N such that

$$\max\{6M^2\bar{t}^2, 8\delta^2\} < \frac{r^2}{2}, \ \delta < \frac{\bar{t}}{N}, \tag{3.5}$$

$$\max\left\{\frac{(M^2 + 4M + 1)\bar{t}^2}{N}, \frac{M^2\bar{t}^2}{N^2} + 2\delta^2\right\} < \frac{\varepsilon^2}{4}. \tag{3.6}$$

We denote by $\pi := \{t_0, t_1, ..., t_N\}$ the uniform partition of the interval $[0, \bar{t}]$. We put $d(\pi) := \max_{0 \leq i \leq N-1} (t_{i+1} - t_i) = \frac{\bar{t}}{N}$ and, by (3.4), we choose an element $s_0^* \in f(x_0) - A(x_0)$ such that $\|s_0^*\| \leq M$. We consider the the function $z_0(t)$, $t \in [t_0, t_1]$ such that

$$\begin{cases} \dot{z}_0(t) = s_0^*, \ t \in [t_0, t_1], \\ z_0(0) = x_0, \end{cases}$$

and denote $z_1 := x_0 + s_0^* t_1$. We pick $\hat{s}_1 \in \Pi_K^\delta(z_1)$. Then there exists a pair (y_1, s_1) such that $s_1 \in K$, $y_1 - s_1 \in \mathrm{N}_K^P(s_1)$ and (see, e.g., [32, 74])

$$\max\{\|y_1 - z_1\|, \|s_1 - \hat{s}_1\|\} \leq \delta, \ \|(y_1 - s_1) - (z_1 - \hat{s}_1)\| \leq 2\delta,$$

as well as (see [5, Lemma 4], also Theorem 2.1)

$$\|s_1 - x_0\|^2 \leq 6\|z_1 - x_0\|^2 + 8\delta^2 = 6t_1^2\|s_0^*\|^2 + 8\delta^2 < 6\bar{t}^2 M^2 + 8\delta^2 < r^2;$$

hence, $s_1 \in \mathrm{int}(\mathrm{B}(x_0, r))$ and, so, $\mathrm{N}_K^P(s_1) = \mathrm{N}_{S_m}^P(s_1)$. Consequently, by the current assumption of the theorem, we find $s_1^* \in (f(s_1) - A(s_1)) \cap \mathrm{B}(\theta, M)$ such that

$$\langle y_1 - s_1, s_1^* \rangle \leq 0.$$

With this vector s_1^* in hand, we consider the function $z_1(t)$, $t \in [t_1, t_2]$, such that

$$\begin{cases} \dot{z}_1(t) = s_1^*, \ t \in [t_1, t_2] \\ z_1(t_1) = z_1. \end{cases}$$

By repeating the arguments used above, for each $i \in \overline{2, N-1}$, we consider the

function $z_i(t)$, $t \in [t_i, t_{i+1}]$, such that

$$\begin{cases} \dot{z}_i(t) = s_i^*, \ t \in [t_i, t_{i+1}] \\ z_i(t_i) = z_{i-1}(t_i) =: z_i, \end{cases}$$

and the corresponding elements $(\hat{s}_i, y_i, s_i, s_i^*)$ such that $\hat{s}_i \in \Pi_K^\delta(z_i)$, $y_i - s_i \in \mathrm{N}_K^P(s_i) = \mathrm{N}_{S_m}^P(s_i)$, $s_i^* \in [f(s_i) - A(s_i)] \cap \mathrm{B}(\theta, M)$,

$$\langle y_i - s_i, s_i^* \rangle \leq 0,$$

$$\max\{\|y_i - z_i\|, \|s_i - \hat{s}_i\|\} \leq \delta, \ \|(y_i - s_i) - (z_i - \hat{s}_i)\| \leq 2\delta.$$

Now, we are going to prove that the absolute continuous trajectory $z(\cdot)$, defined on $[0, \bar{t}]$ as $z(t) := z_i(t) = z_i + (t - t_i)s_i^*$ for $t \in [t_i, t_{i+1}]$, satisfies

$$d_S(z(t)) \leq \varepsilon, \ \forall t \in [0, \bar{t}], \tag{3.7}$$

$$\|s_i - z(t)\| \leq 2\varepsilon, \ \forall t \in [t_i, t_{i+1}]. \tag{3.8}$$

Indeed, for any $1 \leq i \leq N - 1$, one has

$$\begin{aligned} d_K^2(z_{i+1}) &\leq \|z_{i+1} - \hat{s}_i\|^2 = \|z_{i+1} - z_i\|^2 + \|z_i - \hat{s}_i\|^2 + 2\langle z_{i+1} - z_i, z_i - \hat{s}_i \rangle \\ &= \|(t_{i+1} - t_i)s_i^*\|^2 + d_K^2(z_i) + \delta^2 + 2d(\pi)\langle s_i^*, z_i - \hat{s}_i \rangle \\ &\leq M^2 d^2(\pi) + d_K^2(z_i) + \delta^2 + 2d(\pi)\langle s_i^*, y_i - s_i \rangle \\ &\quad + 2d(\pi)\langle s_i^*, (z_i - \hat{s}_i) - (y_i - s_i) \rangle \\ &\leq d_K^2(z_i) + (M^2 + 4M + 1)d(\pi)(t_{i+1} - t_i), \end{aligned}$$

which gives us

$$\begin{aligned} d_K^2(z_{i+1}) &\leq d_K^2(z_1) + (M^2 + 4M + 1)d(\pi)(t_{i+1} - t_1) \\ &\leq \|z_1 - x_0\|^2 + (M^2 + 4M + 1)d(\pi)(t_{i+1} - t_1) \\ &\leq (M^2 + 4M + 1)d(\pi)\bar{t} \leq \frac{(M^2 + 4M + 1)\bar{t}^2}{N} < \frac{\varepsilon^2}{4}. \tag{3.9} \end{aligned}$$

This shows that, for every $t \in [t_i, t_{i+1}]$,

$$\begin{aligned} d_S^2(z(t)) &\leq d_K^2(z(t)) = d_K^2(z_i(t))) = d_K^2(z_i(t_i) + (t - t_i)s_i^*) \\ &\leq 2d_K^2(z_i) + 2(t - t_i)^2 M^2 \leq \frac{\varepsilon^2}{2} + 2d^2(\pi)M^2 \leq \varepsilon^2 \end{aligned}$$

and (3.7) follows. Inequality (3.8) also follows since that for every $t \in [t_i, t_{i+1}]$

$$
\begin{aligned}
\|s_i - z(t)\|^2 &\le 2\|z(t) - z_i\|^2 + 2\|s_i - z_i\|^2 \\
&\le 2(t - t_i)^2 M^2 + 4\|s_i - \hat{s}_i\|^2 + 4\|z_i - \hat{s}_i\|^2 \\
&\le 2(t - t_i)^2 M^2 + 4d_K^2(z_i) + 8\delta^2 \\
&\le 2d^2(\pi) M^2 + \varepsilon^2 + 8\delta^2 \le 2\varepsilon^2,
\end{aligned}
$$

where in the last inequality we used (3.9).

Now, let $x(t)$ be the (strong) solution of (3.1) starting at x_0, and denote $l_i(t) := s_i - z(t)$, $t \in [t_i, t_{i+1}]$, so that $\dot{z}(t) = s_i^* \in f(s_i) - A(s_i) = f(z(t) + l_i(t)) - A(z(t) + l_i(t))$. Hence, by using the monotonicity of A we get

$$
\langle f(z(t) + l_i(t)) - \dot{z}(t) - f(x(t)) + \dot{x}(t), z(t) + l_i(t) - x(t) \rangle \ge 0,
$$

which leads us, using (3.7) and (3.8) together with the l-Lipschitzianity of f, to

$$
\begin{aligned}
\langle \dot{z}(t) - \dot{x}(t), z(t) - x(t) \rangle &\le 2\varepsilon \, \|f(z(t) + l_i(t)) - \dot{z}(t) - f(x(t)) + \dot{x}(t)\| \\
&\quad + \|z(t) - x(t)\| \, \|f(z(t) + l_i(t)) - f(x(t))\| \\
&\le 2\varepsilon \, \|\dot{z}(t) - \dot{x}(t)\| + 2\varepsilon l \, \|z(t) + l_i(t) - x(t)\| \\
&\quad + l \, \|z(t) - x(t)\| \, \|z(t) + l_i(t) - x(t)\|.
\end{aligned}
$$

So, if C is any constant such that $\|\dot{z}(t) - \dot{x}(t)\| \le C$ for all $t \in [0, \bar{t}]$ (as $\|\dot{z}(t)\| \le M$, and $x(\cdot)$ is Lipschitz on $[0, \bar{t}]$), we get

$$
\langle \dot{z}(t) - \dot{x}(t), z(t) - x(t) \rangle \le 2\varepsilon C + 4\varepsilon l \, \|z(t) - x(t)\| + l\|z(t) - x(t)\|^2 + 4\varepsilon^2 l.
$$

Next, by applying Lemma 3.1 to the function $\|z(\cdot) - x(\cdot)\|^2 + \frac{2\varepsilon C + 4\varepsilon^2 l}{l}$ we get, for all $t \in [0, \bar{t}]$

$$
\|z(t) - x(t)\| \le \left(\frac{4\varepsilon^2 l + 2\varepsilon C}{l} \right)^2 e^{lt} + 4\varepsilon(e^{lt} - 1),
$$

implying that, in view of (3.7) and (3.8),

$$
d_S(x(t)) \le d_S(z(t)) + \|z(t) - x(t)\| \le \left(\frac{4\varepsilon^2 l + 2\varepsilon C}{l} \right)^2 e^{l\bar{t}} + 4\varepsilon e^{l\bar{t}}.
$$

Consequently, by the arbitrariness of ε we conclude that $x(t) \in S$ for every $t \in [0, \bar{t}]$. Moreover, as $x(\bar{t}; x_0) \in S \cap \mathrm{dom}A$, by the same argument as above we find $\hat{t} > 0$

such that for every $t \in [0, \hat{t}]$ (recall Proposition 3.3)

$$x(t + \bar{t}; x_0) = x(t; x(\bar{t}; x_0)) \in S \cap \mathrm{dom}A;$$

that is, $x(t) \in S$ for every $t \in [0, \bar{t} + \hat{t}]$. This proves that $x(t) \in S$ for every $t \geq 0$. Finally, if $x_0 \in \overline{S \cap \mathrm{dom}A}$, we take a sequence $(x_k) \subset S \cap \mathrm{dom}A$ such that $x_k \to x_0$. As we have just shown, for every $k \geq 1$ we have that $x(t; x_k) \in S$ for every $t \geq 0$. Thus, since S is closed, as $k \to +\infty$ we deduce that $x(t; x_0) \in S$ for every $t \geq 0$. $\qquad\square$

The proof of Theorem 3.6 shows actually the following:

Corollary 3.7. *Given a closed set $S \subset \overline{\mathrm{dom}A \cap S}$ and $x_0 \in S \cap \mathrm{dom}A$, we assume that for some $m, r > 0$ such that $\left\| \Pi_{A(x)}(f(x_0)) \right\| \leq m$ it holds*

$$\sup_{\xi \in \mathrm{N}^P_{S_m}(y)} \min_{y^* \in A(y) \cap \mathrm{B}(\theta, m)} \langle \xi, f(y) - y^* \rangle \leq 0 \quad \text{for all } y \in \mathrm{B}(x_0, r).$$

Then there exists $\bar{t} > 0$ such that $x(t; x_0) \in S$ for all $t \in [0, \bar{t}]$.

As we show in the corollary below the criterion of Theorem 3.6 becomes necessary if the maximal monotone operator A has a minimal norm section, which is locally bounded relative to its domain. As typical examples of such operators there are normal cones to closed convex sets, and the subdifferential mapping of convex, lower semi-continuous functions, which are Lipschitz relative to their domains. To fix this concept we say that the operator A is locally minimally bounded on S, if for every $x \in S \cap \mathrm{dom}A$ there exist $m, r > 0$ such that

$$\|A^\circ(y)\| \leq m \quad \text{for all } y \in S \cap \mathrm{dom}A \cap \mathrm{B}(x, r). \tag{3.10}$$

This condition is less restrictive compared with the local boundedness of A relative to S, which means that for every $x \in S \cap \mathrm{dom}A$ there exist $m, r > 0$ such that

$$\|y^*\| \leq m, \, \forall y^* \in Ay, \; y \in S \cap \mathrm{dom}A \cap \mathrm{B}(x, r). \tag{3.11}$$

Obviously every locally bounded operator is locally minimally bounded.

Then the following result gives necessary and sufficient simpler criteria for the invariance of closed sets with respect to differential inclusion (3.1), using the normal cone mapping to S, N_S, which stands for either the proximal normal cone N^P_S or the Fréchet normal cone N^F_S.

Corollary 3.8. *Let $S \subset H$ be a closed set satisfying (3.2). Then the following statements are equivalent, provided that A is locally minimally bounded on S,*

(i) S is an invariant set for (3.1);

(ii) for every $x \in S \cap \mathrm{dom}A$

$$f(x) - \Pi_{A(x)}(f(x)) \in \mathrm{T}_S^B(x);$$

(iii) for every $x \in S \cap \mathrm{dom}A$

$$\sup_{\xi \in \mathrm{N}_S(x)} \langle \xi, f(x) - \Pi_{A(x)}(f(x)) \rangle \leq 0;$$

(iv) for every $x \in S \cap \mathrm{dom}A$ and every $m \geq \left\| f(x) - \Pi_{A(x)}(f(x)) \right\|$

$$\sup_{\xi \in \mathrm{N}_S(x)} \inf_{x^* \in (f(x) - A(x)) \cap \mathrm{B}(\theta, m)} \langle \xi, x^* \rangle \leq 0;$$

and the following assertion, when A is locally bounded relative to S,

(v) for every $x \in S \cap \mathrm{dom}A$

$$\sup_{\xi \in \mathrm{N}_S(x)} \inf_{x^* \in f(x) - A(x)} \langle \xi, x^* \rangle \leq 0.$$

Proof. We fix $x \in S \cap \mathrm{dom}A$. The implication $(iii) \implies (iv)$ is immediate, while the implication $(ii) \implies (iii)$ follows because $\mathrm{T}_S^B(x) \subset (\mathrm{N}_S(x))^*$. In the same line, implication $(i) \Rightarrow (ii)$ follows easily by observing that

$$(f(x) - A(x))^\circ = \frac{d^+ x(\cdot; x)}{dt}(0) = \lim_{t \downarrow 0} \frac{x(t; x) - x}{t} \in \mathrm{T}_S^B(x).$$

Thus, we only need to prove that $(iv) \Rightarrow (i)$. If (iv) holds, by the current local boundedness assumption of A° on $S \cap \mathrm{dom}A$ we pick $m, r > 0$ such that $\|(f(y) - A(y))^\circ\| \leq m$ for all $y \in \mathrm{B}(x, 2r) \cap S \cap \mathrm{dom}A$. Hence,

$$\mathrm{B}(x, 2r) \cap S \cap \mathrm{dom}A = S_m \cap \mathrm{B}(x, 2r),$$

and, since $S = \overline{S \cap \mathrm{dom}A}$, for every $y \in \mathrm{B}(x, r) \cap S_m$,

$$\mathrm{N}_{S_m}(y) = \mathrm{N}_{S_m \cap \mathrm{B}(x, 2r)}(y) = \mathrm{N}_{S \cap \mathrm{dom}A \cap \mathrm{B}(x, 2r)}(y) = \mathrm{N}_{S \cap \mathrm{dom}A}(y) = \mathrm{N}_S(y).$$

So (iv) gives us, for every $y \in \mathrm{B}(x, r) \cap S_m$,

$$\sup_{\xi \in \mathrm{N}_{S_m}(y)} \inf_{x^* \in (f(y) - A(y)) \cap \mathrm{B}(\theta, m)} \langle \xi, x^* \rangle \leq 0,$$

and (i) follows, according to Theorem 3.6.

Suppose now that A is locally bounded on $S \cap \mathrm{dom}A$, and consider the intermediate assertion

$(iv)'$ for every $x \in S \cap \mathrm{dom}A$ and every large enough $m \geq \|(f(x) - A(x))^\circ\|$ we have that

$$\sup_{\xi \in \mathrm{N}_S(x)} \inf_{x^* \in (f(x) - A(x)) \cap \mathrm{B}(\theta, m)} \langle \xi, x^* \rangle \leq 0.$$

As we see from the proof above (namely, the implication $(iv) \Rightarrow (i)$), we have that $(iv)' \Rightarrow (i)$, so that $(v) \Rightarrow (iv)' \Rightarrow (i)$. The proof of the corollary is finished because the implication $(iv) \implies (v)$ is immediate. $\qquad \square$

In the following corollary we deduce another sufficient condition for the invariance of closed sets, using the Moreau-Yoshida approximations of A. Observe that we do not require here that set S satisfies condition (3.2).

Corollary 3.9. *Given a closed set $S \subset H$, we suppose that for every bounded subsets B of S*

$$\liminf_{\lambda \downarrow 0} \sup_{y \in B} \sup_{\xi \in \mathrm{N}_S^P(y)} \langle \xi, f(y) - A_\lambda(y) \rangle \leq 0.$$

Then S is invariant set for (3.1).

Proof. Fix an $x \in S$ and let $x(\cdot; x)$ be the corresponding solution of (3.1). Given an $r > 0$ we let λ_k, $k \geq 1$, be such that $\lambda_k \downarrow 0$ and

$$\sup_{\xi \in \mathrm{N}_S^P(y)} \langle \xi, f(y) - A_{\lambda_k}(y) \rangle \leq 0 \text{ for all } k \geq 1 \text{ and } y \in \mathrm{B}(x, r) \cap S. \qquad (3.12)$$

If $\varepsilon < \frac{r}{4}$ and $\bar{t} > 0$ are such that $x(t; x) \in \mathrm{B}(x, \frac{r}{4})$ for all $t \in [0, \bar{t}]$, then for large enough $k \geq 1$ the solution $x_{\lambda_k}(\cdot; x)$ of the differential equation $\dot{x}(t) = f(x(t)) - A_{\lambda_k}(x(t))$, $x(0) = x$, satisfies (see Proposition 3.3)

$$\|x(t; x) - x_{\lambda_k}(t; x)\| \leq \varepsilon < \frac{r}{4}; \qquad (3.13)$$

hence, $x_{\lambda_k}(t; x) \in \mathrm{B}(x, \frac{r}{2})$ for all $t \in [0, \bar{t}]$. On the other hand, since A_{λ_k} is Lipschitz continuous, for large enough $m > 0$ we have $\mathrm{B}(x, r) \cap S = \{z \in \mathrm{B}(x, r) \cap S \mid$

$\|A_{\lambda_k}(z)\| \le m\}$. So, according to Corollary 3.7, (3.12) ensures that for some $\hat{t} > 0$, say $\hat{t} \in (0, \bar{t})$, it holds $x_{\lambda_k}(t; x) \in S$ for all $t \in [0, \hat{t}]$. Since $x_{\lambda_k}(t; x) \in \mathrm{B}(x, \frac{r}{2})$ for all $t \in [0, \bar{t}]$, we infer that $x_{\lambda_k}(t; x) \in \mathrm{B}(x, \frac{r}{2}) \cap S$ for all $t \in [0, \bar{t}]$. Consequently, by (3.13) we get $d_S(x(t; x)) \le \varepsilon$ for all $t \in [0, \bar{t}]$. Then, as $\varepsilon \to 0$, we deduce that $x(t; x) \in S$ for all $t \in [0, \bar{t}]$. Finally, the invariance of S follows by using the semi-group property of the solution $x(\cdot; x)$ (see again Proposition 3.3). $\qquad \square$

We consider now the special case where $f \equiv \theta$, so that our differential inclusion (3.1) takes the simpler form

$$\dot{x}(t) \in -A(x(t)), \quad x(0) = x_0 \in \overline{\mathrm{dom}A}. \tag{3.14}$$

In this case, the criterion of Theorem 3.6 becomes also necessary as the following corollary shows. Here too N_{S_m} stands for either $\mathrm{N}_{S_m}^P$ or $\mathrm{N}_{S_m}^F$.

Corollary 3.10. *Let $S \subset H$ be a closed set satisfying (3.2). Then the following statements are equivalent:*

(i) S is an invariant set of (3.14);

(ii) for every $x \in S \cap \mathrm{dom}A$

$$-A^{\circ}(x) \in \mathrm{T}_{S_m}^B(x) \;\; \textit{for all } m \ge \|A^{\circ}(x)\|\,;$$

(iii) for every $x \in S \cap \mathrm{dom}A$ and for every $m \ge \|A^{\circ}(x)\|$

$$\sup_{\xi \in \mathrm{N}_{S_m}(x)} \langle \xi, -A^{\circ}(x) \rangle \le 0;$$

(iv) for any $x \in S \cap \mathrm{dom}A$ and every $m \ge \|A^{\circ}(x)\|$

$$\sup_{\xi \in \mathrm{N}_{S_m}(x)} \; \inf_{x^* \in (-A(x)) \cap \mathrm{B}(\theta, m)} \langle \xi, x^* \rangle \le 0.$$

Proof. As in the proof of Corollary 3.8, the implications $(ii) \implies (iii)$ and $(iii) \implies (iv$ with $\mathrm{N}_{S_m} = \mathrm{N}_{S_m}^F) \implies (iv$ with $\mathrm{N}_{S_m} = \mathrm{N}_{S_m}^P)$ are immediate. For the implication $(i) \implies (ii)$, we assume that S is an invariant set of (3.14). If $x \in S \cap \mathrm{dom}A$, then for a given $m \ge \|A^{\circ}(x)\|$ we have

$$\|A^{\circ}(x(t; x))\| = \left\| \frac{d^+ x(t; x)}{dt} \right\| \le \left\| \frac{d^+ x(0; x)}{dt} \right\| = \|A^{\circ}(x)\| \le m, \;\text{ for all } t \ge 0.$$

Hence, $x(t; x) \in S_m$ for all $t \ge 0$ and we deduce that $-A^{\circ}(x) = \frac{d^+ x(0; x)}{dt} \in \mathrm{T}_{S_m}^B(y)$,

yielding (ii). Finally, the implication (iv with $\mathrm{N}_{S_m} = \mathrm{N}_{S_m}^P$) $\implies$ (i) is direct from Theorem 3.6. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

To show how can our Theorem 3.6 be applied we consider the following example, which is treated in details in [5] in order to study the existence and the stability of solutions of differential inclusions involving the normal cone to a prox-regular set.

Recall that a closed set $C \subset H$ is said to be uniformly r-prox-regular ($r > 0$) if for every $x \in C$ and $\xi \in \mathrm{N}_C^P(x) \cap \mathrm{B}(\theta, 1)$ we have ([73])

$$\langle \xi, y - x \rangle \leq \frac{1}{2r} \|y - x\|^2 \text{ for all } y \in C.$$

Example 3.11. *Let $C \subset H$ be a uniformly r-prox-regular set and consider the associated differential inclusion*

$$\dot{x}(t) \in g(x(t)) - \mathrm{N}_C(x(t)) \text{ a.e. } t \in [0, T], \ x(0) = x_0 \in C, \qquad (3.15)$$

where g is a Lipschitz mapping on H. According to [5, Lemma 6(c)], let $L : H \rightrightarrows H$ be a maximal monotone operator such that for some $m \geq 0$ it holds, for all $y \in C$,

$$\mathrm{N}_C(y) \cap \mathrm{B}(0, m) + \frac{m}{r}y \subset L(y) \subset \mathrm{N}_C(y) + \frac{m}{r}y,$$

and consider the associated differential inclusion

$$\begin{cases} \dot{x}(t) \in g(x(t)) + \frac{m}{r}x(t) - L(x(t)) \text{ a.e. } t \in [0, T], \\ x(0) = x_0 \in C \ (\subset \mathrm{dom}L). \end{cases} \qquad (3.16)$$

This inclusion perfectly fits the form of differential inclusion (3.1). Then we make appeal to Theorem 3.6 to prove that the set C is invariant for (3.1), so that

$$\dot{x}(t) \in g(x(t)) + \frac{m}{r}x(t) - L(x(t)) \subset g(x(t)) - \mathrm{N}_C(x(t)),$$

providing us with a solution for (3.15). We refer to [5] for more details.

3.4   Lyapunov pairs and functions

In this section, we apply the results of the previous section to derive different criteria for a-Lyapunov pairs with respect to differential inclusion (3.1) :

$$\dot{x}(t) \in f(x(t)) - A(x(t)), \ \ t \in [0, \infty), \ x(0) = x_0 \in \overline{\mathrm{dom}A},$$

whose unique solution is written $x(\cdot; x_0)$. Similar criteria to ours have been established recently in [7, 8] in the case of weakly lower semi-continuous Lyapunov pairs.

Definition 3.12. We say that a pair (V, W) of proper lower semi-continuous functions $V, W : H \to \overline{\mathbb{R}}$ with $W \geq 0$, is (or forms) an a-Lyapunov pair ($a \geq 0$) with respect to system (3.1) if, for every $x_0 \in \overline{\mathrm{dom}A}$,

$$e^{at}V(x(t; x_0)) + \int_s^t W(x(\tau; x_0))d\tau \leq e^{as}V(x(s; x_0)), \quad \text{for all } t \geq s \geq 0.$$

Observe that (V, W) is an a-Lyapunov pair with respect to system (3.1) iff for every $x_0 \in \overline{\mathrm{dom}A}$ there exists a $t > 0$ such that (see, e.g., [7, Proposition 3.2])

$$e^{as}V(x(s; x_0)) + \int_0^s W(x(\tau; x_0))d\tau \leq V(x_0), \quad \text{for all } s \in [0, t].$$

We may assume without loss of generality that W is Lipschitz continuous on every bounded set (see, e.g., [7, Lemma 3.1] or [30, Theorem 1.5.1]). While, concerning function V, one need to suppose the following condition

$$V(x) = \liminf_{y \overset{\mathrm{dom}A}{\longrightarrow} x} V(y) \quad \text{for every } x \in \mathrm{dom}V, \tag{3.17}$$

which is in fact necessary for V to be a Lypunov function in many important cases (for instance, when differential inclusion (3.1) possesses a strong solution).

Theorem 3.13. *Given two proper lower semi-continuous functions $V : H \to \overline{\mathbb{R}}$ satisfying (3.17), $W : H \to \overline{\mathbb{R}}_+$, and a real number $a \geq 0$, we assume that for every $x \in \mathrm{dom}V \cap \mathrm{dom}A$ there are $m, r > 0$ such that $\left\| \Pi_{A(x)}(f(x)) \right\| \leq m$ and, for all $y \in \mathrm{B}(x, r)$,*

$$\sup_{\xi \in \partial_P(V + \mathrm{I}_{A_m})(y)} \inf_{y^* \in A(y) \cap \mathrm{B}(\theta, m)} \langle \xi, f(y) - y^* \rangle + aV(x) + W(x) \leq 0.$$

Then (V, W) forms an a-Lyapunov pair with respect to system (3.1).

Proof. We fix $T > 0$ and $x_0 \in \mathrm{dom}V \cap \mathrm{dom}A$. Following the discussion made before the current theorem we may suppose without loss of generality that W is Lipschitz continuous on every bounded set containing the trajectory $\{x(t; x_0), t \in [0, T]\}$. Let us define the maximal monotone operator $\hat{A} : H \times \mathbb{R}^4 \rightrightarrows H \times \mathbb{R}^4$ and the

Lipschitz function $\hat{f} : H \times \mathbb{R}^4 \to H \times \mathbb{R}^4$ as

$$\hat{A}(x,\mu) := (A(x), \theta_{\mathbb{R}^4}), \ \ \hat{f}(x,\mu) := (f(x), 1, 0, 1, 0),$$

and, given a fixed $\mu_0 \in \mathbb{R}^4$, consider the associated differential inclusion given in $H \times \mathbb{R}^4$ by

$$\dot{y}(t) \in \hat{f}(y(t)) - \hat{A}(y(t)) \ \text{ a.e. } t \in [0, T], \quad y(0) = (x_0, \mu_0), \tag{3.18}$$

whose unique solution is $y(t) := (x(t), t, 0, t, 0) + (\theta, \mu_0)$, $t \in [0, T]$ (with $x(t) := x(t; x_0)$).

For each $n \geq 1$, we consider the lower semi-continuous function $V_n : H \times \mathbb{R}^3 \to \overline{\mathbb{R}}$ defined as

$$V_n(x, \alpha, \beta, \gamma) := e^{a\gamma} V(x) + (\alpha - \beta) g_n(\alpha) + \frac{l'}{2}(\alpha - \beta)^2, \tag{3.19}$$

where g_n is an l'-Lipschitz extension of the function $W(x(\cdot; x_0)) - \frac{1}{n}$ from $[0, T]$ to $[-1, T+1]$; hence,

$$\partial_C g_n(\alpha) \subset \mathrm{B}(0, l') \ \text{ for all } \alpha \in [0, T+1]. \tag{3.20}$$

We denote

$$S := \mathrm{epi} V_n,$$

so that $S = \overline{S \cap \mathrm{dom}\hat{A}}$, by (3.17), and

$$\mathrm{epi}(V_n + \mathrm{I}_{A_m \times \mathbb{R}^3}) = S \cap \hat{A}_m =: S_m. \tag{3.21}$$

We also denote $y_0 := (x_0, \theta_{\mathbb{R}^3}, V(x_0)) \in S \cap \mathrm{dom}\hat{A}$. Let $m, r > 0$ be as in the current assumption, corresponding to x_0, and choose $\bar{r} < r$ small enough such that for all $(x, \alpha, \beta, \gamma) \in \mathrm{B}((x_0, \theta_{\mathbb{R}^3}), \bar{r})$

$$g_n(\alpha) - e^{a\gamma} W(x) + 2l' |\alpha - \beta| \leq \frac{-1}{2n}. \tag{3.22}$$

Take $y := (y_1, \mu_1) \in \mathrm{B}(y_0, \bar{r}) \cap S_m$, with $y_1 := (x_1, \alpha_1, \beta_1, \gamma_1)$, and pick $(\xi, -\kappa) \in \mathrm{N}_{S_m}^P(y)$. Due to (3.21) and [30, Exercise 1.2.1],

$$(\xi, -\kappa) \in \mathrm{N}_{S_m}^P(y) = \mathrm{N}_{\mathrm{epi}(V_n + \mathrm{I}_{A_m \times \mathbb{R}^3})}^P(y) \subset \mathrm{N}_{\mathrm{epi}(V_n + \mathrm{I}_{A_m \times \mathbb{R}^3})}^P(y_1, V_n(y_1));$$

hence, $\kappa \geq 0$. If $\kappa > 0$, say $\kappa = 1$ for simplicity, then $\xi \in \partial_P(V_n + \mathrm{I}_{A_m \times \mathbb{R}^3})(y_1)$ and, thanks to (3.19), we find $\xi_1 \in \partial_P(V + \mathrm{I}_{A_m})(x_1)$ and $\varsigma \in \partial_P g_n(\alpha_1) \subset \partial_C g_n(\alpha_1)$ such that

$$\xi \in \left(e^{a\gamma_1}\xi_1, g_n(\alpha_1) + (\alpha_1 - \beta_1)(\varsigma + l'), -g_n(\alpha_1) + l'(\beta_1 - \alpha_1), ae^{a\gamma_1}V(x_1)\right).$$

Since $y \in \mathrm{B}(y_0, \bar{r}) \cap S_m$ we have that $x_1 \in \mathrm{B}(x_0, \bar{r}) \cap A_m \cap \mathrm{dom}V$ and, so, by the current assumption, there exists an $x_1^* \in A(x_1) \cap \mathrm{B}(\theta, m)$ (this last set being weak*-compact) such that

$$\langle \xi_1, f(x_1) - x_1^* \rangle + aV(x_1) + W(x_1) \leq 0.$$

Then we obtain (recall (3.20) and (3.22))

$$\begin{aligned}
\langle (\xi, -1), (f(x_1) - x_1^*, 1, 0, 1, 0) \rangle &= \langle e^{a\gamma_1}\xi_1, f(x_1) - x_1^* \rangle + g_n(\alpha_1) \\
&\quad + (\alpha_1 - \beta_1)(\varsigma + l') + ae^{a\gamma_1}V(x_1) \\
&= e^{a\gamma_1}\left(\langle \xi_1, f(x_1) - x_1^* \rangle + aV(x_1) + W(x_1)\right) \\
&\quad + g_n(\alpha_1) - e^{a\gamma_1}W(x_1) + (\alpha_1 - \beta_1)(\varsigma + l') \\
&\leq g_n(\alpha_1) - e^{a\gamma_1}W(x_1) + 2l'|\alpha_1 - \beta_1| \leq \frac{-1}{2n}.
\end{aligned}$$

$$(3.23)$$

If $\kappa = 0$, then thanks to (3.19) we find $\xi_2 \in H$ such that $\xi = (\xi_2, \theta_{\mathbb{R}^3})$, with the property that there are sequences $\lambda_k \downarrow 0$, $z_k \xrightarrow{V + \mathrm{I}_{A_m}} x_1$, $\zeta_k \in \partial_P(V + \mathrm{I}_{A_m})(z_k)$ such that $\lambda_k \zeta_k \to \xi_2$ as $k \to \infty$. By the current assumption, for each large enough k so that $z_k \in \mathrm{B}(x_0, r)$ there exists $z_k^* \in A(z_k) \cap \mathrm{B}(\theta, m)$ such that

$$\langle \zeta_k, f(z_k) - z_k^* \rangle + aV(z_k) + W(z_k) \leq 0.$$

Because A is maximal monotone and $(z_k^*)_k$ is bounded, we can find an $x_2^* \in A(x_1) \cap \mathrm{B}(\theta, m)$ such that $\langle \xi_2, f(x_1) - x_2^* \rangle \leq 0$; hence, by multiplying the last inequality above by λ_k and taking the limit as $k \to \infty$,

$$\langle (\xi, 0), (f(x_1) - x_2^*, 1, 0, 1, 0) \rangle = \langle \xi, f(x_1) - x_2^* \rangle \leq 0. \tag{3.24}$$

According to Corollary 3.7, (3.23) and (3.24) imply the existence of some $\bar{t} :=$

$\bar{t}(n) \in (0, T]$ such that for every $t \in [0, \bar{t}]$,

$$(x(t), t, 0, t, V(x_0)) \in S;$$

in other words, $e^{at}V(x(t)) + tg_n(t) + \frac{l'}{2}t^2 \leq V(x_0)$ and, so, for every $t \in [0, \bar{t}]$

$$
\begin{aligned}
e^{at}V(x(t)) + \int_0^t W(x(\tau))d\tau \ &\leq \ e^{at}V(x(t)) + \int_0^t (g(t) + l'(t-\tau))d\tau + \frac{t}{n} \\
&\leq \ V(x_0) + \frac{t}{n}.
\end{aligned}
\tag{3.25}
$$

Now, we claim that for all $t \in [0, T]$

$$e^{at}V(x(t)) + \int_0^t W(x(\tau))d\tau \leq V(x_0) + \frac{e^{(1+a)t}}{n}. \tag{3.26}$$

To prove this claim we define

$$t^* := \sup\{t \in [0, T] \mid \text{inequality (3.26) holds on } [0, t]\}.$$

Indeed, from (3.25) and the lower semi-continuous of V, it follows that (3.26) holds at t^*. If $t^* < T$, we denote $y^* := (x(t^*), \theta_{\mathbb{R}^3}, V(x(t^*)))$ and we easily check that $y^* \in S \cap \text{dom}\hat{A}$. Then, arguing as with y_0 above, we arrive at a relation which is similar to (3.25); that is, there is some $\hat{t} > 0$ such that for all $t \in [0, \hat{t}]$

$$e^{at}V(x(t; x(t^*))) + \int_0^t W(x(\tau; x(t^*)))d\tau \leq V(x(t^*)) + \frac{t}{n}. \tag{3.27}$$

Hence,

$$
\begin{aligned}
e^{a(t+t^*)}&V(x(t+t^*)) + \int_0^{t+t^*} W(x(\tau))d\tau \\
&\leq e^{a(t+t^*)}V(x(t+t^*)) + \int_0^{t+t^*} W(x(\tau))d\tau + (e^{at^*} - 1)\int_0^t W(x(\tau+t^*))d\tau \\
&= e^{at^*}\left(e^{at}V(x(t+t^*)) + \int_0^t W(x(\tau+t^*))d\tau - \frac{t}{n}\right) + \int_0^{t^*} W(x(\tau))d\tau + \frac{e^{at^*}t}{n} \\
&\leq e^{at^*}V(x(t^*)) + \int_0^{t^*} W(x(\tau))d\tau + \frac{e^{at^*}t}{n} \\
&\leq V(x_0) + \frac{e^{(1+a)t^*}}{n} + \frac{e^{at^*}t}{n}
\end{aligned}
$$

Consequently, due to the inequality $e^\gamma \geq 1 + \gamma$, we obtain that for all $t \in [0, \hat{t}]$

$$e^{a(t+t^*)}V(x(t+t^*)) + \int_0^{t+t^*} W(x(\tau))d\tau \leq V(x_0) + \frac{e^{(1+a)(t+t^*)}}{n},$$

leading us to a contradiction with the definition of t^*.

Now, the claim being true, we take the limit in (3.26) as n goes to $+\infty$ to obtain that

$$e^{at}V(x(t)) + \int_0^t W(x(\tau))d\tau \leq V(x_0) \text{ for all } t \in [0, T].$$

Finally, if $x_0 \in \operatorname{dom}V$, then by the current assumption (3.17), there exists a sequence $(x_k)_{k \geq 1} \subset \operatorname{dom}V \cap \operatorname{dom}A$ such that $x_k \overset{V}{\to} x_0$. Thus, from the last inequality above we conclude that

$$e^{at}V(x(t; x_k)) + \int_0^t W(x(\tau; x_k))d\tau \leq V(x_k) \text{ for all } t \in [0, T] \text{ and all } k \geq 1.$$

Hence, as k goes to $+\infty$, the lower semi-continuous of V and Proposition 3.3 ensure that

$$e^{at}V(x(t; x_0)) + \int_0^t W(x(\tau; x_0))d\tau \leq V(x_0) \text{ for all } t \in [0, T],$$

showing that (V, W) is an a-Lyapunov pair. $\qquad \square$

As in the case of the invariance of closed sets, the criterion of Theorem 3.13 takes a more simpler form when the maximal monotone operator A, or its minimal norm section, A°, is locally bounded (see (3.10)). Here, ∂V stands for either $\partial_P V$ or $\partial_F V$.

Corollary 3.14. *Given two proper lower semi-continuous functions $V, W : H \to \overline{\mathbb{R}}$, such that $W \geq 0$ and (3.17) holds, and a number $a \geq 0$, we assume that A is minimally locally bounded relative to $\operatorname{dom}V$. Then the following statements are equivalent.*

(i) (V, W) is an a-Lyapunov pair for (3.1);

(ii) for any $x \in \operatorname{dom}V \cap \operatorname{dom}A$

$$\sup_{\xi \in \partial V(x)} \langle \xi, (f(x) - A(x))^\circ \rangle + aV(x) + W(x) \leq 0;$$

(iii) for any $x \in \operatorname{dom}V \cap \operatorname{dom}A$

$$V'(x; (f(x) - A(x))^\circ) + aV(x) + W(x) \leq 0;$$

Moreover, if in addition, (3.11) holds, then the above statements are also equivalent to

(iv) *for any* $x \in \mathrm{dom}V \cap \mathrm{dom}A$

$$\sup_{\xi \in \partial V(x)} \inf_{x^* \in A(x)} \langle \xi, f(x) - x^* \rangle + aV(x) + W(x) \le 0;$$

(v) *for any* $x \in \mathrm{dom}V \cap \mathrm{dom}A$

$$\inf_{v \in A(x)} V'(x; f(x) - v) + aV(x) + W(x) \le 0.$$

Proof. First, the implications (iii)(with $\partial = \partial_F$)$\Rightarrow$ (iii)(with $\partial = \partial_P$) $\Rightarrow$ (ii) follow since that $\partial_P \subset \partial_F$ and $\text{œ}_{\partial_F V(x)} \le V'(x; \cdot)$.

$(i) \Rightarrow (iii)$. Fix $x_0 \in \mathrm{dom}V \cap \mathrm{dom}A$. Since (V, W) is an a-Lyapunov for (3.1), we have that for all $t > 0$

$$\frac{V(x(t; x_0)) - V(x_0)}{t} + \frac{e^{at} - 1}{t}V(x(t; x_0)) + \frac{1}{t}\int_0^t W(x(\tau; x_0))d\tau \le 0,$$

while Proposition 3.5 ensures that

$$\lim_{t\downarrow 0} \frac{x(t; x_0) - x_0}{t} = \frac{d^+ x(0; x_0)}{dt} = (f(x_0) - A(x_0))°.$$

Hence, using the lower semi-continuous of V together with the continuity of $x(\cdot; x_0)$,

$$V'(x_0; (f(x_0) - A(x_0))°) \le \liminf_{t\downarrow 0} \frac{V(x(t; x_0)) - V(x_0)}{t} \le -aV(x_0) - W(x_0),$$
$$(3.28)$$

leading us to (ii).

(ii)(with $\partial = \partial_P$) $\Rightarrow$ (i). We fix $x_0 \in \mathrm{dom}V \cap \mathrm{dom}A$. From the one hand, by the boundedness assumption of $A°$, for a large $m \ge 0$ there exists an $r > 0$ such that

$$\mathrm{B}(x_0, r) \cap \mathrm{dom}V \cap \mathrm{dom}A \subset A_m. \qquad (3.29)$$

On the other hand, we have that

$$\partial_P(V + \mathrm{I}_{A_m})(x) \subset \partial_P V(x) \quad \text{for all } x \in \mathrm{B}(x_0, \frac{r}{2}). \qquad (3.30)$$

Indeed, if $\xi \in \partial_P(V + I_{A_m})(x)$ for $x \in \mathrm{B}(x_0, \frac{r}{2})$, there exist $\delta > 0$ and $\rho \in (0, \frac{r}{2})$ such that

$$(V + \mathrm{I}_{A_m})(z) \ge V(x) + \langle \xi, z - x \rangle - \delta\|z - x\|^2 \quad \forall z \in \mathrm{B}(x, \rho).$$

Take $z \in \mathrm{B}(x, \frac{\rho}{4}) \cap \mathrm{dom}V \ (\subset \mathrm{B}(x_0, r))$. By (3.17) together with (3.29), there exists a sequence $(z_n)_n \subset \mathrm{B}(x, \rho) \cap \mathrm{dom}V \cap A_m$ such that $z_n \to z$ and $V(z_n) \to V(z)$. Since each z_n satisfies the last inequality above, by taking the limit as $n \to \infty$ we arrive at $V(z) \geq V(x) + \langle \xi, z - x \rangle - \delta \|z - x\|^2$ and the inclusion (3.30) follows.

At this stage, from (3.29) and the Lipschitzianity of f there exists some $M \geq m$ such that, for all $x \in \mathrm{B}(x_0, r)$,

$$\left\| \Pi_{A(x)}(f(x)) \right\| \leq \|f(x)\| + \|A^\circ(x)\| \leq \|f(x)\| + m \leq M,$$

which shows that $(f(x) - A(x))^\circ \in f(x) - A(x) \cap \mathrm{B}(\theta, M)$. Since $\partial_P(V + \mathrm{I}_{A_M}) \subset \partial_P(V + \mathrm{I}_{A_m})$, in view of (3.30), assumption (ii)(with $\partial = \partial_P$) implies that, for every $x \in \mathrm{B}(x_0, \frac{r}{2})$

$$\sup_{\xi \in \partial_P(V + \mathrm{I}_{A_M})(x)} \inf_{x^* \in A(x) \cap \mathrm{B}(\theta, M)} \langle \xi, f(x) - x^* \rangle + aV(x) + W(x) \leq$$

$$\sup_{\xi \in \partial_P V(x)} \langle \xi, (f(x) - A(x))^\circ \rangle + aV(x) + W(x) \leq 0.$$

Thus, (i) follows from Theorem 3.13.

Finally, if A is locally bounded on $\mathrm{dom}V$, then from the first part of the proof one only needs to verify the implication $(iv) \implies (i)$, the proof of which is similar to the one of "$(ii) \Rightarrow (i)$" that we did above. $\qquad \square$

In the following corollary we provide criteria for a-Lyapunov pairs, which use the Moreau-Yoshida approximation of A.

Corollary 3.15. *Let V, W and a be as in Corollary 3.14, and let ∂ be such that $\partial_P \subset \partial \subset \partial_C$. If there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0]$*

$$\sup_{\xi \in \partial V(x)} \langle \xi, f(x) - A_\lambda(x) \rangle + aV(x) + W(x) \leq 0 \ \ \forall x \in \mathrm{dom}V,$$

then (V, W) is an a-Lyapunov pair for (3.1).

Proof. Fix $x_0 \in \mathrm{dom}V$ and $t \geq 0$. If $x_\lambda(\cdot; x_0)$ is the solution of the differential equation

$$\dot{x}_\lambda(t) = f(x_\lambda(t)) - A_\lambda(x_\lambda(t)), \ \ x_\lambda(0) = x_0 \ (\lambda \in (0, \lambda_0]), \qquad (3.31)$$

then, according to Corollary 3.14(ii), the pair (V, W) is an a-Lyapunov pair of (3.31); that is,

$$e^{at}V(x_\lambda(t)) + \int_0^t W(x_\lambda(\tau))d\tau \leq V(x_0) \text{ for all } t \geq 0.$$

Hence, the conclusion follows as $\lambda \downarrow 0$. $\qquad\qquad\square$

We consider now the case when $f \equiv 0$ so that differential inclusion (3.1) reads

$$\dot{x}(t) \in -A(x(t)), \ x(0) = x_0 \in \overline{\mathrm{dom}A}. \qquad (3.32)$$

In the following theorem ∂ stands for either ∂_P or ∂_F.

Corollary 3.16. *Let $V, W : H \to \overline{\mathbb{R}}$ be two proper lower semi-continuous functions, such that $W \geq 0$ and (3.17) holds, and let $a \geq 0$. Then the following statements are equivalent:*

(i) *(V, W) is an a-Lyapunov pair for (3.32);*

(ii) *for every $x \in \mathrm{dom}V \cap \mathrm{dom}A$ and every $m \geq \|A^\circ(x)\|$*

$$\sup_{\xi \in \partial(V + \mathrm{I}_{A_m})(x)} \langle \xi, -A^\circ(x) \rangle + aV(x) + W(x) \leq 0;$$

(iii) *for every x and m as in (ii)*

$$\sup_{\xi \in \partial(V + \mathrm{I}_{A_m})(x)} \inf_{x^* \in -A(x) \cap \mathrm{B}(\theta, m)} \langle \xi, x^* \rangle + aV(x) + W(x) \leq 0;$$

(iv) *for every x and m as in (ii)*

$$(V + \mathrm{I}_{A_m})'(x; -A^\circ(x)) + aV(x) + W(x) \leq 0;$$

(v) *for every x and m as in (ii)*

$$\inf_{v \in -A(x) \cap \mathrm{B}(\theta, m)} (V + \mathrm{I}_{A_m})'(x; v) + aV(x) + W(x) \leq 0.$$

Proof. The implications $(ii) \Rightarrow (iii)$, $(iv) \Rightarrow (v)$, $(iv) \Rightarrow (ii)$, and $(v) \Rightarrow (iii)$ are immediate. To prove that $(i) \Rightarrow (iv)$, we fix $x_0 \in \mathrm{dom}V \cap \mathrm{dom}A$ and $m \geq \|A^\circ(x_0)\|$. According to Proposition 3.3, for any $t \geq 0$ we have that

$$\|-A^\circ(x(t; x_0))\| = \left\| \frac{d^+ x(t; x_0)}{dt} \right\| \leq \left\| \frac{d^+ x(0; x_0)}{dt} \right\| = \|-A^\circ(x_0)\| \leq m;$$

that is, $x(t, x_0) \in A_m$ for all $t \geq 0$. Hence, since $\frac{x(t, x_0) - x_0}{t} \to -A^\circ(x_0)$ as $t \downarrow 0$, provided that (V, W) is an a-Lyapunov pair for (3.32) we obtain, by arguing as in

the proof of (3.28),

$$
\begin{aligned}
(V + \mathrm{I}_{A_m})'(x_0; -A^\circ(x_0)) &\leq \liminf_{t \downarrow 0} \frac{(V + \mathrm{I}_{A_m})(x(t; x_0)) - (V + \mathrm{I}_{A_m})(x_0)}{t} \\
&= \liminf_{t \downarrow 0} \frac{V(x(t; x_0)) - V(x_0)}{t} \leq -aV(x) - W(x),
\end{aligned}
$$

giving rise to (iv).

Finally, the conclusion of the corollary follows because the implication $(iii) \Rightarrow (i)$ holds according to Theorem 3.13. $\qquad \square$

We obtain the following corollary, which can be find in [54]; the original version of this result was established in [70]

Corollary 3.17. *Let* $V, W : H \to \overline{\mathbb{R}}$ *be two proper lower semi-continuous functions, such that* $W \geq 0$, *and let* $a \geq 0$. *If condition* (3.17) *and, for every* $x \in \mathrm{dom}V$,

$$
\liminf_{\lambda \downarrow 0} \frac{V(J_\lambda(x)) - V(x)}{\lambda} + aV(x) + W(x) \leq 0,
$$

then (V, W) *is an a-Lyapunov pair for* (3.32).

Proof. We fix $x \in \mathrm{dom}V \cap A_m$ for some large $m \geq 1$. Since $A_\lambda(x) \in A(J_\lambda x)$ and $\|A_\lambda(x)\| \leq \|A^\circ(x)\| \leq m$, we infer that $J_\lambda(x) \in A_m$ and, so, using the current assumption,

$$
(V + \mathrm{I}_{A_m})'(x_0; -A^\circ(x_0)) \leq \liminf_{t \downarrow 0} \frac{V(J_\lambda(x)) - V(x)}{t} \leq -aV(x) - W(x).
$$

The conclusion follows then from Corollary 3.16(iv). $\qquad \square$

Corollary 3.14 obviously covers the case when A is the null operator, where (3.1) becomes a usual differential equation stated in the Hilbert space H as

$$
\dot{x}(t) = f(x(t)) \text{ a.e. } t \geq 0, \quad x(0) = x_0 \in H. \tag{3.33}
$$

The following characterization is known when ∂ is the viscosity subdifferential as defined in [54, Definition 2.7], while the case of weakly lower semi-continuous a-Lyapunov pairs can be found in [7].

Corollary 3.18. *Let V, W and a be as in Corollary 3.14, and let ∂ be such that $\partial_P \subset \partial \subset \partial_C$. Then the following statements are equivalent:*

(i) *(V, W) is an a-Lyapunov pair for differential equation (3.33),*

(ii) *for every $x \in \text{dom}V$*

$$\sup_{\xi \in \partial V(x)} \langle \xi, f(x) \rangle + aV(x) + W(x) \leq 0, \tag{3.34}$$

(iii) *for every $x \in \text{dom}V$*

$$V'(x; f(x)) + aV(x) + W(x) \leq 0.$$

Proof. In view of Corollary 3.14, we only need to check that $(i) \implies (ii)$ (with $\partial = \partial_C$), and this easily follows from the relation $\partial_C V = \overline{\text{co}}(\partial_L V + \partial_\infty V)$. Indeed, assume that (i) holds and take $\xi \in \partial_L V(x)$ and $\zeta \in \partial_\infty V(x)$. By the definition of $\partial_L V(x)$ we choose sequences $\xi_k \in \partial_P V(x_k)$ such that $x_k \xrightarrow{V} x$ and $\xi_k \rightharpoonup \xi$. Then, by (i),

$$\langle \xi_k, f(x_k) \rangle + aV(x_k) + W(x_k) \leq 0 \text{ for all } k \geq 1,$$

and, so, as $k \to \infty$, we deduce that $\langle \xi, f(x) \rangle + aV(x) + W(x) \leq 0$. Similarly, we choose sequences $x_k \xrightarrow{V} x$ and $\lambda_k \downarrow 0$ such that $\zeta_k \in \partial_P V(x_k)$ and $\lambda_k \zeta_k \rightharpoonup \zeta$. Then, by arguing as above we deduce that $\langle \zeta, f(x) \rangle \leq 0$, which in turn yields

$$\langle \xi + \zeta, f(x) \rangle + aV(x) + W(x) \leq 0,$$

and this gives us (ii) (with $\partial = \partial_C$) by convexification. $\square$

We close this section by analyzing a typical example of Lyapunov pairs.

Example 3.19. Assume that a function $V : H \to \overline{\mathbb{R}}$ is a proper, convex and lower semi-continuous, and consider the differential inclusion

$$\dot{x}(t) \in -\partial V(x(t)).$$

Then the pair $(V, \|(\partial V)^\circ\|^2)$ is a Lyapunov pair, so that for every $x_0 \in \overline{\text{dom}V}$

$$V(x(t; x_0)) + \int_0^t \|\dot{x}(\tau; x_0)\|^2 d\tau \leq V(x_0) \text{ for all } t > 0.$$

To see this fact we fix $x \in \text{dom}A \cap \text{dom}\partial V$. Since $A_\lambda(x) \in A(J_\lambda(x))$ for every $\lambda > 0$

$(A = \partial V)$, condition (3.17) holds and one has that

$$V(J_\lambda(x)) - V(x) \leq -\langle A_\lambda(x), x - J_\lambda(x) \rangle = -\frac{1}{\lambda} \|x - J_\lambda(x)\|^2.$$

Hence,

$$\liminf_{\lambda \downarrow 0} \frac{V(J_\lambda(x)) - V(x)}{\lambda} + \|A^\circ(x)\|^2$$
$$\leq \liminf_{\lambda \downarrow 0} \left(\frac{V(J_\lambda(x)) - V(x)}{\lambda} + \frac{1}{\lambda^2} \|x - J_\lambda(x)\|^2 \right) \leq 0 \quad,$$

and Corollary 3.17 (together with Proposition 3.3) applies.

3.5 Conclusion and further research

We gave different conditions for the invariance of closed sets, which only involve the input data, represented by the maximal monotone operator and the Lipschitz mapping. These conditions are applicable to a large variety of closed sets which do not need to be convex or smooth. The current work extends and improves some of the results given in [7, 8] and dealing with weakly closed invariant sets and weakly lower semi-continuous a-Lypunov pairs. It will be our aim in a forthcoming work to apply the current results to specific differential equations/inclusions where the underlying maximal monotone operator is not known explicitly. This will make the access to the corresponding semi-group more easier, namely regarding the behavior at infinity of trajectories.

Chapter 4

A convex approach to differential inclusions with prox-regular sets: Stability analysis and observer design

We study the existence and stability of solutions for differential inclusions governed by the normal cone to a prox-regular set and subject to a Lipschitz perturbation. We prove that such, apparently, more general systems can be indeed remodeled into the classical theory of differential inclusions involving maximal monotone operators. This result is new in the literature and permits us to make use of the rich and abundant achievements in this class of monotone operators to derive the desired existence result and stability analysis, as well as the continuity and differentiability properties of the solutions. This going back and forth between these two models of differential inclusions is made possible thanks to a viability result for maximal monotone operators. As an application, we study a Luenberger-like observer, which is shown to converge exponentially to the actual state when the initial value of the state's estimation remains in a neighborhood of the initial value of the original system.

4.1 Introduction

We consider in this paper the existence and stability of solutions problem of the following differential inclusion, given in a Hilbert space H,

$$\begin{cases} \dot{x}(t) \in f(x(t)) - \mathrm{N}_C(x(t)) & \text{for almost every } t \geq 0, \\ x(0; x_0) = x_0 \in C, \end{cases} \quad (4.1)$$

where N_C is the normal cone to an r-uniformly prox-regular closed subset C of H. The dynamical system driven by the set C is subject to a l-Lipschitz continuous perturbation mapping f defined on H. By a solution of (4.1) we mean an absolutely continuous function $x(\cdot; x_0) : [0, +\infty) \to H$, with $x(0; x_0) = x_0$, which satisfies (4.1) for almost every (a.e.) $t \geq 0$; hence, in particular, $x(t) \in C$ for all $t \geq 0$. Indeed, such a solution is necessarily Lipschitz continuous on each interval of the form $[0, T]$ for $T \geq 0$ (see Theorem 4.14). Differential inclusion (4.1) appears in the modeling of many concrete problems in economics, unilateral mechanics, electrical engineering as well as optimal control (see eg. [1, 33, 62, 82] and references therein.)

It was recently shown in [63] and [62] that (4.1) has one and only one (absolutely continuous) solution, which satisfies the imposed initial condition. These authors employed a regularization approach based on the Moreau-Yosida approximation, and use the nice properties of uniform prox-regularity to show that the approximate scheme converges to the required solution. In this way, such an approach repeats those arguments of approximation ideas which, previously, were extensively used in the setting of differential inclusions with maximal monotone operators.

Problems dealing with the stability of solutions of (4.1), namely the characterization of weakly lower semi-continuous Lyapunov pairs and functions, have been developed in [61] following the same strategy, also based on Moreau-Yosida approximations. Most of works on these problems use indeed this natural approximation approach; see, e.g. [61–63].

In this paper, at a first glance we provide a different, but quite direct, approach to tackle this problem. We prove that problem (4.1) can be equivalently written as a differential inclusion given in the current Hilbert setting under the form

$$\begin{cases} \dot{x}(t) \in g(x(t)) - A(x(t)) & \text{a.e.} \quad t \in [0, T], \\ x(0; x_0) = x_0 \in \mathrm{dom}A, \end{cases} \quad (4.2)$$

where $A : H \rightrightarrows H$ is an appropriate maximal monotone operator defined on H,

and $g : H \to H$ is a Lipschitz continuous mapping. Then, it will be sufficient to apply the classical theory of maximal monotone operators ([21]; see, also, [7, 8]) to analyze the existence and the stability of solutions for differential inclusion (4.1). The concept of invariant sets will be the key tool to go back and forth between inclusions (4.1) and (4.2). Invariant sets with respect to differential inclusions governed by maximal monotone operators have been studied and characterized in [7, 8]. Other references for invariant sets, also referred to as viable sets, and the related theory of Lyapunov stability are [12, 26, 54, 70] among others. We also refer to [39] for an interesting criterion for weakly invariant sets, which is established in the finite-dimensional setting for differential inclusions governed by one side-Lipschitz multivalued mappings with nonempty convex and compact values. This result has been used in [33], always in finite dimensions, to provide weakly and strongly invariance criteria for closed sets with respect to more general differential inclusions where the set C in (4.1) is time dependent and f is a Lipschitzian multivalued mapping.

We shall also provide different criteria for the so-called a-Lyapunov pairs of lower semi-continuous functions to extend some of the results given in [7, 8, 61] to the current setting. It is worth to observe that the assumption of uniformly prox-regularity is required to obtain global solutions of (4.1), which are defined on the whole interval $[0, T]$. However, our analysis also works in the same way when the set C is prox-regular at x_0 rather than being a uniformly prox-regular set; but, in this case, we only obtain a local solution defined around x_0.

This paper is organized as follows. After giving the necessary notations and preliminary results in Section 2, we review and study in Section 3 different aspects of the theory of differential inclusions governed by maximal monotone operators, including the existence of solutions, and we provide a stability results dealing with the invariance of closed sets with respect to such differential inclusions. In Sections 4, we provide the new proof of the existence of solutions for differential inclusions involving normal cones to r-uniformly prox-regular sets. Section 5 is devoted to the characterization of lower semi-continuous a-Lyapunov pairs and functions. Inspired from the recent paper [78], we give in section 6 an application of our result to a Luenberger-like observer.

4.2 Preliminaries and examples

4.2.1 Preliminary results

In this paper, H is a Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and an associated norm $||\cdot||$. The strong and weak convergences in H are denoted by $\to$ and $\rightharpoonup$, resp. We denote by $\mathrm{B}(x, \rho)$ the closed ball centered at $x \in H$ of radius $\rho > 0$, and particularly we use $\mathbb{B}$ for the closed unit ball. The null vector in H is written θ. Given a set $S \subset H$, by $\mathrm{co}S$, $\mathrm{cone}S$ and $\overline{S}$ we respectively denote the *convex hull, the conic hull* and the *closure* of S. The dual cone of S is the set

$$S^* := \{x^* \in H \mid \langle x^*, x \rangle \leq 0 \text{ for all } x \in S\}.$$

The *indicator* and the *distance functions* are respectively given by

$$\mathrm{I}_S(x) := 0 \text{ if } x \in S; +\infty \text{ otherwise}, \quad d_S(x) := \inf\{||x - y||\colon y \in S\}$$

(in the sequel we shall adopt the convention $\inf_\emptyset = +\infty$). We shall write $\overset{S}{\to}$ for the convergence when restricted to the set S. For $\delta \geq 0$, we denote Π_S^δ the (*orthogonal) δ-projection mapping* onto S defined as

$$\Pi_S^\delta(x) := \{y \in S : \ ||x - y||^2 \leq d_S^2(x) + \delta^2\}.$$

For $\delta = 0$, we simply write $\Pi_S(x) := \Pi_S^0(x)$. It is known that Π_S is nonempty-valued on a dense subset of $H \setminus S$ ([29]).

For an extended real-valued function $\varphi : H \to \overline{\mathbb{R}}$, we denote $\mathrm{dom}\varphi := \{x \in H \mid \varphi(x) < +\infty\}$ and $\mathrm{epi}\varphi := \{(x, \alpha) \in H \times \mathbb{R} \mid \varphi(x) \leq \alpha\}$. Function φ is lower semi-continuous if $\mathrm{epi}\varphi$ is closed. The *contingent directional derivative* of φ at $x \in \mathrm{dom}\varphi$ in the direction $v \in H$ is

$$\varphi'(x, v) := \liminf_{t \to 0^+, w \to v} \frac{\varphi(x + tw) - \varphi(x)}{t}.$$

A vector $\xi \in H$ is called a *proximal subgradient* of φ at $x \in H$, written $\xi \in \partial_P \varphi(x)$, if there are $\rho > 0$ and $\sigma \geq 0$ such that

$$\varphi(y) \geq \varphi(x) + \langle \xi, y - x \rangle - \sigma ||y - x||^2 \ \forall \ y \in B_\rho(x);$$

a *Fréchet subgradient* of φ at x, written $\xi \in \partial_F \varphi(x)$, if

$$\varphi(y) \geq \varphi(x) + \langle \xi, y - x \rangle + o(\|y - x\|) \ \forall \ y \in H;$$

and a *basic (or Limiting) subdifferential* of φ at x, written $\xi \in \partial_L \varphi(x)$, if there exist sequences $(x_k)_k$ and $(\xi_k)_k$ such that

$$x_k \xrightarrow{\varphi} x, \ (\text{i.e., } x_k \to x \text{ and } \varphi(x_k) \to \varphi(x)), \ \xi_k \in \partial_P \varphi(x_k), \ \xi_k \rightharpoonup \xi.$$

If $x \notin \operatorname{dom}\varphi$, we write $\partial_P \varphi(x) = \partial_F \varphi(x) = \partial_L \varphi(x) = \emptyset$. In particular, if S is a closed set and $s \in S$, we define the *proximal normal cone* to S at s as $\mathrm{N}_S^P(s) = \partial_P \mathrm{I}_S(s)$, the *Fréchet normal* to S at s as $\mathrm{N}_S^F(s) = \partial_F \mathrm{I}_S(s)$, the *limiting normal cone* to S at s as $\mathrm{N}_S^F(s) = \partial_F \mathrm{I}_S(s)$, and *the Clarke normal cone* to S at s as $\mathrm{N}_S^C(s) = \overline{\mathrm{co}}(\mathrm{N}_S^L(s))$. Equivalently, we have that $\mathrm{N}_S^P(s) = \operatorname{cone}(\Pi_S^{-1}(s) - s)$, where $\Pi_S^{-1}(s) := \{x \in H \mid s \in \Pi_S(x)\}$. The Bouligand and weak *Bouligand tangent cones to S at x* are defined as

$$\mathrm{T}_S^B(x) := \left\{ v \in H \mid \exists \ x_k \in S, \exists \ t_k \to 0, \ \mathrm{st} \ t_k^{-1}(x_k - x) \to v \text{ as } k \to +\infty \right\}$$
$$\mathrm{T}_S^{\mathrm{w}}(x) := \left\{ v \in H \mid \exists \ x_k \in S, \exists \ t_k \to 0, \ \mathrm{st} \ t_k^{-1}(x_k - x) \rightharpoonup v \text{ as } k \to +\infty \right\}, \ \text{resp.}$$

We also define the *Clarke subgradient* of φ at x, written $\partial_C \varphi(x)$, as the vectors $\xi \in H$ such that $(\xi, -1) \in \mathrm{N}_{\mathrm{epi}\varphi}^C(x, \varphi(x))$, and the *singular subgradient* of φ at x, written $\partial_\infty \varphi(x)$, as the vectors $\xi \in H$ such that $(\xi, 0) \in \mathrm{N}_{\mathrm{epi}\varphi}^P(x, \varphi(x))$; in particular, if $\xi \in \partial_\infty \varphi(x)$, then there are sequences $x_k \xrightarrow{\varphi} x$, $\xi_k \in \partial_P \varphi(x_k)$, and $\lambda_k \to 0^+$ such that $\lambda_k \xi_k \to \xi$. Observe that $\partial_P \varphi(x) \subset \partial_F \varphi(x) \subset \partial_L \varphi(x) \subset \partial_C \varphi(x)$. For all these concepts and their properties we refer to the book [64].

We shall frequently use the following version of Gronwall's Lemma:

Lemma 4.1. (Gronwall's Lemma; see, e.g., [2]) Let $T > 0$ and $a, b \in L^1(t_0, t_0 + T; \mathbb{R})$ such that $b(t) \geq 0$ a.e. $t \in [t_0, t_0 + T]$. If, for some $0 \leq \alpha < 1$, an absolutely continuous function $w : [t_0, t_0 + T] \to \mathbb{R}_+$ satisfies

$$(1 - \alpha)w'(t) \leq a(t)w(t) + b(t)w^\alpha(t) \quad \text{a.e. } t \in [t_0, t_0 + T],$$

then

$$w^{1-\alpha}(t) \leq w^{1-\alpha}(t_0)e^{\int_{t_0}^t a(\tau)d\tau} + \int_{t_0}^t e^{\int_s^t a(\tau)d\tau} b(s)ds, \ \forall t \in [t_0, t_0 + T].$$

4.2.2 Some examples

Example 4.2. (*Parabolic Variational Inequalities*). Let $\Omega \subset \mathbb{R}^N$ be an open bounded subset with a smooth boundary $\partial\Omega$. Let us consider the following boundary value problem, with Signorini conditions, of finding a function $(t,x) \mapsto u = u(t,x)$ such that

$$(P) \begin{cases} \frac{\partial u}{\partial t} - \Delta u = f, & (t,x) \in [0,T] \times \Omega, \\ u(0,x) = u_0(x), & x \in \Omega \text{ (initial condition)} \\ u \geq 0, \ \frac{\partial u}{\partial n} \geq 0 \text{ and } u\frac{\partial u}{\partial n} = 0 \text{ for } (t,x) \in [0,T] \times \partial\Omega. \end{cases}$$

It is well-known that the weak formulation of problem (P) is given by the following parabolic variational inequalities

$$(VI) \begin{cases} \text{Find } u \in \mathcal{C} \text{ such that} \\ \int_\Omega u'(t)(v(t)-u(t))dx + \int_\Omega \nabla u(t) \cdot \nabla(v(t)-u(t))dx \geq \\ \qquad \int_\Omega f(t)(v(t)-u(t))dx, \ \forall v \in \mathcal{C}, \text{ a.e. } t \in [0,T]. \end{cases}$$

Here, $\mathcal{C} = \{v \in L^2(0,T;H^1(\Omega)) : v(t) \in C \text{ for a.e. } t \in [0,T]\}$, where $C = \{v \in H^1(\Omega) : v \geq 0 \text{ on } \partial\Omega\}$. It is easy to see that the parabolic variational inequality (VI) is of the form (4.1). The convexity structure of the set C (since it is a closed convex cone) makes the problem (VI) standard and may be straightforward. Let us consider now a function $g : \mathbb{R} \to \mathbb{R}$ and define the new set C with the associated set $\mathcal{C}$

$$C = \{v \in H^1(\Omega) : g(v(x)) \geq 0 \text{ for } x \in \partial\Omega\}.$$

The set C is no more convex and some sufficient conditions on the function g are necessary to ensure the prox-regularity of the sets C and $\mathcal{C}$ (see [4] for more details).

Example 4.3. (*Nonlinear Differential Complementarity Systems*). Let us consider the following ordinary differential equation, coupled with a complementarity condition,

$$(NDCS) \begin{cases} \dot{x}(t) = f(x(t)) + \lambda(t), \ t \in [0,T] \\ \lambda(t), g(x(t)) \geq 0, \ \langle \lambda(t), g(x(t)) \rangle = 0, \end{cases}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^m$ are of class C^1 and $\lambda : [0,T] \to \mathbb{R}^m$ is a Lagrange

multiplier (unknown function). We have that

$$\lambda(t), g(x(t)) \geq 0, \ \langle \lambda(t), g(x(t)) \rangle = 0 \iff -\lambda(t) \in \mathrm{N}_{\mathbb{R}^m_+}(g(x(t))).$$

Hence, (NDCS) is written as

$$\dot{x}(t) \in f(x(t)) - \mathrm{N}_{\mathbb{R}^m_+}(g(x(t))),$$

with $\mathrm{N}_{\mathbb{R}^m_+}(g(x(t))) = \partial \mathrm{I}_{\mathbb{R}^m_+}(g(x(t)))$, where ∂ denotes the subdifferential in the sense of convex analysis. If we suppose a qualification condition such as, e.g., ∇g is surjective, then, using classical chain rules for Clarke generalized subdifferential (see e.g. [76]), we get

$$\partial(\mathrm{I}_{\mathbb{R}^m_+} \circ g)(x) = \nabla g(x)^T \mathrm{N}_{\mathbb{R}^m_+}(g(x)).$$

By setting $C = \{x \in \mathbb{R}^n : g(x) \geq 0\}$, it is easy to see that problem (NCDS) is equivalent to the following differential inclusion

$$\dot{x}(t) \in f(x(t)) - \mathrm{N}_C(x(t)),$$

which is of the form of (4.1). Under some sufficient conditions on the vectorial function g (see [4, Theorem 3.5]), we show that the set C is r-prox-regular.

Many problems in power converters electronics and unilateral mechanics can be modeled by nonlinear differential complementarity problems of the form (NDCS) (see e.g. [1] and [82])).

4.3 Differential inclusions involving maximal monotone operators

We review in this section some aspects of the theory of differential inclusions involving maximal monotone operators. Namely, we provide an invariance result for associated closed sets that we use in the sequel.

Given a set-valued operator $A : H \rightrightarrows H$, which we identify with its graph, we denote its *domain* by $\mathrm{dom}A := \{x \in H \mid A(x) \neq \emptyset\}$. Operator A is *monotone* if

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0 \quad \text{for all} \ (x_1, y_1), \ (x_2, y_2) \in A,$$

and α-*hypomonotone* for $\alpha \geq 0$ if the operator $A + \alpha\,\mathrm{id}$ is monotone, where id is the identity mapping. We say that A is *maximal monotone* if A is monotone and coincides with every monotone operator containing its graph. In such a case, it is known that $A(x)$ is convex and closed for every $x \in H$. We shall denote by $(A(x))^\circ$, $x \in \mathrm{dom}A$, the set of minimal norm vectors in $A(x)$; i.e., $(A(x))^\circ := \{y \in A(x) \,|\, \|y\| = \min_{z \in A(x)} \|z\|\}$; hence, for any vector $x \in \mathrm{dom}A$ and $y \in H$, the set $\Pi_{A(x)}(y)$ is a singleton and we have that $(y - A(x))^\circ = y - \Pi_{A(x)}(y)$.

We consider the following differential inclusion

$$\dot{x}(t) \in f(x(t)) - A(x(t)) \quad t \in [0, \infty), \ x(0; x_0) = x_0 \in \overline{\mathrm{dom}A}, \qquad (4.3)$$

governed by a maximal monotone operator $A : H \rightrightarrows H$, which is subject to a perturbation by a $(l-)$Lipschitz continuous mapping $f : H \to H$. By a strong solution of (4.3) starting at $x_0 \in \overline{\mathrm{dom}A}$ we refer to an absolutely continuous function $x(\cdot; x_0)$ which satisfies (4.3) for a.e. $t \geq 0$, together with the initial condition $x(0; x_0) = x_0$. It is known that (4.3) processes a unique strong solution whenever $x_0 \in \mathrm{dom}A$, H is finite-dimensional, $\mathrm{int}(\mathrm{dom}A) \neq \emptyset$, or A is the subdifferential of convex, proper, and lower semi-continuous function. More generally, we call $x(\cdot; x_0)$ a weak solution of (4.3) starting at $x_0 \in \overline{\mathrm{dom}A}$, the unique continuous function which is the uniform limit of strong solutions $x(\cdot; x_k)$ with $(x_k) \subset \mathrm{dom}A$ converging to x_0.

The following result provides other properties of the solutions of (4.3); for more details we refer to the book [14, 21]. To denote the right-derivative whenever it exists we use the notation

$$\frac{d^+ x(t; x_0)}{dt} := \lim_{h \downarrow 0} \frac{x(t + h; x_0) - x(t)}{h}.$$

Proposition 4.4. *Fix $x_0, y_0 \in \overline{\mathrm{dom}A}$. Then system (4.3) has a unique continuous solution $x(t) \equiv x(t; x_0)$, $t \geq 0$, such that, for all $s, t \geq 0$*

$$x(s; x(t; x_0)) = x(t + s; x_0), \quad \|x(t; x_0) - x(t; y_0)\| \leq e^{lt} \|x_0 - y_0\|.$$

Moreover, if $x_0 \in \mathrm{dom}A$, then

$$\frac{d^+ x(t; x_0)}{dt} = [f(x(t; x_0)) - A(x(t; x_0))]^\circ = f(x(t; x_0)) - \Pi_{A(x(t; x_0))}(f(x(t; x_0))),$$

and the function $t \to \frac{d^+ x(t)}{dt}$ is right-continuous at every $t \geq 0$ with

$$\left\| \frac{d^+ x(t)}{dt} \right\| \leq e^{lt} \left\| \frac{d^+ x(0)}{dt} \right\|. \tag{4.4}$$

We are going to characterize those closed sets which are invariant with respect to differential inclusion (4.3).

Definition 4.5. A closed set $S \subset H$ is strongly invariant for (4.3) if every solution of (4.3) starting in S remains in this set for all time $t \geq 0$.
The set $S \subset H$ is weakly invariant for (4.3) if for every $x_0 \in S$, there exists a solution $x(\cdot; x_0)$ of (4.3) such that $x(t; x_0) \in S$ for all time $t \geq 0$.
When differential inclusion (4.3) has a unique solution for every given initial condition, both notions coincide, and we simply say in this case that S is invariant.

Due to the semigroup property in Proposition 4.4, it is immediately seen that S is invariant iff every solution of (4.3) starting in S remains in this set for all sufficiently small time $t \geq 0$. The issue with these sets, also referred to as *viable sets* for (4.3); see, [12], is to find good characterizations via explicit criteria, which do not require an a-priori computation of the solution of (4.3). An extensive research has been done to solve this problem for different kinds of differential inclusions and equations ([29, 32]). Complete primal and dual characterizations are given in [7, 8].

Theorem 4.7 provides a criterion for the invariance of closed sets satisfying the relation

$$S = \overline{S \cap \mathrm{dom} A}. \tag{4.5}$$

Such a property is almost necessary for the invariance of set S; indeed, it is necessary whenever (4.3) admits strong solutions, as is the case in the finite-dimensional setting.

We start by recalling the following lemma (see, e.g., [32, 74]), which is a consequence of Ekeland's Variational principle [40].

Lemma 4.6. *Suppose that S is closed. Then, for any $x \in H$ and any $s \in \Pi_S^\delta(x)$, with $\delta > 0$, there exist $s_\delta \in S$ and $y \in H$ such that*

$$\begin{cases} y - s_\delta \in \mathrm{N}_S^P(s_\delta), \\ \|y - s_\delta - (x - s)\| \leq 2\delta, \\ \|s - s_\delta\| \leq \delta, \quad \|x - y\| \leq \delta. \end{cases} \tag{4.6}$$

In addition, if $x \in \mathrm{B}(x_0, \sigma)$ for some $x_0 \in S$ and $\sigma > 0$, then s_δ satisfies

$$\|s_\delta - x_0\| \leq 2\sigma + \delta. \tag{4.7}$$

Theorem 4.7. *Let $S \subset H$ be as in (4.5) and take $x_0 \in S \cap \mathrm{dom}A$. Assume there are $m, \rho > 0$ such that for all $x \in S \cap \mathrm{dom}A \cap \mathrm{B}(x_0, \rho)$*

$$\sup_{\xi \in \mathrm{N}_S^P(x)} \min_{x^* \in A(x) \cap \mathrm{B}(\theta, m)} \langle \xi, f(x) - x^* \rangle \leq 0. \tag{4.8}$$

Then there exists $T > 0$ such that the solution $x(\cdot; x_0)$ of (4.3) satisfies

$$x(t; x_0) \in S \quad \text{for all } t \in [0, T].$$

Consequently, if for every $x \in S \cap \mathrm{dom}A$ inequality (4.8) holds for some $m(x), \rho(x) > 0$, then S is invariant for (4.3).

Proof. Fix $\varepsilon > 0$, and let positive real numbers m, ρ be as in (4.8). We start by showing that the set

$$K := S \cap \mathrm{dom}A \cap \mathrm{B}(x_0, \rho)$$

is closed and satisfies

$$\mathrm{N}_K^P(x) = \mathrm{N}_S^P(x) \text{ for all } x \in \mathrm{int}(\mathrm{B}(x_0, \rho)) \cap K. \tag{4.9}$$

Suppose that $x_n \to x$ for some $\{x_n\} \subset K$; hence, $x \in S \cap \mathrm{B}(x_0, \rho)$ as a consequence of (4.5). From the inequality of the current assumption one has that $A(x_n) \cap \mathrm{B}(0, m) \neq \emptyset$ for all n. Take $x_n^* \in A(x_n) \cap \mathrm{B}(0, m)$ that we assume (w.l.o.g.) weak converging to some $x^* \in \mathrm{B}(\theta, m)$. Since A is maximal monotone, it is norm-weak upper semi-continuous, and so we get $x^* \in A(x)$; that is, $x \in K$.

Relation (4.9) follows since, for all $x \in K$ such that $\|x - x_0\| < \rho$, one has (ecall (4.5))

$$\mathrm{N}_K^P(x) = \mathrm{N}_{\mathrm{B}(x_0, \rho) \cap S \cap \mathrm{dom}A}^P(x)$$
$$= \mathrm{N}_{S \cap \mathrm{dom}A}^P(x) = \mathrm{N}_{\overline{S \cap \mathrm{dom}A}}^P(x) = \mathrm{N}_S^P(x).$$

Let us also observe that in view of condition (4.8), and using the Lipschitzianity of f, there is a constant $M > 0$ such that

$$\emptyset \neq f(x) - (A(x) \cap \mathrm{B}(0, m)) \subset \mathrm{B}(0, M) \quad \forall x \in K. \tag{4.10}$$

We choose positive numbers $\bar{t}, \delta$ and positive integer N such that

$$4M\bar{t} < \rho, \ \max\{4\bar{t}(M^2 + 4M + 1)\varepsilon^{-2}, 2M\bar{t}\varepsilon^{-1}\} < N, 4N\delta < \min\{\bar{t}, N\rho, N\varepsilon\},$$
(4.11)

and denote by $\pi := \{t_0, t_1, ..., t_N\}$ the uniform partition of the interval $[0, \bar{t}]$; hence,

$$d(\pi) := \max_{0 \leq i \leq N-1}(t_{i+1} - t_i) = \frac{\bar{t}}{N}.$$

We claim the existence of vectors z_i, for $0 \leq i \leq N$, with $z_0 = x_0$, and $\hat{s}_i \in H$, $y_i \in H$, $s_i \in K$, $s_i^* \in H$, for $0 \leq i \leq N - 1$, such that, for all $0 \leq i \leq N - 1$

$$\hat{s}_i \in \Pi_K^\delta(z_i),$$
(4.12)

$$y_i - s_i \in \mathrm{N}_K^P(s_i),$$
(4.13)

$$\|z_i - \hat{s}_i - (y_i - s_i)\| \leq 2\delta,$$
(4.14)

$$\langle s_i^*, y_i - s_i \rangle \leq 0,$$
(4.15)

$$s_i^* \in (f(s_i) - A(s_i)) \cap \mathrm{B}(\theta, M),$$
(4.16)

together with a function $Z_i(\cdot)$, defined on $[t_i, t_{i+1}]$ and satisfying

$$\dot{Z}_i(t) = s_i^*, \ \forall t \in (t_i, t_{i+1}); \ Z_i(t_i) = z_i, \ Z_i(t_{i+1}) = z_{i+1}.$$
(4.17)

We proceed by finite induction on $k = 0, 1, \cdots, N$:

The claim is true for all $0 \leq i \leq k$ when $k = 0$: indeed, it suffices to take $z_0 = \hat{s}_0 = y_0 = s_0 = x_0$ and $z_1 = (t_1 - t_0)s_0^* + z_0$. Then the existence of s_0^* comes from (4.10) and the fact that $s_0 = x_0 \in K \subset \mathrm{dom}A$.

We suppose that the claim is true for all $0 \leq i \leq k$, and we shall prove it for all $0 \leq i \leq k + 1$; we may suppose that $k < N - 1$, because, for otherwise, we are done. To proceed we first observe that the vector z_{k+1}, which is already defined at the induction hypothesis, satisfies

$$
\begin{aligned}
\|z_{k+1} - x_0\| &\leq \sum_{i=0}^{k}\|z_{i+1} - z_i\| = \sum_{i=0}^{k}\|Z_i(t_{i+1}) - Z_i(t_i)\| \\
&= \sum_{i=0}^{k}\|(t_{i+1} - t_i)s_i^*\| \quad \text{(by (4.17))} \\
&\leq M(t_{k+1} - t_0) \leq M\bar{t}.
\end{aligned}
$$

Then, by choosing any element $\hat{s}_{k+1} \in \Pi_K^{\delta}(z_{k+1})$, Lemma 4.6 applied with z_{k+1} and $\hat{s}_{k+1}$ yields a pair $(s_{k+1}, y_{k+1}) \in K \times H$ such that

$$
\begin{cases}
y_{k+1} - s_{k+1} \in \mathrm{N}_K^P(s_{k+1}), \\[2mm]
\|(y_{k+1} - s_{k+1}) - (z_{k+1} - \hat{s}_{k+1})\| \leq 2\delta, \\[2mm]
\|y_{k+1} - z_{k+1}\| \leq \delta, \\[2mm]
\|\hat{s}_{k+1} - s_{k+1}\| \leq \delta, \\[2mm]
\|s_{k+1} - x_0\| \leq 2M\bar{t} + \delta \leq 2(\delta + \bar{t}M) < \rho \quad \text{(by (4.11))}.
\end{cases}
$$

Next, from the current hypothesis (4.8), together with (4.9) and (4.10), we find $s_{k+1}^* \in (f(s_{k+1}) - A(s_{k+1})) \cap \mathrm{B}(\theta, M)$ such that

$$
\left\langle y_{k+1} - s_{k+1}, s_{k+1}^* \right\rangle \leq 0. \tag{4.18}
$$

With this vector s_{k+1}^* in hand, we define the function $Z_{k+1}(\cdot)$ on $[t_{k+1}, t_{k+2}]$ as the unique solution of the following differential equation

$$
\begin{cases}
\dot{Z}_{k+1}(t) = s_{k+1}^*, \quad \text{a.e.} \ \ t \in [t_{k+1}, t_{k+2}], \\[2mm]
Z_{k+1}(t_{k+1}) = z_{k+1}.
\end{cases}
$$

We also introduce the vector

$$
z_{k+2} := Z_{k+1}(t_{k+2}).
$$

So, the vectors z_i, $0 \leq i \leq k+2$, and $\hat{s}_i \in H$, $y_i \in H$, $s_i \in K$, $s_i^* \in H$, for $0 \leq i \leq k+1$, together with the functions $Z_i(\cdot)$, for $0 \leq i \leq k+1$, accomplish with requirements of the claim.

At this stage, based on the claim above, we introduce the continuous piecewise linear function $Z(\cdot)$ defined on $[0, \bar{t}]$ as

$$
Z(t) := Z_i(t) \ \ \text{for } t \in [t_i, t_{i+1}], \ i = 0, ..., N-1.
$$

We are going to verify that

$$
d(Z(t), K) \leq \varepsilon \ \ \text{for all } t \in [0, \bar{t}], \tag{4.19}
$$

$$
\|s_i - Z(t)\| \leq 2\varepsilon \quad \text{for all } i = 0, ..., N-1 \text{ and } t \in [t_i, t_{i+1}], \tag{4.20}
$$

where s_i is defined in (4.13). Indeed, for every $j = 0, ..., N - 1$ one has

$$
\begin{aligned}
d_K^2(Z_{j+1}(t_{j+1})) = d_K^2(Z_j(t_{j+1})) & \\
\leq \|Z_j(t_{j+1}) - \hat{s}_j\|^2 \quad & (\text{since } \hat{s}_j \in \Pi_K^\delta(z_j) \subset K \text{ by } (4.12)) \\
= \|Z_j(t_{j+1}) - Z_j(t_j)\|^2 + \|Z_j(t_j) - \hat{s}_j\|^2 & \\
+ 2 \langle Z_j(t_{j+1}) - Z_j(t_j), Z_j(t_j) - \hat{s}_j \rangle & \\
\leq (t_{j+1} - t_j)^2 M^2 + \|z_j - \hat{s}_j\|^2 + 2(t_{j+1} - t_j) \langle s_j^*, z_j - \hat{s}_j \rangle & (\text{by } (4.17)) \\
\leq (t_{j+1} - t_j)^2 M^2 + d_K^2(z_j) + \delta^2 & (\text{by } (4.12)) \\
+ 2(t_{j+1} - t_j) \underbrace{\langle s_j^*, z_j - \hat{s}_j - (y_j - s_j) \rangle}_{\leq 2\delta M \ (\text{by } (4.14) \text{ and } (4.16))} + \underbrace{2(t_{j+1} - t_j) \langle s_j^*, y_j - s_j \rangle}_{\leq 0 \ (\text{by } (4.15))} & \\
\leq (t_{j+1} - t_j)^2 M^2 + d_K^2(Z_j(t_j)) + \delta^2 + 4\delta M (t_{j+1} - t_j) & \\
\leq (t_{j+1} - t_j)^2 M^2 + d_K^2(Z_j(t_j)) + d(\pi)(1 + 4M(t_{j+1} - t_j)) & \\
\leq d_K^2(Z_j(t_j)) + (t_{j+1} - t_j) d(\pi)(M^2 + 4M + 1), &
\end{aligned}
$$

which by summing up over $j = 0, \cdots, i - 1$ (when $i = 1, ..., N - 1$), and taking into account that $x_0 \in K$, gives us

$$
d_K^2(Z_i(t_i)) \leq (t_i - t_0) d(\pi)(M^2 + 4M + 1) \leq d(\pi) t_i (M^2 + 4M + 1) \leq \frac{\varepsilon^2}{4}. \quad (4.21)
$$

This inequality also holds when $i = 0$, because $d_K^2(Z_0(t_0)) = d_K^2(x_0) = 0$. Then, for every $i = 0, ..., N - 1$ and $t \in (t_i, t_{i+1})$ we write

$$
\begin{aligned}
d_K^2(Z(t)) = d_K^2(Z_i(t)) = d_K^2(s_i^*(t - t_i) + Z_i(t_i)) \quad & (\text{by } (4.17)) \\
\leq (\|s_i^*(t - t_i)\| + d_K(Z_i(t_i)))^2 & \\
\leq 2 \|s_i^*(t - t_i)\|^2 + 2 d_K^2(Z_i(t_i)) & \\
\leq 2(t - t_i)^2 M^2 + 2 d_K^2(Z_i(t_i)) \quad & (\text{by } (4.16)) \\
\leq 2 d(\pi)^2 M^2 + 2 d_K^2(Z_i(t_i)) \leq 2 d(\pi)^2 M^2 + \frac{\varepsilon^2}{2} \leq \varepsilon^2, & (4.22)
\end{aligned}
$$

and (4.19) follows. As for relation (4.20), it follows from the following inequalities,

for $i = 0, ..., N - 1$ and $t \in [t_i, t_{i+1}]$,

$$
\begin{aligned}
\|s_i - Z(t)\|^2 &= \|s_i - Z_i(t)\|^2 \\
&\leq 2\|s_i - Z_i(t_i)\|^2 + 2\|Z_i(t_i) - Z_i(t)\|^2 \\
&\leq 2(2\|s_i - \hat{s}_i\|^2 + 2\|\hat{s}_i - Z_i(t_i)\|^2) + 2(t - t_i)^2 M^2 \\
&\leq 2(t - t_i)^2 M^2 + 4\delta^2 + 4\|\hat{s}_i - z_i\|^2 \\
&\leq 2(t - t_i)^2 M^2 + 8\delta^2 + 4d_K^2(z_i) \\
&= 2(t - t_i)^2 M^2 + 8\delta^2 + 4d_K^2(Z_i(t_i)) \\
&\leq 2d(\pi)^2 M^2 + \varepsilon^2 + 8\delta^2 < 2\varepsilon^2 \quad \text{(by (4.21))}.
\end{aligned}
$$

Now, we consider the (strong) solution $x(t) := x(\cdot; x_0)$ of differential inclusion (4.3), and define the absolutely continuous function $\eta : [0, \bar{t}] \to \mathbb{R}$ as

$$
\eta(t) := \|Z(t) - x(t)\|^2.
$$

Let us prove that for $i = 0, ..., N - 1$ and t in a full-measure subset of $[t_i, t_{i+1}]$ it holds

$$
\frac{1}{2}\dot{\eta}(t) \leq (l + 1)\eta(t) + 2(4\varepsilon l)^2 + 4l\varepsilon(2\varepsilon + c), \tag{4.23}
$$

where $c := l^{-1} \sup_{t \in [0, \bar{t}]} \|\dot{x}(t) - s_i^*\|$ (c is finite, due to Proposition 4.4).

Indeed, by using the Lipschitz condition of f and relation (4.20),

$$
\begin{aligned}
\frac{1}{2}\dot{\eta}(t) &= \langle \dot{Z}(t) - \dot{x}(t), Z(t) - x(t) \rangle \\
&= \langle s_i^* - f(s_i) - (\dot{x}(t) - f(x(t))), Z(t) - s_i \rangle \\
&\quad + \langle f(s_i) - f(x(t)), Z(t) - x(t) \rangle \\
&\quad + \underbrace{\langle s_i - f(s_i) - (\dot{x}(t) - f(x(t))), s_i - x(t) \rangle}_{\leq 0 \text{ (by (4.16) and the monotonicity of } A)} \\
&\leq \langle s_i^* - f(s_i) - (\dot{x}(t) - f(x(t))), Z(t) - s_i \rangle \\
&\quad + \langle f(s_i) - f(x(t)), Z(t) - x(t) \rangle \\
&\leq \|s_i - Z(t)\| \, \|f(s_i) - s_i^* - f(x(t)) + \dot{x}(t)\| \\
&\quad + \|Z(t) - x(t)\| \, \|f(s_i) - f(x(t))\| \\
&\leq 2\varepsilon l \, \|s_i - x(t)\| + 2\varepsilon \, \|s_i^* - \dot{x}(t)\| \quad \text{(by (4.20))} \\
&\quad + l \, \|Z(t) - x(t)\| \, \|s_i - x(t)\| \\
&\leq 2\varepsilon l (\|s_i - Z(t)\| + \|Z(t) - x(t)\|) + 2\varepsilon \, \|s_i^* - \dot{x}(t)\| \\
&\quad + l \, \|Z(t) - x(t)\| \, (\|s_i - Z(t)\| + \|Z(t) - x(t)\|) \\
&\leq l\|Z(t) - x(t)\|^2 + 4\varepsilon l \, \|Z(t) - x(t)\| + 2l\varepsilon(2\varepsilon + c) \quad \text{(by (4.20)} \\
&\leq (l+1)\|Z(t) - x(t)\|^2 + (4\varepsilon l)^2 + 2l\varepsilon(2\varepsilon + c) \\
&= (l+1)\eta(t) + 2(4\varepsilon l)^2 + 4l\varepsilon(2\varepsilon + c),
\end{aligned}
$$

and (4.23) follows. Hence, by Gronwall's lemma (Lemma 4.1), we obtain that for very $t \in [0, \bar{t}]$,

$$
\begin{aligned}
\eta(t) &\leq \eta(0)e^{(l+1)t} + (2(4\varepsilon l)^2 + 4l\varepsilon(2\varepsilon + c)) \int_0^t e^{(l+1)(t-s)} ds \\
&= (2(4\varepsilon l)^2 + 4l\varepsilon(2\varepsilon + c)) \int_0^t e^{(l+1)(t-s)} ds,
\end{aligned}
$$

and, consequently,

$$
\|Z(t) - x(t)\| = \eta^{\frac{1}{2}}(t) \leq \left(2t(4\varepsilon l)^2 + 4l\varepsilon t(2\varepsilon + c) \right)^{\frac{1}{2}} e^{(l+1)t}.
$$

By combining this with inequality (4.19), we infer that for every $t \in [0, \bar{t}]$

$$
\begin{aligned}
d_S(x(t)) &\leq d_K(x(t)) \\
&\leq d_K(Z(t)) + \|Z(t) - x(t)\| \\
&\leq \left(2\bar{t}(4\varepsilon l)^2 + 4l\varepsilon\bar{t}(2\varepsilon + c) \right)^{\frac{1}{2}} e^{(l+1)\bar{t}} + \varepsilon.
\end{aligned}
$$

Since this inequality holds for any $\varepsilon > 0$, and $\bar{t}$ does not depend on ε, we conclude that $x(t) \in S$ for any $t \in [0, \bar{t}]$. The first part of the theorem is proved.

We now suppose that for any $x \in S \cap \mathrm{dom}A$, there exist positive numbers m, ρ (depending on x) such that inequality (4.8) holds.

Let us fix $x_0 \in S \cap \mathrm{dom}A$, so that $x(t; x_0) \in \mathrm{dom}A$ for every $t \geq 0$ (Proposition 4.4). From the first assertion of the theorem, there exists $\bar{t} > 0$ such that $x(t; \bar{x}) \in S \cap \mathrm{dom}A$ for any $t \in [0, \bar{t})$; moreover, since S is closed, we also have that $x(\bar{t}; x_0) \in S \cap \mathrm{dom}A$. By applying the first assertion of the current theorem, and taking into account the semi-group property (again by Proposition 4.4), we find $\hat{t} > 0$ such that for any $t \in [0, \hat{t}]$, one has

$$x(t + \bar{t}; x_0) = x(t; x(\bar{t}; x_0)) \in S \cap \mathrm{dom}A.$$

Thus, we prove that $x(t; x_0) \in S$ for every $t \geq 0$.

Assume now that $x_0 \in \overline{S \cap \mathrm{dom}A} \setminus \mathrm{dom}A$, and by (4.5) let $(x_k) \subset S \cap \mathrm{dom}A$ be such that $x_k \to \bar{x}$. Then, by arguing as in the last paragraph, for each $k \geq 1$ we have that $x_k(t; x_k) \in S$ for every $t \geq 0$. But $x(\cdot; x_k)$ converges uniformly to $x(\cdot; x_0)$ on each interval $[0, t]$ (see [6–8]), and so $x(\cdot; x_0)$ also stays in S. The proof of the theorem is complete. $\qquad\square$

The invariance criterion of Theorem 4.7 takes a simple form when the domain of operator A has a nonempty interior, and $S \subset \mathrm{int}(\overline{\mathrm{dom}A})$. In this case, because A is locally bounded on $\mathrm{int}(\overline{\mathrm{dom}A})$, the number m is dropped from inequality (4.8).

Corollary 4.8. *Let S be a nonempty closed subset of $\mathrm{int}(\overline{\mathrm{dom}A})$ such that*

$$\sup_{\xi \in \mathrm{N}_S^P(x)} \min_{x^* \in A(x)} \langle \xi, f(x) - x^* \rangle \leq 0 \ \forall x \in S.$$

Then S is invariant for system (4.3).

The following corollary will be useful in the proof of Theorem 4.19.

Corollary 4.9. *Assume that A is a monotone operator, and let S be a closed subset of $\mathrm{dom}A$. Suppose that $x(\cdot)$ is an absolutely continuous function such that*

$$\dot{x}(t) \in f(x(t)) - A(x(t)) \ a.e. \ t \in [0, T], x(0) = x_0 \in S.$$

If there are some numbers $m, \rho > 0$ such that

$$\sup_{\xi \in N_S^P(x)} \min_{x^* \in A(x) \cap B(\theta, m)} \langle \xi, f(x) - x^* \rangle \leq 0 \ \ \forall x \in S \cap B(x_0, \rho),$$

then there is some $T^* \in (0, T]$ such that $x(t) \in S$ for all $t \in [0, T^*]$.

Proof. According to [21], there exists a maximal monotone extension of A that we denote in the same way. By the current hypothesis, for every $x \in S \cap B(x_0, \rho)$ we have that $A(x) \cap B(\theta, m) \neq \emptyset$. Since $x(\cdot)$ is a unique solution, we apply Theorem 4.7, and the conclusion of the corollary follows. $\square$

4.4 The existence result

In this section, we use tools from convex and variational analysis to prove the existence of a solution for the differential inclusion (4.1),

$$\begin{cases} \dot{x}(t) \in f(x(t)) - N_C(x(t)) & \text{a.e.} \ \ t \geq 0, \\ x(0, x_0) = x_0 \in C, \end{cases}$$

where N_C is the proximal, or, equivalently, the limiting, normal cone to an r-uniformly prox-regular closed subset C of H, and f is a Lipschitz continuous mapping. We shall denote by $x(\cdot; x_0)$ the solution of this inclusion.

Definition 4.10. (see [63, 73]) For positive numbers r and α, a closed set S is said to be (r, α)-prox-regular at $\overline{x} \in S$ provided that one has $x = \Pi_S(x + v)$, for all $x \in S \cap B(\overline{x}, \alpha)$ and all $v \in N_S^P(x)$ such that $||v|| < r$.
The set S is r-prox-regular (resp., prox-regular) at $\overline{x}$ when it is (r, α)-prox-regular at $\overline{x}$ for some real $\alpha > 0$ (resp., for some numbers $r, \alpha > 0$). The set S is said to be r-uniformly prox-regular when $\alpha = +\infty$.

It is well-known and easy to check that when S is r-uniformly prox-regular, then for every $x \in S$, $N_S^P(x) = N_S^C(x)$; thus, for such sets we will simply write $N_S(x)$ to refer to each one of these cones, and write $T_S(x)$ to refer to the Bouligand tangent cone $T_S^B(x) = (N_S(x))^*$.

We have the following property of r-uniformly prox-regular sets, which can be easily checked.

Proposition 4.11. *Let S be a closed subset of H. If S is r-uniformly prox-regular, then the set-valued mapping defined by $x \mapsto N_S^P(x) \cap \mathbb{B}$ is $\frac{1}{r}$-hypomonotone.*

Before we state the main theorem of this section we give a useful characterization of prox-regularity.

Lemma 4.12. *The following statements are equivalent for every closed set $C \subset H$ and every $m > 0$,*

    (a) *C is r'-uniformly prox-regular for every $r' < r$,*

    (b) *the mapping $\mathrm{N}_C^P \cap \mathrm{B}(\theta, m) + \frac{m}{r}\,\mathrm{id}$ is monotone,*

    (c) *there exists a maximal monotone operator A defined on H such that*

$$\mathrm{N}_C^P(x) \cap \mathrm{B}(\theta, m) + \frac{m}{r}x \subset A(x) \subset \mathrm{N}_C^P(x) + \frac{m}{r}x \quad \text{for every } x \in C.$$

Proof. The equivalence $(a) \iff (b)$ is given in [73, Theorem 4.1], while the implication $(c) \implies (b)$ is immediate. Then we only have to prove that $(b) \implies (c)$. If (b) holds, we choose a maximal monotone operator A, which extends the monotone mapping $\mathrm{N}_C^P \cap \mathrm{B}(\theta, m) + \frac{m}{r}\mathrm{id}$, such that $C \subset \mathrm{dom}A \subset \overline{\mathrm{co}}C$ (see, e.g., [21]). Moreover, we have that

$$\mathrm{N}_C^P(x) \cap \mathrm{B}(\theta, m) + \frac{m}{r}x \subset A(x) \subset \mathrm{N}_C^P(x) + \frac{m}{r}x, \quad \forall x \in C. \qquad (4.24)$$

Indeed, the first inclusion is obvious. If $x \in C$ and $\xi \in A(x)$, then for any $y \in C$ we have $\frac{m}{r}y \in A(y)$ (since $0 \in \mathrm{N}_C^P(y) \cap \mathrm{B}(\theta, m)$) and, so, $\langle \xi - \frac{m}{r}y, x - y \rangle \geq 0$. This implies

$$\langle \xi - \frac{m}{r}x, y - x \rangle \leq \frac{m}{r}\|y - x\|^2,$$

which proves that $\xi - \frac{m}{r}x \in \mathrm{N}_C(x)$, for every $\xi \in A(x)$. Hence, $A(x) \subset \mathrm{N}_C^P(x) + \frac{m}{r}x$. $\qquad \square$

We also need some properties of the solution of (4.1). As pointed out by one of the reviewers, the assertions of the following lemma are very natural and may have already appeared in the literature. For the convenience of the reader, we give a complete proof.

Lemma 4.13. *If $x(\cdot; x_0)$ is a solution of (4.1), then for a.e. $t \in [0, T]$ we have*

$$\langle \dot{x}(t), f(x(t)) - \dot{x}(t) \rangle = 0, \qquad (4.25)$$

$$\|f(x(t)) - \dot{x}(t)\| \leq \|f(x(t))\|, \qquad (4.26)$$

$$\|\dot{x}(t)\| \leq \min\{\|f(x(t))\|, \|f(x_0)\|e^{lt}\}, \quad \|x(t) - x_0\| \leq t\|f(x_0)\|e^{lt}. \qquad (4.27)$$

Consequently, $x(\cdot; x_0)$ is the unique solution of (4.1) on $[0, T]$.

Proof. Let $t \in (0, T]$ be a differentiability point of the solution $x(\cdot)$. Then there is some $\delta > 0$ such that

$$\langle f(x(t)) - \dot{x}(t), x(s) - x(t) \rangle \leq \delta ||x(s) - x(t)||^2, \text{ for all } s \in [0, T],$$

and, so, by dividing on $s - t$ and taking the limit as $s \downarrow t$ we derive that

$$\langle f(x(t)) - \dot{x}(t), \dot{x}(t) \rangle \leq 0.$$

Similarly, when $s \uparrow t$ we get $\langle f(x(t)) - \dot{x}(t), \dot{x}(t) \rangle \geq 0$, which yields (4.25). Since $f(x(t)) - \dot{x}(t) \in \mathrm{N}_C(x(t))$ and $\dot{x}(t) \in \mathrm{T}_C^B(x(t))$, statement (4.25) means that $f(x(t)) - \dot{x}(t) = \Pi_{\mathrm{N}_C(x(t))}(f(x(t)))$ and this yields (4.26), $||f(x(t)) - \dot{x}(t)|| \leq ||f(x(t))||$. Moreover, using (4.25), we have (for a.e. $t \in [0, T]$)

$$||\dot{x}(t)||^2 = \langle \dot{x}(t), \dot{x}(t) \rangle = \langle \dot{x}(t), f(x(t)) \rangle \leq ||\dot{x}(t)|| \, ||f(x(t))||, \qquad (4.28)$$

which gives us $||\dot{x}(t)|| \leq ||f(x(t))||$. Then

$$\frac{d}{dt} ||x(t) - x_0||^2 = 2\langle x(t) - x_0, \dot{x}(t) \rangle \leq 2||x(t) - x_0|| \, ||f(x(t))||$$
$$\leq 2||x(t) - x_0||(||f(x_0)|| + l||x(t) - x_0||)$$
$$= 2||f(x_0)|| \, ||x(t) - x_0|| + 2l||x(t) - x_0||^2,$$

which by Lemma 4.1 gives us

$$||x(t) - x_0|| \leq \frac{||f(x_0)||}{l}(e^{lt} - 1) \leq ||f(x_0)||te^{lt}, \qquad (4.29)$$

so that, using the inequality of the middle together with (4.28),

$$||\dot{x}(t)|| \leq ||f(x(t))|| \leq ||f(x_0)|| + l||x(t) - x_0||$$
$$\leq ||f(x_0)|| + ||f(x_0)||(e^{lt} - 1) = ||f(x_0)||e^{lt}.$$

This proves (4.26) and (4.27).

To finish we need to check the uniqueness of the solution. Proceeding by contradiction, we assume that $y(\cdot)$ is another solution on $[0, T]$ of (4.1). Then for all $t \in [0, T]$ such that $||f(x(t))|| + ||f(y(t))|| > 0$ and $f(y(t)) - \dot{y}(t) \in \mathrm{N}_C(y(t))$ we have

$$\frac{f(y(t)) - \dot{y}(t)}{||f(x(t))|| + ||f(y(t))||} \in \mathrm{N}_C(y(t)) \cap \mathbb{B},$$

and similarly for $x(\cdot)$. Then, by the r-uniformly prox-regularity hypothesis on C,

$$\langle \dot{x}(t) - \dot{y}(t), x(t) - y(t) \rangle \leq \left(l + \frac{1}{r}(||f(x(t))|| + ||f(y(t))||) \right) ||x(t) - y(t)||^2; \quad (4.30)$$

this inequality also holds when $||f(x(t))|| + ||f(y(t))|| = 0$ as a consequence of (4.28). By applying Gronwall's Lemma (Lemma 4.1) with the function $\frac{1}{2}||x(t) - y(t)||^2$, and observing that $x(0) = y(0) = x_0$, it follows that $x(t) = y(t)$ for every $t \in [0, T]$. $\quad\square$

The main result is given in the following theorem, using a convex analysis approach, while Theorem 4.15 below provides more properties of the solution, which will be used later on.

Theorem 4.14. *System* (4.1) *has a unique solution* $x(\cdot, x_0)$ *starting at* $x_0 \in C$, *which is Lipschitz on every bounded interval.*

Proof. We fix a sufficiently large $m > 0$ and choose a $T_0 > 0$ such that

$$||f(x_0)|| + l(||f(x_0)|| T_0 e^{(l + \frac{m}{r})T_0} + 1) \leq m. \quad (4.31)$$

By Lemma 4.12(c) we consider a maximal monotone extension A such that, for all $x \in C$,

$$\mathrm{N}_C(x) \cap \mathrm{B}(\theta, m) + \frac{m}{r}x \subset A(x) \subset \mathrm{N}_C(x) + \frac{m}{r}x. \quad (4.32)$$

According to [14, 21], the differential inclusion

$$\begin{cases} \dot{x}(t) \in f(x(t)) + \frac{m}{r}x(t) - A(x(t)), & \text{a.e. } t \in [0, T_0] \\ x(0) = x_0 \in C, \end{cases} \quad (4.33)$$

has a unique solution $x(\cdot)$ such that $x(t) \in \mathrm{dom}A \ (\subset \overline{\mathrm{co}}(C))$ for all $t \in [0, T_0]$, as well as (see, e.g., [7])

$$\left\| \frac{d^+ x(t)}{dt} \right\| \leq e^{(l + \frac{m}{r})t} \left\| \frac{d^+ x(0)}{dt} \right\| \leq e^{(l + \frac{m}{r})t} ||\Pi_{A(x_0)}(f(x_0) + \frac{m}{r}x_0)||.$$

Moreover, since $\frac{m}{r}x_0 \in A(x_0)$ (due to (4.32)), for all $t \in [0, T_0]$

$$\left\| \frac{d^+ x(t)}{dt} \right\| \leq e^{(l + \frac{m}{r})t} ||f(x_0)|| \leq e^{(l + \frac{m}{r})T_0} ||f(x_0)|| =: k,$$

and, hence,

$$||x(t) - x_0|| \leq kT_0, \quad (4.34)$$

$$||f(x(t))|| \leq ||f(x_0)||+l||x(t)-x_0|| \leq ||f(x_0)||+lkT_0;$$

in particular, $x(\cdot)$ is k-Lipschitz on $[0, T_0]$.

Next, we want to show that $x(t) \in C$ for every $t \in [0, T_0]$. For this aim we shall apply Theorem 4.7. Given $y \in C \cap B(x_0, kT_0 + 1)$ and $\xi \in N_C(y)$, we define $z := \Pi_{N_C(y)}(f(y)) \in N_C(y)$ (z is well defined since $N_C(y)$ is closed (and convex)). It is easy to see that

$$||z|| \leq ||f(y)|| \leq ||f(x_0)||+l||y-x_0|| \leq ||f(x_0)||+l(kT_0 + 1) \leq m.$$

Hence, according to (4.32), we derive that $y^* := z + \frac{m}{r}y \in N_C(y) \cap B(\theta, m) + \frac{m}{r}y \subset A(y)$, with $\|y^*\| \leq \overline{m} := m(1 + \frac{1}{r}(\|x_0\| + kT_0 + 1))$.

Now, since $f(y) - z \in T_C(y)$ we obtain that $\langle \xi, f(y) - z \rangle \leq 0$, which shows that

$$\inf_{v^* \in A(y) \cap B(\theta, \overline{m})} \left\langle \xi, f(y) + \frac{m}{r}y - v^* \right\rangle \leq \langle \xi, f(y) + \frac{m}{r}y - y^* \rangle \leq 0. \qquad (4.35)$$

Consequently, according to Theorem 4.7, there is a positive number $T' \in (0, T_0)$ such that $x(t) \in C$ for every $t \in [0, T']$. Moreover, due to (4.34), for all $t \in [0, T_0]$ we have that $x(t) \in B(x_0, kT_0 + 1)$ and, so, from the argument above we infer that $x(t) \in C$ for all $t \in [0, T_0]$. Whence, since $x(t) \in C$ for $t \in [0, T_0]$, (4.32) implies that

$$\dot{x}(t) \in f(x(t)) + \frac{m}{r}x(t) - A(x(t)) \subset f(x(t)) + \frac{m}{r}x(t) - N_C(x(t)) - \frac{m}{r}x(t)$$
$$= f(x(t)) - N_C(x(t));$$

that is, $x(\cdot)$ is a solution of (4.1) on $[0, T_0]$.

Now, we set

$$T := \sup \{T' > 0 \text{ such that system (4.1) has a solution } x(\cdot; x_0) \text{ on } [0, T']\};$$

so, $T > 0$ from the paragraph above. If T is finite, then we take a sequence (T_n) such that $T_n \uparrow T$, and denote $x_n(\cdot; x_0)$ the corresponding solution of (4.1), which is defined on $[0, T_n]$. Let function $x(\cdot; x_0) : [0, T) \to H$ be defined as

$$x(t; x_0) = x_n(t) \text{ if } t \leq T_n.$$

According to Lemma 4.13 (relation (4.27)), this function is a well-defined Lipschitz continuous function on $[0, T)$, with Lipschitz constant equal to $||f(x_0)||e^{lT}$. Thus,

we can extend continuously function $x(\cdot; x_0)$ to $[0, T]$ by setting $x(T) := \lim_{n \to \infty} x(T_n)$. Since $x(T) \in C$, from the first paragraph we find a $T_1 > 0$ and a solution of (4.1) on $[0, T + T_1]$ which coincides with $x(\cdot; x_0)$ on $[0, T]$, contradicting the finiteness of T–this is to say that $T = \infty$. $\qquad\square$

An immediate consequence of (the proof of) Theorem 4.14 is that the solution of differential inclusion (4.1) satisfies the so-called semi-group property,

$$x(t; x(s; x_0)) = x(t + s; x_0) \text{ for all } t, s \geq 0 \text{ and } x_0 \in C. \qquad (4.36)$$

The following theorem gathers further properties of the solution of (4.1), that we shall use in the sequel. Relation (4.37) below on the derivative of the solution reinforces the statement of Lemma 4.13.

Theorem 4.15. *Let $x(\cdot; x_0)$, $x_0 \in C$, be the solution of (4.1). Then the following statements hold true:*

(a) *For every $t \geq 0$, $x(\cdot; x_0)$ is right-derivable at t with*

$$\begin{aligned} \frac{d^+ x(t)}{dt} &= (f(x(t)) - \mathrm{N}_C(x(t)))^\circ \\ &= f(x(t)) - \Pi_{\mathrm{N}_C(x(t))}(f(x(t)) = \Pi_{\mathrm{T}_C(x(t))}(f(x(t))), \end{aligned} \qquad (4.37)$$

$$\left\| \frac{d^+ x(t)}{dt} \right\| \leq \min\{\|f(x(t))\|, \|f(x_0)\| e^{lt}\}, \qquad (4.38)$$

$$\left\| \frac{d^+ x(t)}{dt} \right\| \leq \left\| \frac{d^+ x(0)}{dt} \right\| e^{lt + \frac{2\|f(x_0)\|}{lr}(e^{lt}-1)}. \qquad (4.39)$$

(b) *The mapping $t \to \frac{d^+ x(t)}{dt}$ is right-continuous on $[0, T)$.*

(c) *If $y(\cdot; y_0)$, $y_0 \in C$, is the corresponding solution of (4.1), then for every $t \geq 0$*

$$\|x(t) - y(t)\| \leq \|x_0 - y_0\| \, e^{lt + \frac{\|f(x_0)\| + \|f(y_0)\|}{lr}(e^{lt}-1)}.$$

Proof. We fix $t \geq 0$ (we may suppose that $t = 0$). From the argument used in the proof of Theorem 4.14 we know that for some $m > \|f(x_0)\| + l$ (l is the Lipschitz constant of f) there exists a maximal monotone operator A such that $x(\cdot) := x(\cdot; x_0)$ is the solution of the following differential inclusion on some interval $[0, \delta]$, $\delta > 0$,

$$\dot{x}(t) \in f(x(t)) + \frac{m}{r} x(t) - A(x(t)), \ x(0) = x_0,$$

where r comes from the r-uniform prox-regularity of C. W.l.o.g. we may suppose that $\|f(x(t))\|+l < m$ for all $t \in [0, \delta]$ so that (see Proposition 4.4), for every $t \in [0, \delta]$,

$$\frac{d^+ x(t)}{dt} = (f(x(t)) + \frac{m}{r} x(t) - A(x(t)))^\circ. \tag{4.40}$$

Since $f(x(t)) \in \mathrm{B}(\theta, m)$ we have that

$$\begin{aligned}
(f(x(t)) - \mathrm{N}_C(x(t)))^\circ &= f(x(t)) - \Pi_{\mathrm{N}_C(x(t))}(f(x(t))) \\
&= f(x(t)) - \Pi_{\mathrm{N}_C(x(t)) \cap \mathrm{B}(\theta, m)}(f(x(t))) \\
&= (f(x(t)) - \mathrm{N}_C(x(t)) \cap \mathrm{B}(\theta, m))^\circ,
\end{aligned}$$

and, so, due to (4.40), and the inclusions (4.32):

$$f(x(t)) - \mathrm{N}_C(x(t)) \cap \mathrm{B}(\theta, m) \subset f(x(t)) + \frac{m}{r} x(t) - A(x(t)) \subset f(x(t)) - \mathrm{N}_C(x(t)),$$

we get the first equality in (4.37). The other two equalities in (4.37) easily follow from the definition of the orthogonal projection. Moreover, statement (b) is also a consequence of Proposition 4.4. Thus, (4.37) follows from Lemma 4.13. Finally, (4.39) and statement (c) follow easily using relation (4.30) (and Lemma 4.1). $\square$

The main idea behind the previous existence theorems, Theorems 4.14 and 4.15, as well as the forthcoming results on Lyapunov stability in the next section, is that differential inclusion (4.1) is in some sense equivalent to a differential inclusion governed by a (Lipschitz continuous perturbation of a) maximal monotone operator. This fact is highlighted in the following corollary. Recall, by Lemma 4.12(c), that for every $m > 0$ the r-uniformly prox-regularity of the set C yields the existence of a maximal monotone operator A_C such that

$$\mathrm{N}_C(x) \cap \mathrm{B}(\theta, m) + \frac{m}{r} x \subset A_C(x) \subset \mathrm{N}_C(x) + \frac{m}{r} x \quad \text{for every } x \in C. \tag{4.41}$$

Corollary 4.16. *An absolutely continuous function $x(t)$ is a solution of (4.1) on $[0, T]$; that is,*

$$\begin{cases} \dot{x}(t) \in f(x(t)) - \mathrm{N}_C(x(t)) \ \text{a.e. } t \in [0, T] \\ x(0) = x_0 \in C, \end{cases}$$

if and only if it is (the unique) solution of the following differential inclusion, for

some $m > 0$,

$$(DIM) \quad \begin{cases} \dot{x}(t) \in f(x(t)) + \frac{m}{r}x(t) - A_C(x(t)) \ \text{a.e.} \ t \in [0, T] \\ x(0) = x_0 \in C, \end{cases}$$

where the maximal monotone operator $A_C : H \rightrightarrows H$ is defined in (4.41).

Proof. According to Theorems 4.14 and 4.15 (namely, (4.38)), differential inclusion (101) has a unique (absolutely continuous) solution $x(t) := x(t; x_0)$ which satisfies $||\frac{d^+x(t)}{dt}|| \leq ||f(x_0)||e^{lT}$ for a.e. $t \in [0, T]$. Then, we find an $m > 0$ such that

$$\dot{x}(t) \in f(x(t)) - \mathrm{N}_C(x(t)) \cap \mathrm{B}(\theta, m),$$

and, so, by the definition of A_C above (see (4.41)) we conclude that $x(t)$ is also the solution of differential inclusion (DIM).

Conversely, if $x(t)$ is a solution of differential inclusion (DIM) for some $m > 0$, then, as it follows from the proof of Theorem 4.14, we get that $x(t) \in C$ for all $t \in [0, T_0]$ for some $T_0 > 0$. Hence, once again by (4.41), we conclude that $x(t)$ is also a solution of (101) on $[0, T_0]$. Taking into account Lemma 4.14 we show, also as in the proof of Theorem 4.14, that T_0 can be taken to be T. $\qquad \square$

4.5 Lyapunov stability analysis

In this section, we give explicit characterizations for lower semi-continuous a-Lyapunov pairs, Lyapunov functions, and invariant sets associated to differential inclusion (4.1). Recall that $x(\cdot; x_0)$ (or $x(\cdot)$, when any confusion is excluded) refers to the unique solution of (4.1), which satisfies $x(0; x_0) = x_0$.

Definition 4.17. Let functions $V, W : H \to \overline{\mathbb{R}}$ be lower semi-continuous, with $W \geq 0$, and let an $a \geq 0$. We say that (V, W) is (or forms) an a-Lyapunov pair for differential inclusion (4.1) if, for all $x_0 \in C$,

$$e^{at}V(x(t; x_0)) + \int_0^t W(x(\tau; x_0))d\tau \leq V(x_0) \ \text{ for all } t \geq 0. \qquad (4.42)$$

In particular, if $a = 0$, we say that (V, W) is a Lyapunov pair. If, in addition, $W = 0$, then V is said to be a Lyapunov function.

A closed set $S \subset C$ is said to be invariant for (4.1) if the function δ_S is a Lyapunov function.

Equivalently, using (4.36), it is not difficult to show that a-Lyapunov pairs are those pairs of functions $V, W : H \to \overline{\mathbb{R}}$ such that the mapping $t \to e^{at}V(x(t;x_0)) + \int_0^t W(x(\tau,x_0))d\tau$ is nonincreasing. In other words (see, e.g. [7, Proposition 3.2]), for any $x_0 \in C$, there exists $t > 0$ such that

$$e^{as}V(x(s;x_0)) + \int_0^s W(x(\tau;x_0))d\tau \leq V(x_0) \text{ for all } s \in [0,t]. \tag{4.43}$$

The failure of *regularity* in our Lyapunov candidate-like pairs is mainly carried out by the function V, since the function W can be always regularized to a Lipschitz continuous function on every bounded subset of H as the following lemma shows (see, e.g., [29]).

Lemma 4.18. *Let V, W and a be as in Definition* 4.17. *Then there exists a sequence of lower semi-continuous functions $W_k : H \to \mathbb{R}$, $k \geq 1$, converging pointwisely to W (for instance, $W_k \nearrow W$) such that W_k is Lipschitz continuous on every bounded subset of H. Consequently, (V, W) forms an a-Lyapunov pair for* (4.1) *if and only if each (V, W_k) does.*

Now, we give the main theorem of this section, which characterizes lower semi-continuous a-Lyapunov pairs associated to differential inclusion (4.1).

Theorem 4.19. *Let functions $V, W : H \to \overline{\mathbb{R}}$ be lower semi-continuous, with $W \geq 0$ and $\mathrm{dom}V \subset C$, $a \geq 0$, and let $x_0 \in \mathrm{dom}V$. If there is $\rho > 0$ such that, for any $x \in \mathrm{B}(x_0, \rho)$,*

$$\sup_{\xi \in \partial_P V(x)} \min_{x^* \in \mathrm{N}_C(x) \cap \mathrm{B}(\theta, \|f(x)\|)} \langle \xi, f(x) - x^* \rangle + aV(x) + W(x) \leq 0, \tag{4.44}$$

then there is some $T^* > 0$ such that

$$e^{at}V(x(t;x_0)) + \int_0^t W(x(\tau;x_0))d\tau \leq V(x_0) \ \forall t \in [0, T^*].$$

Consequently, the following statements are equivalent provided that either $\partial \equiv \partial_P$ or $\partial \equiv \partial_F$:

(i) (V, W) is an a-Lyapunov pair for (4.1);

(ii) for every $x \in \mathrm{dom}V$ and $\xi \in \partial V(x)$;

$$\langle \xi, (f(x) - \mathrm{N}_C(x))^\circ \rangle + aV(x) + W(x) \leq 0;$$

(*iii*) *for every* $x \in \text{dom}V$ *and* $\xi \in \partial V(x)$;

$$\min_{x^* \in \text{N}_C(x) \cap \text{B}(\theta, ||f(x)||)} \langle \xi, f(x) - x^* \rangle + aV(x) + W(x) \leq 0;$$

(*iv*) *for every* $x \in \text{dom}V$;

$$V'(x; (f(x) - \text{N}_C(x))^\circ) + aV(x) + W(x) \leq 0;$$

(*v*) *for every* $x \in \text{dom}V$;

$$\inf_{x^* \in \text{N}_C(x) \cap \text{B}(\theta, ||f(x)||)} V'(x; f(x) - x^*) + aV(x) + W(x) \leq 0.$$

Moreover, when H is finite-dimensional, all the statements above except (ii) are equivalent when $\partial = \partial_L$.

Proof. Let us start with the first part of the theorem. We choose $T > 0$ such that

$$T||f(x_0)||e^{lT} \leq \frac{\rho}{2},$$

and put

$$k := 2\max\{||f(x_0)||e^{lT}, ||f(x_0)||+lTe^{lT}||f(x_0)||+l+1\};$$
$$m := k + \frac{k}{r}(||x_0||+\rho).$$

Thanks to Lemma 4.18 we shall assume in what follows that W is Lipschitz continuous on $\text{B}(x_0, \rho)$. As before we denote $x(\cdot)$ the solution of (4.1) on $[0, T]$ satisfying $x(0) = x_0$. According to Theorem 4.15, for a.e. $t \in [0, T]$ we have $||\dot{x}(t)|| \leq ||f(x(t))||$ and, due to the l-Lipschitzianity of f,

$$2||f(x(t))|| \leq 2||f(x_0)||+2l||x(t) - x_0||$$
$$< 2\max\{||f(x_0)||e^{lT}, ||f(x_0)||+lTe^{lT}||f(x_0)||+l+1\} = k;$$

that is, $\dot{x}(t) \in f(x(t)) - (\text{N}_C(x(t)) \cap \text{B}(\theta, k))$. Hence, if $A : H \rightrightarrows H$ is the monotone operator defined as

$$A(x) := \begin{cases} \text{N}_C(x) \cap \text{B}(\theta, k) + \frac{k}{r}x & \text{if } x \in C, \\ \emptyset & \text{otherwise,} \end{cases}$$

then it is immediately seen that $x(\cdot)$ is also the unique solution of the following

differential inclusion,

$$\begin{cases} \dot{x}(t) \in f(x(t)) + \frac{k}{r}x(t) - A(x(t)) \text{ a.e. } t \in [0, T], \\ x(0) = x_0, \end{cases}$$

where r comes from the r-uniform prox-regularity of the set C. We introduce now the monotone operator $\widetilde{A} : H \times \mathbb{R}^4 \rightrightarrows H \times \mathbb{R}^4$ and the l-Lipschitz function $\tilde{f} : H \times \mathbb{R}^4 \to H \times \mathbb{R}^4$ defined as

$$\widetilde{A}(x, \mu) := (A(x), 0_{\mathbb{R}^4}) \text{ and } \tilde{f}(x, \mu) := (f(x) + \frac{k}{r}x, 1, 0, 1, 0),$$

and consider the associated differential inclusion

$$\begin{cases} \dot{y}(t) \in \tilde{f}(y(t)) - \widetilde{A}(y(t)), \text{ a.e. } t \in [0, T] \\ y(0) = (x_0, \mu_0) \in C \times \mathbb{R}^4, \end{cases} \tag{4.45}$$

whose unique solution is given by $y(t) := (x(t), t, 0, t, 0) + (0, \mu_0) \subset C \times \mathbb{R}^4$. We define the lower semi-continuous functions $V_n : H \times \mathbb{R}^3 \to \overline{\mathbb{R}}$, $n \geq 1$, as

$$V_n(x, \alpha, \beta, \gamma) := e^{a\gamma}V(x) + (\alpha - \beta)g_n(\alpha) + \frac{l'}{2}(\alpha - \beta)^2,$$

where g_n is an l'-Lipschitz extension of the (Lipschitz) function $W(x(\cdot)) - \frac{1}{n}$ from $[0, T]$ to $[-1, T+1]$; hence,

$$\partial_C g_n(\alpha) \subset \mathrm{B}(\theta, l') \text{ for all } \alpha \in [0, T+1].$$

Observe that $\mathrm{epi}V_n \subset \mathrm{dom}\widetilde{A}$ and, for every $(x, \alpha, \beta, \gamma) \in \mathrm{dom}V_n$, we have that $\partial_{P,\infty}V_n(x, \alpha, \beta, \gamma) \subset (e^{a\gamma}\partial_{P,\infty}V(x), 0, 0, 0)$ and

$$\partial_P V_n(x, \alpha, \beta, \gamma) \subset (e^{a\gamma}\partial_P V(x), g_n(\alpha), -g_n(\alpha), ae^{a\gamma}V(x))$$
$$+ (0, (\alpha - \beta)\partial_C g_n(\alpha) + l'(\alpha - \beta), l'(\beta - \alpha), 0). \tag{4.46}$$

At this step, we pick $t \in [0, T)$ and fix $n \geq 1$ such that $\frac{1}{n} \leq \frac{\rho}{2}$.
We denote $y_0 := (x(t), t, t, 0, V_n(x(t), t, t, 0))$, and choose $\varepsilon > 0$ such that (recall that W is Lipschitz continuous on $\mathrm{B}(x_0, \rho)$)

$$g_n(\alpha') + 2l'|\alpha' - \beta'| - e^{a\gamma'}W(x') \leq \frac{-1}{2n}, \tag{4.47}$$

for any $(y', \mu') := (x', \alpha', \beta', \gamma', \mu') \in U_\varepsilon := B(y_0, \varepsilon) \cap \mathrm{epi}V_n$. Take vectors $(y, \mu) := (x, \alpha, \beta, \gamma, \mu) \in U_\varepsilon$ and $\xi \in \mathrm{N}^P_{\mathrm{epi}V_n}(y, \mu)$. Then $x \in \mathrm{dom}V \subset C$ and (recall Lemma 4.13 (relation (4.27)))

$$||x - x_0|| \leq ||x - x(t)|| + ||x(t) - x_0|| \leq \varepsilon + ||f(x_0)||Te^{lT} < \rho;$$

hence, $x \in \mathrm{dom}V \cap \mathrm{int}(B(x_0, \rho))$ and

$$||f(x)|| \leq ||f(x_0)|| + l||x - x_0|| < ||f(x_0)|| + l(1 + ||f(x_0)||Te^{lT}) \leq \frac{k}{2};$$

thus,

$$f(x) - (\mathrm{N}_C(x) \cap B(0, ||f(x)||)) \subset f(x) - (\mathrm{N}_C(x) \cap B(\theta, k)) = f(x) + \frac{k}{r}x - A(x). \tag{4.48}$$

Also, we have that $\xi \in \mathrm{N}^P_{\mathrm{epi}V_n}(y, V_n(y))$ (see, e.g., [29, Exercise 2.1]) and, so, due to relation (4.46), either

$$\xi = \iota\left(e^{a\gamma}\xi_1, g_n(\alpha) + (\alpha - \beta)(\varsigma + l'), -g_n(\alpha) + l'(\beta - \alpha), ae^{a\gamma}V(x), -1\right), \tag{4.49}$$

for some $\xi_1 \in \partial_P V(x), \varsigma \in \partial_C g_n(\alpha)$ and $\iota \geq 0$, or

$$\xi = \iota(e^{a\gamma}\xi_2, 0) \tag{4.50}$$

with $\xi_2 \in \partial_{P,\infty}V(x)$.

If (4.49) holds, by the current assumption, there exists an $x^* \in \mathrm{N}_C(x) \cap B(\theta, ||f(x)||)$ such that

$$\langle \xi_1, f(x) - x^* \rangle + aV(x) + W(x) \leq 0.$$

Hence, due to (4.48), $y^* := (x^* + \frac{k}{r}x, 0_{\mathbb{R}^4})$ belongs to $\widetilde{A}(y, \mu) \cap B(0, m)$ (since $||x^* + \frac{k}{r}x|| \leq ||x^*|| + \frac{k}{r}||x|| \leq \frac{k}{2} + \frac{k}{r}(||x_0|| + \rho) \leq m$), and satisfies

$$\begin{aligned}
\langle \xi, \tilde{f}(y, \mu) - y^* \rangle &= \iota\left(\langle e^{a\gamma}\xi_1, f(x) - x^* \rangle + g_n(\alpha) + (\alpha - \beta)(\varsigma + l') + ae^{a\gamma}V(x)\right) \\
&= \iota e^{a\gamma}\left(\langle \xi_1, f(x) - x^* \rangle + aV(x) + W(x)\right) \\
&\quad + \iota\left(g_n(\alpha) + (\alpha - \beta)(\varsigma + l') - e^{a\gamma}W(x)\right) \\
&\leq \iota\left(g_n(\alpha) + 2l'|\alpha - \beta| - e^{a\gamma}W(x)\right) \\
&\leq -\frac{\iota}{2n} \leq 0,
\end{aligned}$$

where in the last inequality we used (4.47).

We now consider the case when ξ satisfies (4.50). Let sequences $x_k \xrightarrow{V} x$, $\nu_k \in \partial_P V(x_k)$ and $\alpha_k \to 0^+$ be such that $\alpha_k \nu_k \to \xi_2$ (see [64, Lemma 2.37]). Since $x \in \text{int}B(x_0, \rho)$, we may assume that $x_k \in B(x_0, \rho) \cap \text{dom}V$ for all $k = 1, 2, \dots$ By hypothesis, for every k, there exists $x_k^* \in N_C(x_k) \cap B(\theta, \|f(x_k)\|)$ such that

$$\langle \nu_k, f(x_k) - x_k^* \rangle + aV(x_k) + W(x_k) \le 0.$$

Since f is Lipschitz, the sequence (x_k^*) is bounded and we may suppose (w.l.o.g.) that it weakly converges to some $x^* \in N_C(x) \cap B(\theta, \|f(x)\|)$, due to the r-uniform prox-regularity of C. Consequently, by multiplying the last inequality above by α_k and next passing to the limit on k we obtain $\langle \xi_2, f(x) - x^* \rangle \le 0$.

Hence, the vector $z^* := (x^* + \frac{k}{r}x, 0_{\mathbb{R}^4})$ belongs to $\widetilde{A}(y, \mu) \cap B(\theta_{H \times \mathbb{R}^4}, m)$ and satisfies

$$\langle \xi, \tilde{f}(y, \mu) - z^* \rangle = \iota \langle (e^{a\gamma}\xi_2, 0, 0, 0, 0), (f(x) - x^*, 1, 0, 1, 0) \rangle = \iota e^{a\gamma} \langle \xi_2, f(x) - x^* \rangle \le 0.$$

Consequently, according to Corollary 4.9, there is a $T_0 > 0$ such that the solution of (4.45) on $[0, T_0]$ starting at $(x(t), t, t, 0, V(x(t)))$, which is given by $y(s) = (x(s + t), s + t, t, s, V(x(t)))$, lies in $\text{epi}V_n$; that is, for every $s \in [0, T_0]$,

$$V_n(x(s+t), s+t, t, s) = e^{as}V(x(t+s)) + sg_n(s+t) + \frac{l'}{2}s^2 \le V(x(t)),$$

implying that,

$$e^{as}V(x(t+s)) + \int_0^s g_n(t+\tau)d\tau \le V(x(t)). \tag{4.51}$$

We claim that

$$e^{as}V(x(t+s)) + \int_0^s g_n(t+\tau)d\tau \le V(x(t)) + \frac{2}{n}e^{\max\{1,a\}s} \quad \forall s \in [0, T - t]. \tag{4.52}$$

Indeed, if

$$T^* := \sup\{T' > 0, \text{ such that (4.52) holds on } [0, T']\},$$

then from (4.51) we have that $T^* \ge T_0 > 0$, while the lower semi-continuous of V and the continuity of g_n yield that (4.52) also holds at T^*. If $T^* < T - t$, by (4.51) there exits $\delta > 0$ such that for all $s \in [0, \delta]$

$$e^{as}V(x(t + T^* + s)) + \int_0^s g_n(t + T^* + \tau)d\tau \le V(x(t + T^*)).$$

Therefore, for all $s \in [0, \delta]$ we obtain

$$
e^{a(T^*+s)} V(x(t+T^*+s)) + \int_0^{T^*+s} g_n(t+\tau) d\tau
$$

$$
\leq e^{aT^*} \left(V(x(t+T^*)) - \int_0^s g_n(t+T^*+\tau) d\tau \right) + \int_0^{T^*} g_n(t+\tau) d\tau
$$

$$
+ \int_0^s g_n(t+T^*+\tau) d\tau
$$

$$
\leq V(x(t)) + \frac{2}{n} e^{\max\{1,a\}T^*} + (1 - e^{aT^*}) \int_0^s g_n(t+T^*+\tau) d\tau
$$

$$
\leq V(x(t)) + \frac{2}{n} e^{\max\{1,a\}T^*} + \frac{2\delta(e^{aT^*}-1)}{n} \leq V(x(t)) + \frac{2}{n} e^{\max\{1,a\}(T^*+s)},
$$

where in the last inequality we used the fact that $1 + \lambda \leq e^\lambda$ for all $\lambda > 0$. This contradicts the definition of T^*, and so (4.52) holds true; that is (evaluating at $t = 0$), for all $s \in [0, T]$

$$
e^{as} V(x(s)) + \int_0^s W(x(\tau)) d\tau - \frac{s}{n} = e^{as} V(x(s)) + \int_0^s g_n(\tau) d\tau \leq V(x_0) + \frac{2}{n} e^{\max\{1,a\}s};
$$

hence, as n goes to ∞, we get $e^{as} V(x(s)) + \int_0^s W(x(\tau)) d\tau \leq V(x_0)$ for all $s \in [0, T]$, and the first part of the theorem is proved.

We turn now to the second part of the theorem. Implications $(iv) \Rightarrow (v)$ and $(ii) \Rightarrow (iii)$ follow from the relation $(f(x) - \mathrm{N}_C(x))^\circ = f(x) - \Pi_{\mathrm{N}_C(x)}(f(x))$, $x \in C$, and the fact that $\left\| \Pi_{\mathrm{N}_C(x)}(f(x)) \right\| \leq \|f(x)\|$.

$(i) \Rightarrow (iv)$. Assuming that (V, W) is an a-Lyapunov pair, for any $x_0 \in \mathrm{dom}V$ and $t > 0$ the solution $x(\cdot) = x(\cdot; x_0)$ satisfies

$$
0 \geq t^{-1}(V(x(t)) - V(x_0)) + t^{-1}(e^{at}-1)V(x(t)) + \int_0^t t^{-1} W(x(\tau)) d\tau. \tag{4.53}
$$

Thus, observing that $\frac{x(t)-x_0}{t} \to [f(x_0) - \mathrm{N}_C(x_0)]^\circ$ (recall Theorem 4.15(a)), and using the lower semi-continuous of V and W, as $t \downarrow 0$ in the last inequality we get

$$
V'(x_0, (f(x_0) - \mathrm{N}_C(x_0))^\circ) = \liminf_{\substack{t \downarrow 0 \\ w \to (f(x_0)-N_C(x_0))^\circ}} \frac{V(x_0 + tw) - V(x_0)}{t}
$$

$$
\leq \liminf_{t \downarrow 0} t^{-1}(V(x(t)) - V(x_0)) \leq -aV(x_0) - W(x_0).
$$

$(iv) \Rightarrow (ii)$ and $(v) \Rightarrow (iii)$, when $\partial = \partial_F$ or $\partial = \partial_P$. These implications follow due to the relation $\langle \xi, v \rangle \leq V'(x, v)$ for all $\xi \in \partial_F V(x)$, $x \in \mathrm{dom}V$, and $v \in H$.

$(iii) \Rightarrow (i)$ is an immediate consequence of the first part of the theorem together with (4.43).

Finally, to prove the last statement of the theorem when $\partial = \partial_L$ in the finite-dimensional case, we first check that $(i) \implies (iii)$. Assume that (i) holds and take $x \in \mathrm{dom}V$ together with $\xi \in \partial_L V(x)$, and let sequences $x_k \xrightarrow{V} x$ together with $\xi_k \in \partial_P V(x_k)$ such that $\xi_k \to \xi$. Since (iii) holds for $\partial = \partial_P$, for each k there exists $x_k^* \in \mathrm{N}_C(x_k) \cap \mathrm{B}(\theta, \|f(x_k)\|)$ such that

$$\langle \xi_k, f(x_k) - x_k^* \rangle + aV(x_k) + W(x_k) \le 0.$$

We may assume that (x_k^*) converges to some $x^* \in \mathrm{N}_C(x) \cap \mathrm{B}(\theta, \|f(x)\|)$ (thanks to the r-uniform prox-regularity of C), which then satisfies $\langle \xi, f(x) - x^* \rangle + aV(x) + W(x) \le 0$ (using the lower semi-continuous of the involved functions), showing that (iii) holds. Thus, since (iii) (with $\partial = \partial_P$) $\implies (i)$, we deduce that $(i) \iff (iii)$. This suffices to get the conclusion of the theorem. $\qquad \square$

Because the solution $x(\cdot)$ of differential inclusion (4.1) naturally lives in C, it is immediate that a (lower semi-continuous) function $V : H \to \overline{\overline{\mathbb{R}}}$ is Lyapunov for (4.1) iff the function $V + \mathrm{I}_C$ is Lyapunov. Hence, Theorem 4.19 also provides the characterization of Lyapunov functions without any restriction on their domains; for instance, accordingly to Theorem 4.19(iii), V is Lyapunov for (4.1) iff for every $x \in \mathrm{dom}V \cap C$ and $\xi \in \partial(V + \mathrm{I}_C)(x)$ it holds

$$\min_{x^* \in \mathrm{N}_C(x) \cap \mathrm{B}(\theta, \|f(x)\|)} \langle \xi, f(x) - x^* \rangle + aV(x) + W(x) \le 0.$$

The point here is that this condition is not completely written by means exclusively of the subdifferential of V. Nevertheless, this condition becomes more explicit in each time one can decompose the subdifferential set $\partial(V + \mathrm{I}_C)(x)$. For instance, this is the case, if V is locally Lipschitz and lower regular (particularly convex, see [64, Definition 1.91]). This fact is considered in Corollary 4.21 below. However, the following example shows that we can not get rid of the condition $\mathrm{dom}V \subset C$, in general.

Remark 4.20. We consider the differential inclusion (4.1) in $\mathbb{R}^2$, with $C := \mathbb{B}$ and $f(x, y) = (-y, x)$, whose unique solution such that $x(0) = (1, 0)$ is $x(t) = (\cos t, \sin t)$. We take $V = \mathrm{I}_S$, where

$$S := \{(1, y) : y \in [0, 1]\},$$

so that $\mathrm{dom}V \cap C = \{(1,0)\}$. For $\overline{x} := (1,0)$ and $\xi := (x,y) \in \partial_P V(\overline{x}) = \{(x,y)|\ y \leq 0\}$ we have that

$$\min_{x^* \in \mathrm{N}_C(\overline{x}) \cap \mathrm{B}(\theta, \|f(\overline{x}\|)} \langle \xi, f(\overline{x}) - x^* \rangle \leq \langle \xi, f(\overline{x}) \rangle = \langle (x,y), (0,1) \rangle = y \leq 0,$$

which shows that condition (iii) of Theorem 4.19 holds. However, it is clear that V is not a Lyapunov function of (4.1).

Corollary 4.21. *Let V, W and a be as in Theorem 4.19. Then the following assertions hold:*

(i) *If V is Fréchet differentiable on $\mathrm{dom}V \cap C$, then (V,W) is an a-Lyapunov pair of differential inclusion (4.1) iff for every $x \in \mathrm{dom}V \cap C$*

$$\langle \nabla V(x), (f(x) - \mathrm{N}_C(x))^\circ \rangle + aV(x) + W(x) \leq 0.$$

(ii) *If V is locally Lipschitz on $\mathrm{dom}V \cap C$, then (V,W) is an a-Lyapunov pair for differential inclusion (4.1) if for every $x \in \mathrm{dom}V \cap C$*

$$\langle \xi, (f(x) - \mathrm{N}_C(x))^\circ \rangle + aV(x) + W(x) \leq 0 \ \ \forall \xi \in \partial_L V(x).$$

(iii) *If H is of finite dimension and V is regular and locally Lipschitz on $\mathrm{dom}V \cap C$, then (V,W) is an a-Lyapunov pair for differential inclusion (4.1) iff for every $x \in \mathrm{dom}V \cap C$,*

$$\langle \xi, (f(x) - \mathrm{N}_C(x))^\circ \rangle + aV(x) + W(x) \leq 0 \ \ \forall \xi \in \partial_L V(x).$$

Proof. (i). Since $x(t) \in C$ for every $t \geq 0$, we have that (V,W) forms an a-Lyapunov pair for (4.1) iff the pair $(V + \mathrm{I}_C, W)$ does. Thus, since $\partial_F (V + \mathrm{I}_C)(x) = \nabla V(x) + \mathrm{N}_C(x)$ for every $x \in \mathrm{dom}V \cap C$, according to Proposition 1.107 in [64], Theorem 4.19 ensures that (V,W) is an a-Lyapunov pair of (4.1) iff for every $x \in \mathrm{dom}V \cap C$ and $\xi \in \mathrm{N}_C(x)$

$$\langle \nabla V(x) + \xi, (f(x) - \mathrm{N}_C(x))^\circ \rangle + aV(x) + W(x) \leq 0. \tag{4.54}$$

Because $0 \in \mathrm{N}_C(x)$ and $(f(x) - \mathrm{N}_C(x))^\circ \in \mathrm{T}_C^B(x) = (\mathrm{N}_C(x))^*$, it follow that this last inequality is equivalent to $\langle \nabla V(x), (f(x) - \mathrm{N}_C(x))^\circ \rangle + aV(x) + W(x) \leq 0$.

(ii). Under the current assumption, for every $x \in V \cap C$ we have that $\partial_L(V + \mathrm{I}_C)(x) \subset \partial_L V(x) + \mathrm{N}_C(x)$, and we argue as in the proof of statement (i).

(iii). In this case, we argue as above but using the relation $\partial_L(V + \mathrm{I}_C)(x) =$

$\partial_L V(x) + \mathrm{N}_C(x)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

It the result below, Theorem 4.19 is rewritten in order to characterize invariant sets associated to differential inclusion (4.1) (see Definition 4.5). Criterion (iii) below is of the same nature as the one used in [33].

Theorem 4.22. *Given a closed set $S \subset C$ we denote by N_S either N_S^P or N_S^F, and by T_S either T_S^B, T_S^w, $\overline{co}\,\mathrm{T}_S^w$, or $(\mathrm{N}_S)^*$. Then S is an invariant set for (4.1) iff one of the following equivalent statements hold:*

(i) $(f(x) - \mathrm{N}_C(x))^\circ \in \mathrm{T}_S(x) \ \forall x \in S$;

(ii) $[f(x) - \mathrm{N}_C(x)] \cap \mathrm{T}_S(x) \cap \mathrm{B}(\theta, \|f(x)\|) \neq \emptyset \ \forall x \in S$;

(iii) $\displaystyle\inf_{x^* \in [f(x) - \mathrm{N}_C(x)] \cap \mathrm{B}(\theta, \|f(x)\|)} \langle \xi, x^* \rangle \leq 0 \ \forall x \in S, \ \forall \xi \in \mathrm{N}_S(x)$.

Proof. Under the invariance of S we write (recall Theorem 4.15)

$$(f(x) - \mathrm{N}_C(x)))^\circ = \frac{d^+ x(0; x)}{dt} = \lim_{t \searrow 0} \frac{x(t) - x}{t} \in \mathrm{T}_S^B(x),$$

showing that (i) with $\mathrm{T}_S(x) = \mathrm{T}_S^B(x)$ holds. The rest of the implications follows by applying Theorem 4.19 with the use of the following equalities

$$\mathrm{T}_S^B(x) \subset \mathrm{T}_S^w(x) \subset \overline{co}(\mathrm{T}_S^w(x)) \subset (\mathrm{N}_S^F(x))^* \subset (\mathrm{N}_S^P(x))^*, \ x \in S,$$

where the star in the superscript refers to the dual cone. $\qquad\qquad\qquad$ □

4.6 Stability and observer designs

In this section, we give an application of the results developed in the previous sections, to study the stability and observer design for Lur'e systems involving nonmonotone set-valued nonlinearities. The state of the system is constrained to evolve inside a time-independent prox-regular set. More precisely, let us consider the following problem

$$\dot{x}(t) = Ax(t) + Bu(t), \text{ a.e. } t \in [0, \infty), \qquad\qquad (4.55\mathrm{a})$$

$$y(t) = Dx(t), \ \forall t \geq 0, \qquad\qquad\qquad\qquad\qquad (4.55\mathrm{b})$$

$$u(t) \in -\mathrm{N}_S(y(t)) \ \forall \, t \geq 0, \qquad\qquad\qquad\qquad (4.55\mathrm{c})$$

$$x(0) = x_0 \in D^{-1}(S); \qquad\qquad\qquad\qquad\qquad (4.55\mathrm{d})$$

where $x(t) \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times l}, D \in \mathbb{R}^{l \times n}, l \leq n$, and $S \subset \mathbb{R}^l$ is a uniformly-prox-regular set.

Using (4.55b) and (4.55c), and putting the resulting equation in (4.55a), we get the following differential inclusion

$$\dot{x}(t) \in Ax(t) - B\mathrm{N}_S(Dx(t)) \text{ a.e. } t \in [0, \infty), x(0) = x_0 \in D^{-1}(S). \qquad (4.56)$$

It is well-known that if $D : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping and S is a convex subset of $\mathbb{R}^m$, then the set

$$D^{-1}(S) := \{x \in \mathbb{R}^n : D(x) \in S\}$$

is always convex. This fails when S is prox-regular (see Example 2 in [4] for a counterexample). The following lemma provides a sufficient condition to ensure that $D^{-1}(S)$ is still prox-regular.

Lemma 4.23 ([78]). *Consider a nonempty, closed, r-prox-regular set S such that S is contained in the range space of a linear mapping $D : \mathbb{R}^n \to \mathbb{R}^l$. Then the set $D^{-1}(S)$ is r'-uniformly prox-regular with $r' := \frac{r\delta_D^+}{\|D\|^2}$, where δ_D^+ denote the least positive singular value of the matrix D.*

The following proposition shows that system (4.55), or equivalently (4.56), can be transformed into a differential inclusion of the form (4.1).

Proposition 4.24. *Let us consider system (4.55). Assume that S is contained in the range space of D and there exists a symmetric positive definite matrix P such that $PB = D^T$. Then every solution of (4.55) is also a solution of the following system*

$$\dot{z}(t) \in f(z(t)) - \mathrm{N}_{S'}(z(t)), \ \ a.e. \ t \geq 0, z(0) \in S',$$

with $z(t) = P^{\frac{1}{2}}x(t), f = P^{\frac{1}{2}}AP^{-\frac{1}{2}}$ and $S' = (DP^{-\frac{1}{2}})^{-1}(S)$.

Proof. We set $R := P^{\frac{1}{2}}$. According to Lemma 4.23, the set S' is r'-uniformly prox-regular with $r' := \frac{r\delta_{DR^{-1}}^+}{\|DR^{-1}\|^2}$. Combining this with the basic chain rule (see Theorem 10.6, [76]), for any $x \in \mathbb{R}^n$, one has

$$(DR^{-1})^T\mathrm{N}_S(DR^{-1}x) = (DR^{-1})^T\partial\mathrm{I}_S(DR^{-1}x) \subset \partial(\mathrm{I}_S \circ (DR^{-1}))(x)$$
$$= \partial\mathrm{I}_{S'}(x) = \mathrm{N}_{S'}(x).$$

By the hypothesis $PB = C^T$, we deduce that $DR^{-1} = (RB)^T$. From the above inclusion, it is easy to see that for a.e. $t \geq 0$, one has

$$\begin{aligned}
\dot{z}(t) &\in RAR^{-1}z(t) - RB\mathrm{N}_S(DR^{-1}z(t)) \\
&= RAR^{-1}z(t) - (DR^{-1})^T\mathrm{N}_S(DR^{-1}z(t)) \subset RAR^{-1}z(t) - \mathrm{N}_{S'}(z(t)).
\end{aligned} \qquad (4.57)$$

The proof of Proposition 4.24 is thereby completed. $\square$

The above Proposition proves that under some assumptions, system (4.55) can be studied within the framework of (4.1). Let us now investigate the asymptotic stability of differential inclusion (4.1)

$$\begin{cases} \dot{x}(t) \in f(x(t)) - N_C(x(t)) & \text{a.e. } t \geq 0, \\ x(0; x_0) = x_0 \in C, \end{cases}$$

at the equilibrium point 0, with the assumption $0 \in C$ and $f(0) = \theta$.
Recall that the set C is an r-uniformaly prox-regular set ($r > 0$), and that f is a Lipschitz continuous mapping with Lipschitz constant L.

We have the following result which provides a partial extension of [78, Theorem 3.2] (here, we are considering the case where the set C is time-independent).

Theorem 4.25. *Assume that $0 \in C, f(0) = \theta$. If there exist $\varepsilon, \delta > 0$ such that*

$$\langle x, f(x) \rangle + \delta ||x||^2 \leq 0 \quad \forall x \in C \cap B(\theta, \varepsilon). \tag{4.58}$$

Then

$$\lim_{t \to \infty} x(t, x_0) = 0 \quad \text{for all } x_0 \in \text{int}(B(\theta, \min\{r\delta l^{-1}, \varepsilon\})) \cap C.$$

Proof. We shall verify that the (lower semi-continuous proper) function $V : H \to \mathbb{R} \cup \{+\infty\}$, defined by $V(x) := \frac{1}{2}||x||^2 + I_C(x)$, satisfies the assumption of Theorem 4.19 (when $W \equiv 0$ and $a = \delta$). We fix $\eta \in (0, \min\{r\delta L^{-1}, \varepsilon\})$, $x \in B(\theta, \eta) \cap C$ and $\xi \in \partial_P V(x) \subset x + N_C(x)$ ([32, Ch. 1, Proposition 2.11]); hence, since $(f(x) - N_C(x))° = \Pi_{T_C(x)}(f(x)) \in T_C(x)$ we obtain

$$\langle \xi, (f(x) - N_C(x))° \rangle \leq \langle x, (f(x) - N_C(x))° \rangle = \langle x, f(x) - \Pi_{N_C(x)}(f(x)) \rangle,$$

so that, by (4.58),

$$\langle \xi, (f(x) - N_C(x))° \rangle \leq -\langle x, \Pi_{N_C(x)}(f(x)) \rangle - 2\delta V(x). \tag{4.59}$$

Moreover, because $\Pi_{N_C(x)}(f(x)) \in N_C(x)$ and $\theta \in C$, from the r-uniformaly prox-regularity of the set C we have

$$\langle \Pi_{N_C(x)}(f(x)), -x \rangle \leq \frac{\left\| \Pi_{N_C(x)}(f(x)) \right\|}{r} V(x) \leq \frac{\|f(x)\|}{r} V(x),$$

and we get, using (4.59),

$$\langle \xi, f(x) - \Pi_{\mathrm{N}_C(x)}(f(x)) \rangle + \delta V(x) = \langle \xi, (f(x) - \mathrm{N}_C(x))^\circ \rangle + \delta V(x)$$
$$\leq (r^{-1} \|f(x)\| - \delta) V(x).$$

But, by the choice of η we have $\|f(x)\| = \|f(x) - f(0)\| \leq l \|x\| \leq L\eta \leq r\delta$, and so,

$$\langle \xi, f(x) - \Pi_{\mathrm{N}_C(x)}(f(x)) \rangle + \delta V(x) \leq 0.$$

Consequently, observing that $\Pi_{\mathrm{N}_C(x)}(f(x)) \in \mathrm{N}_C(x) \cap \mathrm{B}(\theta, \|f(x)\|)$, by Theorem 4.19 we deduce that for every $x_0 \in C \cap \mathrm{int}(\mathrm{B}(\theta, \eta))$, there exists $t_0 > 0$ such that

$$e^{\delta t} V(x(t; x_0)) \leq V(x_0) \ \ \forall t \in [0, t_0];$$

hence, in particular, $\frac{1}{2}\|x(t; x_0)\|^2 \leq \frac{1}{2}\|x_0\|^2$ and $x(t_0; x_0) \in C \cap \mathrm{int}(\mathrm{B}(\theta, \eta))$. This proves that

$$\hat{t}_0 := \sup\{t > 0 \mid e^{\delta t} V(x(s; x_0)) \leq V(x_0) \ \forall s \in [0, t]\} = +\infty,$$

and we conclude that

$$e^{\delta t} V(x(t; x_0)) \leq V(x_0) \ \ \forall t \geq 0,$$

which leads us to the desired conclusion. $\qquad \square$

Corollary 4.26. *Let us consider system* (4.55). *Assume that S is uniformly prox-regular set such that S is contained in the rank of D. If there exists a symmetric positive definite matrix P and $\delta > 0$ such that*

$$A^T P + PA \leq -\delta P, \ PB = D^T, \tag{4.60}$$

then

$$\lim_{t \to \infty} x(t; x_0) = 0 \ \ \text{for all } x_0 \in \mathrm{int}(\mathrm{B}(\theta, \rho)) \cap S,$$

where $\rho := (2\|R^{-1}\| \, \|DR^{-1}\| \, \|RAR^{-1}\|)^{-1} \delta r \delta_{DR^{-1}}^+$.

Proof. Firstly we will show that for any $x \in \mathbb{R}^n$, one has

$$\langle RAR^{-1}x, x \rangle + \frac{\delta}{2}\|x\|^2 \leq 0.$$

Indeed, by the first inequality of (4.60), for every $x \in \mathbb{R}^n$, one has

$$\langle(A^T P + PA + \delta P)x, x\rangle = \langle(A^T R^2 + R^2 A + \delta R^2)x, x\rangle \leq 0.$$

Since R is positive definite, for any $z = R^{-1}x$, one has

$$\begin{aligned}
0 &\geq \langle(A^T P + PA + \delta P)R^{-1}x, R^{-1}x\rangle = \langle(A^T R + PAR^{-1} + \delta R)x, R^{-1}x\rangle \\
&= \langle(R^{-T}A^T R + RAR^{-1} + \delta I_n)x, x\rangle = 2\langle RAR^{-1}x, x\rangle + \delta||x||^2.
\end{aligned} \tag{4.61}$$

Applying Theorem 4.25 to system (4.57) with $f = RAR^{-1}, C = S', r = r'$, we get

$$\lim_{t\to\infty} z(t; z_0) = 0,$$

for every $z_0 \in \text{int}[\text{B}(\theta, \frac{1}{2}||R^{-1}AR||^{-1}r'\delta)] \cap S'$. Combining this with the fact that $x(t) = R^{-1}z(t)$, the conclusion of Corollary 4.26 follows because

$$z = Rx \in \text{int}\left(\text{B}(0, \frac{1}{2}||R^{-1}AR||^{-1}r'\delta)\right),$$

for any $x \in \text{int}[\text{B}(0, \rho)]$. $\qquad\square$

Next let us remind the Luenberger-like observer associated to differential inclusion (4.55). Given $x_0 \in D^{-1}(S)$, we assume that the output equation associated with differential inclusion (4.55) is

$$y(t) = G(x(t; x_0))$$

where $G \in \mathbb{R}^{p\times n}$ with $p \leq n$.

The Luenberger-like observer associated to differential inclusion (4.55) has the following form

$$\begin{aligned}
\dot{\hat{x}}(t) &= (A - LG)\hat{x}(t) + Ly(t) + B\hat{u}(t), & \text{(4.62a)} \\
\hat{y}(t) &= D\hat{x}(t), & \text{(4.62b)} \\
\hat{u}(t) &\in -\text{N}_S(\hat{y}(t)), & \text{(4.62c)} \\
\hat{x}(0) &= z_0, & \text{(4.62d)}
\end{aligned}$$

where $L \in \mathbb{R}^{n\times p}$ is the observer gain. This differential inclusion always has a unique solution, denoted by $\hat{x}(\cdot; z_0)$. We want to find the gain L for the basic

observer such that

$$\lim_{t\to\infty} ||\hat{x}(t;z_0)-x(t;x_0)||= 0, \text{ for all } z_0 \in \mathrm{B}(x_0,\rho)\cap D^{-1}(S) \text{ for some } \rho > 0. \quad (4.63)$$

We see that if $\hat{x}(\cdot) := \hat{x}(\cdot;z_0)$ is the solution of (4.62), then it is also the solution of the differential inclusion

$$\dot{\hat{x}}(t) \in (A - LG)\hat{x}(t) + Ly(t) - B\mathrm{N}_S(D\hat{x}(t)) \text{ a.e. } t \geq 0, \hat{x}(0) = z_0. \quad (4.64)$$

Under the hypothesis

$$\exists\, P \text{ symmetric positive definite, such that } PB = D^T, \quad (4.65)$$

similarly to the proof of Proposition 4.24, we have

$$\dot{\hat{z}}(t) \in (RAR^{-1} - RLG')\hat{z}(t) + RLG'z(t) - \mathrm{I}_{S'}(\hat{z}(t)),$$

where $G'^{-1}, \hat{z}(t) := R\hat{x}(t;z_0)$ and $z(t) = Rx(t;x_0), S'^{-1})^{-1}(S)$.
On the other hand, one has

$$||R||^{-1}\, ||\hat{z}(t) - z(t)||\leq ||\hat{x}(t) - x(t)||\leq ||R^{-1}||\, ||\hat{z}(t) - z(t)||,$$

which means that $||\hat{z}(t) - z(t)||\to 0$ as $t \to \infty$ if and only if $||\hat{x}(t) - x(t)||$ does.

Next, we investigate a general Luenberger-like observer associated to our differential inclusion (4.1). Following the same idea as above, we assume that $x_0 \in C$ and the output equation associated with differential inclusion (4.1) is

$$y(t) = \mathcal{G}(x(t;x_0)),$$

where $\mathcal{G} : H \to H$ is a Lipschitz mapping. We want to find a Lipschitz mapping $\mathcal{L} : H \to H$ such that the solution $\hat{x}(\cdot;z_0)$ of the differential inclusion

$$\begin{cases} \dot{\hat{x}}(t) \in f(\hat{x}(t)) - \mathcal{L}(\mathcal{G}(\hat{x}(t)) + \mathcal{L}(y(t)) - \mathrm{N}_C(\hat{x}(t)) & \text{a.e. } t \geq 0 \\ \hat{x}(0) = z_0 \in C, \end{cases} \quad (4.66)$$

satisfies, for some $\rho > 0$,

$$\lim_{t\to\infty} ||\hat{x}(t;z_0) - x(t;x_0)||= 0, \text{ for all } z_0 \in \mathrm{B}(x_0,\rho) \cap C.$$

To solve this problem we consider the Lipschitz mapping $\tilde{f} : H \times H \to H \times H$, defined as

$$\tilde{f}(z, x) := \Big(f(z) - \mathcal{L}(\mathcal{G}(z)) + \mathcal{L}(\mathcal{G}(x)), f(x) \Big), \tag{4.67}$$

together with the set $S := C \times C$; hence, $\mathrm{N}_S^P(x, y) = \mathrm{N}_C(x) \times \mathrm{N}_C(y)$, for every $(x, y) \in S$, so that S is also an r-uniformly prox-regular set. Consequently, we easily check that $y(t) := (\hat{x}(t; z_0), x(t; x_0))$ is the unique solution of the differential inclusion

$$\dot{y}(t) \in \tilde{f}(y(t)) - \mathrm{N}_S(y(t)) \text{ a.e. } t \geq 0, \ y(0) = (z_0, x_0) \in S.$$

We have the following result, which extends [78, Proposition 3.5] in the case where the set C does not depend on the time variable.

Theorem 4.27. *Fix $(z_0, x_0) \in C \times C$ and assume that the solution of (4.1), $x(t; x_0)$, is bounded, say $||x(t; x_0)|| \leq m$ for all $t \geq 0$. If $M := \sup\{||f(x)||, \ x \in \mathrm{B}(\theta, m)\}$, we choose a Lipschitz continuous mapping $\mathcal{L}$ together with positive numbers $\delta, \varepsilon, \eta > 0$ such that $\varepsilon < \delta r - M$, $\eta \leq (6l)^{-1}\varepsilon$, and*

$$||x - y|| \leq 3\eta, \ x, y \in H \implies ||\mathcal{L}(\mathcal{C}(x)) - \mathcal{L}(\mathcal{C}(y))|| \leq \varepsilon, \tag{4.68}$$

at the same time as, for all $x, y \in \mathrm{B}(\theta, m + 3\eta)$,

$$\langle x - y, (f - \mathcal{L} \circ \mathcal{G})(x) - (f - \mathcal{L} \circ \mathcal{C})(y) \rangle \leq -\delta ||x - y||^2. \tag{4.69}$$

Then for every $z_0 \in \mathrm{B}(x_0, \eta)$ we have that

$$||\hat{x}(t; z_0) - x(t; x_0)|| \leq e^{\frac{-(\delta - \frac{M+\varepsilon}{r})}{2}t} ||z_0 - x_0||,$$

and, consequently,

$$||\hat{x}(t; z_0) - x(t; x_0)|| \to 0 \text{ as } t \to +\infty.$$

Proof. For every $z, y \in \mathrm{B}(\theta, m + 3\eta) \cap C$ such that $||z - y|| \leq 3\eta$ we have that

$$\max\{||f(z)||, ||f(y)||\} \leq M + 3\eta l \leq M + \frac{\varepsilon}{2},$$

$$||\mathcal{L}(\mathcal{G}(z)) - \mathcal{L}(\mathcal{G}(y))|| \leq \varepsilon.$$

We consider the (C^1-) function $V : H \times H \to \mathbb{R}$ defined as $V(z, y) := \frac{1}{2}||z - y||^2$.

If $\beta := \delta - \frac{M+\varepsilon}{r}$, then by definition (4.67), we obtain

$$
\begin{aligned}
&\langle V'(z,y), (\tilde{f}(z,y) - \mathrm{N}_S(z,y))^\circ\rangle + \beta V(z,y) \\
&\quad = \langle z - y, f(z) - \mathcal{L}(\mathcal{G}(z)) + \mathcal{L}(\mathcal{G}(y)) - \Pi_{\mathrm{N}_C(z)}(f(z) - \mathcal{L}(\mathcal{G}(z)) + \mathcal{L}(\mathcal{G}(y)))\rangle \\
&\qquad + \langle y - z, f(y) - \Pi_{\mathrm{N}_C(y)}(f(y))\rangle + \frac{\beta}{2}||z - y||^2 \\
&\quad = \langle z - y, f(z) - f(y) - \mathcal{L}(\mathcal{G}(z)) + \mathcal{L}(\mathcal{G}(y))\rangle + \langle z - y, \Pi_{\mathrm{N}_C(y)}(f(y))\rangle \\
&\qquad - \langle z - y, \Pi_{\mathrm{N}_C(z)}(f(z) - \mathcal{L}(\mathcal{G}(z)) + \mathcal{L}(\mathcal{G}(y)))\rangle + \frac{\beta}{2}||z - y||^2.
\end{aligned}
$$

Since $\Pi_{\mathrm{N}_C(y)}(f(y)) \in \mathrm{N}_C(y)$ and $\left\|\Pi_{\mathrm{N}_C(y)}(f(y))\right\| \le \|f(y)\|$, and similarly for $\Pi_{\mathrm{N}_C(z)}(f(z) - \mathcal{L}(\mathcal{G}(z)) + \mathcal{L}(\mathcal{G}(y)))$, the last equality yields

$$
\begin{aligned}
&\langle V'(z,y), (\tilde{f}(z,y) - \mathrm{N}_S(z,y))^\circ\rangle + \beta V(z,y) \\
&\quad \le \langle z - y, f(z) - f(y) - \mathcal{L}(\mathcal{G}(z)) + \mathcal{L}(\mathcal{G}(y))\rangle + \frac{||f(y)||}{2r}||z - y||^2 \\
&\qquad + \frac{||f(z) - \mathcal{L}(\mathcal{G}(z)) + \mathcal{L}(\mathcal{G}(y))||}{2r}||z - y||^2 + \frac{\beta}{2}||z - y||^2,
\end{aligned}
$$

which by assumptions (4.68) and (4.69) gives us

$$
\begin{aligned}
&\langle V'(z,y), (\tilde{f}(z,y) - \mathrm{N}_S(z,y))^\circ\rangle + \beta V(z,y) \\
&\quad \le \langle z - y, f(z) - f(y) - \mathcal{L}(\mathcal{G}(z)) + \mathcal{L}(\mathcal{G}(y))\rangle + \frac{||f(z)|| + ||f(y)||}{2r}||z - y||^2 \\
&\qquad + \frac{||\mathcal{L}(\mathcal{G}(z)) - \mathcal{L}(\mathcal{G}(y))||}{2r}||z - y||^2 + \frac{\beta}{2}||z - y||^2 \\
&\quad \le \langle z - y, f(z) - f(y) - \mathcal{L}(\mathcal{G}(z)) + \mathcal{L}(\mathcal{G}(y))\rangle + \frac{M + \varepsilon}{r}||z - y||^2 + \frac{\beta}{2}||z - y||^2 \\
&\quad \le -\delta||z - y||^2 + (\frac{M + \varepsilon}{r} + \frac{\beta}{2})||z - y||^2 \le 0. \tag{4.70}
\end{aligned}
$$

Now we choose $z_0 \in \mathrm{B}(x_0, \eta) \cap C$, so that

$$
\mathrm{B}(z_0, \eta) \times \mathrm{B}(x_0, \eta) \subset [\mathrm{B}(\theta, m + 3\eta) \times \mathrm{B}(\theta, m + 3\eta)] \cap \{(z, y) \in H \times H : ||z - y|| \le 3\eta\}.
$$

Then, thanks to (4.70), we can apply Corollary 4.21(i) to find some $t_0 > 0$ such that for every $t \in [0, t_0]$

$$
e^{\beta t}V(\hat{x}(t; z_0), x(t; x_0)) \le V(z_0, x_0);
$$

that is,

$$||\hat{x}(t; z_0) - x(t; x_0)|| \leq e^{\frac{-\beta t}{2}} ||z_0 - x_0||.$$

Moreover, since $||\hat{x}(t_0; z_0) - x(t_0; x_0)|| \leq \eta$ and $\hat{x}(t_0; z_0) \in \mathrm{B}(\theta, m + 2\eta) \cap C$, we can also find $t_1 > 0$ such that for any $t \in [0, t_1]$

$$
\begin{aligned}
||\hat{x}(t + t_0; z_0) - x(t + t_0; x_0)|| &\leq e^{\frac{-\beta t}{2}} ||\hat{x}(t_0; z_0) - x(t_0; x_0)|| \\
&\leq e^{\frac{-\beta t}{2}} e^{\frac{-\beta t_0}{2}} ||z_0 - x_0|| = e^{\frac{-\beta(t+t_0)}{2}} ||z_0 - x_0||.
\end{aligned}
$$

Consequently, we deduce that for every $t \geq 0$

$$||\hat{x}(t; z_0) - x(t; x_0)|| \leq e^{\frac{-\beta t}{2}} ||z_0 - x_0||,$$

which completes the proof. $\qquad\square$

To close this section we consider the special case of linear Luenberger-like , where the assumption of Theorem 4.27 takes a simpler form. In this case (4.66) is written as

$$
\begin{cases}
\dot{\hat{x}}(t) \in (A - LG)\hat{x}(t) + LGx(t) - \mathrm{N}_C(\hat{x}(t)) & \text{a.e. } t \geq 0 \\
\hat{x}(0) = z_0 \in C,
\end{cases}
$$

where $A, L, G : H \to H$ are linear continuous mappings; A^* and G^* will denote the corresponding adjoints mappings. Assume that $x(\cdot) := x(\cdot; x_0)$, $x_0 \in C$, is the solution of (4.1) (corresponding to $f = A$).

Corollary 4.28. *Fix* $(z_0, x_0) \in C \times C$ *and assume that the solution of (4.1) (corresponding to* $f = A$*),* $x(t; x_0)$*, is bounded, say* $||x(t; x_0)|| \leq m$ *for all* $t \geq 0$*. Let* $\delta, \varepsilon, \rho > 0$ *be such that*

$$r^{-1}(m||f|| + \varepsilon) < \delta, \text{ and } \frac{1}{2}(A + A^*) - \rho G^*G \leq -\delta\mathrm{id}.$$

If $L := \rho G^*$*,* $\eta := \min\{(6||A||)^{-1}\varepsilon, (3||LG||)^{-1}\varepsilon\}$*, and* $\beta := \delta - r^{-1}(m||A|| + \varepsilon)$*, then for every* $z_0 \in \mathrm{B}(x_0, \eta)$ *we have that, for all* $t \geq 0$*,*

$$||\hat{x}(t; z_0) - x(t; x_0)|| \leq e^{\frac{-\beta t}{2}} ||z_0 - x_0||.$$

Proof. The proof is similar as the one of Theorem 4.27, by observing that for every

$x \in H$, we have

$$\langle x, (A - LG)x \rangle = \frac{\langle x, (A - LG)x \rangle + \langle x, (A^* - G^*L^*)x \rangle}{2}.$$

$\square$

4.7 Concluding remarks

In this paper, we proved that a differential variational inequality involving a prox-regular set can be equivalently written as a differential inclusion governed by a maximal monotone operator. Therefore, the existence result and the stability analysis can be conducted in a classical way. We also give a characterization of lower semi-continuous a-Lyapunov pairs and functions. An application to a Luenberger-like observer is proposed. These new results will open new perspectives from both the numerical and applications points of view. An other interesting problem dealing with sweeping processes was introduced by J.J. Moreau in the seventies, which is of a great interest in applications. This problem is obtained by replacing the fixed set C by a moving set $C(t)$, $t \in [0, T]$. It will be interesting to extend the ideas developed in this current work to the sweeping process involving prox-regular sets. Many other issues require further investigation including the study of numerical methods for problem (4.1) and the extension to second-order dynamical systems. This is out of the scope of the present paper and will be the subject of a future project of research.

Chapter 5

Lyapunov stability of differential inclusions with Lipschitz Cusco perturbations of maximal monotone operators

We characterize weak and strong invariant closed sets with respect to differential inclusions given in $\mathbb{R}^n$ and governed by Lipschitz Cusco perturbations of maximal monotone operators. Correspondingly, we provide different characterizations for Lyapunov functions and pairs for such differential inclusions. Our criteria of invariance and Lyapunov functions/pairs only depend on the data of the system and the geometry of the involved candidates for invariant sets and Lyapunov functions, and thus, no need to explicit calculus of the solutions, nor to calculus on the semi-group generated by the underlying maximal monotone operator.

5.1 Introduction

Our main purpose in this paper is to give explicit characterizations for weak and strong invariant closed sets with respect to the following differential inclusion, given $\in \mathbb{R}^n$ as

$$\dot{x}(t) \in F(x(t)) - A(x(t)) \quad \text{a.e. } t \geq 0, \ x(0) = x_0 \in \overline{\mathrm{dom}A}, \qquad (5.1)$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a Lipschitz Cusco multifunction; that is, a Lipschitz set-valued mapping with nonempty, convex and compact values, and $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a

maximal monotone operator. There is no restriction on the initial condition x_0 that can be any point in the closure of the domain of A, possibly not a point of definition of A. We also characterize weak and strong Lyapunov functions and, more generally, a-Lyapunov pairs associated to the differential inclusion above. Our criteria are given by means only of the data of the system; that is, the multifunction F and the operator A, together with first-order approximations of the invariant sets candidates, using Bouligand tangent cones, or, equivalently, Fréchet or proximal normal cones, and first-order (general) derivatives of Lyapunov functions candidates, using directional derivatives, Fréchet or proximal subdifferentials.

Our analysis aims at gathering two different kinds of dynamic systems in one, that were studied separately in the literature, at least in what concerns Lyapunov stability. The first kind of these dynamic systems is governed exclusively by Cusco multifunctions, and gives rise to a natural extension of the classical differential equations, given in the form

$$\dot{x}(t) \in F(x(t)) \quad \text{a.e. } t \geq 0, \ x(0) = x_0 \in \mathbb{R}^n. \tag{5.2}$$

The consideration of differential inclusions rather than differential equations allows more useful existence theorems, as revealed by Filippov's theory for differential equation with discontinuous right-hand-sides [42]. Stability of such systems, namely, the study of Lyapunov functions and invariant sets, has been extensively studied and investigated especially during the nineties by many authors; see, for example, [30, 32, 39], as well as [10, 13, 43] (see, also, the references therein). For instance, complete weak and strong invariance characterizations for closed sets can be found in [30] in the finite-dimensional setting, and in [32] for Hilbert spaces. It is worth recalling that only the upper semi-continuity of the Cusco mapping F is required for the weak invariance, while Lipschitzianity is used for the strong invariance (see [32]). Invariance characterizations of a same nature have been done in [39] for one-side Lipschitz (not necessary Lipschitz) and compact valued multifunctions. These results have been adapted in [33] to the following more general differential inclusion (for $T \in [0, +\infty]$)

$$\dot{x}(t) \in F(t, x(t)) - \mathrm{N}_{C(t)}(x(t)) \text{ a.e. } t \in [0, T], \ x(0) = x_0 \in C(0), \tag{5.3}$$

where $C(t)$ is a uniformly prox-regular sets in $\mathbb{R}^n$ and $\mathrm{N}_{C(t)}$ is the associated normal cone. Observe here that the right-hand-side may be unbounded, but however, in the case when $T < +\infty$, the last differential inclusion above is equivalent to the

following one, for some positive constant $M > 0$,

$$\dot{x}(t) \in F(t, x(t)) - \mathrm{N}_C(t)(x(t)) \cap \mathrm{B}(\theta, M) \ \text{ a.e. } t \in [0, T], x(0) = x_0 \in C(0),$$

giving rise to a differential inclusion in the form of (5.2).

The other kind of systems that is covered by (5.1) concerns differential inclusions governed by maximal monotone operators, or, more generally, (single-valued) Lipschitz perturbations of these operators, that we write as

$$\dot{x}(t) \in f(x(t)) - A(x(t)) \ \text{ a.e. } t \geq 0, \ x(0) = x_0 \in \overline{\mathrm{dom}A}. \tag{5.4}$$

This system can be seen as perturbations of the ordinary differential equation $\dot{x}(t) = f(x(t))$, where A could represent some associated control action. In this single-valued Lipschitzian setting, weak and strong invariance coincide since differential inclusion (5.1) possesses unique solutions. Compared to (5.2) the right-hand-side in this differential inclusion can be unbounded, or even empty. Typical examples of (5.4) involve the Fenchel subdifferential of proper, lower semi-continuous convex functions ([2]). System (5.4) has been extensively studied, namely, regarding existence, regularity and properties of the solutions [21], while Lyapunov stability of such systems have been initiated in [70]; see, also, [6–8] for recent contributions on the subject. Different criteria using the semi-group generated by the operator A can also be found in [54], where Lyapunov functions are characterized as viscosity-type solutions of Hamilton-Jacobi equations, and in [26], using implicit tangent cones associated to the invariant sets candidates.

It is worth observing that (5.1) is a special case of the following more general differential inclusion

$$\dot{x}(t) \in F(t, x(t)) - A(t)(x(t)) \ \text{ a.e. } t \geq 0, \ x(0) = x_0 \in \overline{\mathrm{dom}A(0, \cdot)}, \tag{5.5}$$

where A and F are also allowed to move in an appropriate way with respect to the time, satisfying some natural continuity and measurability conditions. Existence of solution of (5.5) have been also studied in [9, 59, 84] among others. In particular, [9] considered in a Hilbert setting similar systems as the one in (5.1), but with requiring strong assumptions on the multifunction F. In [84] the authors assume that F is a single-valued mapping, that is Lipschitz with respect to the second variable, while the minimal section mapping of the maximal monotone operators $A(t)$ is uniformly bounded.

In this paper, we study and characterize strong and weak invariant closed

subsets of the closure of the domain of A, dom$A$, with respect to differential inclusion (5.1). We shall assume in our analysis that the invariant sets candidates $S \subset \mathbb{R}^n$ satisfy the following condition

$$\Pi_S(x) \subset S \cap \text{dom}A \ \ \forall x \in \text{dom}A, \tag{5.6}$$

where Π_S refers to the projection operator on S. This condition has been used in many works; see, for instance, [15], where the author is concerned with flow invariance characterizations for differential equations, with right-hand-sides given by nonlinear semigroup generators in the sense of Crandall- Liggett (see [37]). It is clear that condition (5.6) holds whenever $S \subset \text{dom}A$. When dealing with weak invariant closed sets, we shall require some usual boundedness conditions on the invariant set, relying on the minimal norm section of the maximal monotone operator A. The invariance criteria are then used to characterize weak and strong a-Lypaunov pairs of extended-real-valued proper lower semi-continuous functions (V, W) associated to (5.1), such that dom$V \subset$dom$A$. These results are specified to differential inclusions involving normal cones to a uniformly prox-regular set C, given in the form

$$\dot{x}(t) \in F(x(t)) - \text{N}_C(x(t)) \ \text{a.e.} \ t \in [0, T], \ x(0) = x_0 \in C,$$

where we provide existence and properties of the solutions as well as different characterizations for invariant closed subsets and Lyapunov functions/pairs, all written by means of the multifunction F and the set C.

The paper is organized as follows: After Section 2, reserved to give the necessary notations and present the main tools, we make in Section 3 a review of the existence theorems of differential inclusion (5.1), and establish some first properties of the solutions. In Section 4 we characterize weak and strong invariant closed sets with respect to (5.1), while in Section 5 strong and weak Lyapunov pairs are provided. In Section 6 we apply the previous results to study differential inclusions involving normal cones to uniformly prox-regular sets.

5.2   Notation and main tools

In this paper, $\mathbb{R}^n$ is a (real) finite-dimensional Hilbert space with the null vector is denoted by θ, the notations $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ are the inner product and the norm, respectively. For each $x \in \mathbb{R}^n$ and $\rho \geq 0$, $\text{B}(x, \rho)$ is the closed with center x and

radius ρ and $B_r := B(\theta, r)$.

Given a nonempty set S, the notation $\overline{S}$ is closure of S. We denote $\|S\|$ is real positive number define by

$$\|S\| := \sup\{\|v\| : \ v \in S\}.$$

The *distance function* to S is defined by

$$d_S(x) := \inf\{\|x - s\|, s \in S\},$$

and *orthogonal projection mapping* onto S defined as

$$\Pi_S(x) := \{s \in S : \ \|x - s\| = d_S(x)\}.$$

If S is a closed set then $\Pi_S(x) \neq \emptyset$ for every $x \in \mathbb{R}^n$, we denote $S^\circ := \Pi_S(\theta)$ is minimal norm in S. The *indicator function* of S is defined as

$$I_S(x) := \begin{cases} 0 & \text{if } x \in S \\ +\infty & \text{if } x \notin S, \end{cases}$$

and the *support function* of S is defined as

$$\sigma_S(x) := \sup\{\langle x, s \rangle : s \in S\},$$

with the convention that $\sigma_\emptyset = -\infty$. Given a function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, its *domain* and *epigraph* are defined by

$$\text{dom}\varphi := \{x \in \mathbb{R}^n : \varphi(x) < +\infty\};$$
$$\text{epi}\varphi := \{(x, \alpha) \in \mathbb{R}^{n+1} : \varphi(x) \leq \alpha\}.$$

We say φ is proper if $\text{dom}\varphi \neq \emptyset$; lower semi-continuous , if $\text{epi}\varphi$ is closed. Notation $\mathcal{F}(\mathbb{R}^n)$ is the set all proper, lower semi-continuous functions.

We now introduce some basic concepts of nonsmooth and variational analysis. Let $\varphi \in \mathcal{F}(\mathbb{R}^n)$ and $x \in \text{dom}\varphi$. We call $\xi \in \mathbb{R}^n$ is a *proximal subgradient* of φ at x, written $\xi \in \partial_P \varphi(x)$ if

$$\liminf_{y \to x, y \neq x} \frac{\varphi(y) - \varphi(x) - \langle \xi, y - x \rangle}{\|y - x\|^2} > -\infty.$$

A vector $\xi \in \mathbb{R}^n$ is said to be a *Fréhet subgradient* of φ at x, written $\xi \in \partial_P \varphi(x)$ if

$$\liminf_{y \to x, y \neq x} \frac{\varphi(y) - \varphi(x) - \langle \xi, y - x \rangle}{\|y - x\|} \geq 0.$$

The *limiting subdifferential* of φ at x is defined as

$$\partial_L \varphi(x) := \{ \lim_{n \to \infty} \xi_n \mid \xi_n \in \partial_P \varphi(x_n), x_n \to x, f(x_n) \to f(x) \};$$

and the singular subdifferential of φ at x is defined as

$$\partial_\infty \varphi(x) := \{ \lim_{n \to \infty} \alpha_n \xi_n \mid \xi_n \in \partial_P \varphi(x_n), x_n \to x, f(x_n) \to f(x), \alpha_n \downarrow 0 \};$$

The *Clarke subdifferential* is defined as

$$\partial_C \varphi(x) := \overline{\mathrm{co}}(\partial_L \varphi(x) + \partial_\infty \varphi(x)).$$

In the case $x \notin \mathrm{dom}\varphi$, then by convention, we set $\partial_P \varphi(x) = \partial_F \varphi(x) = \partial_L \varphi(x) = \emptyset$. We have the classical inclusions $\partial_P \varphi(x) \subset \partial_F \varphi(x) \subset \partial_L \varphi(x)$.

If φ is locally Lipschitz around x, then $\partial_\infty \varphi(x) = \{\theta\}$ and

$$\partial_C \varphi(x) = \overline{\mathrm{co}}(\partial_L \varphi(x)).$$

The *generalized directional derivative* of φ at x in the direction v which defined by

$$\varphi^0(x; v) := \limsup_{y \to x, t \downarrow 0} \frac{\varphi(y + tv) - \varphi(y)}{t}.$$

We have that

$$\varphi^0(x; v) = \sup_{\xi \in \partial_C \varphi(x)} \langle \xi, v \rangle \ \forall v \in \mathbb{R}^n.$$

We also remind the *contingent directional derivative* of φ at $x \in \mathrm{dom}\varphi$ in the direction $v \in \mathbb{R}^n$ is

$$\varphi'(x; v) := \liminf_{t \to 0^+, w \to v} \frac{\varphi(x + tw) - \varphi(x)}{t}.$$

From definitions of proximal subgradient, Fréhet subgradient, it is easy to prove that

$$\sigma_{\partial_P V(x)}(\cdot) \leq \sigma_{\partial_F V(x)}(\cdot) \leq V'(x; \cdot) \ \forall x \in \mathrm{dom}V. \tag{5.7}$$

The *proximal, Fréhet, limiting normal cone* are defined, respectively, by

$$\mathrm{N}_S^P(x) := \partial_P \delta_S(x), \mathrm{N}_S^F(x) := \partial_F \delta_S(x), \mathrm{N}_S^L(x) := \partial_L \delta_S(x).$$

We also define singular prox-subdifferential $\partial_{P,\infty}\varphi(x)$ of φ at x as follows

$$(\xi, 0) \in \mathrm{N}_{\mathrm{epi}\varphi}^P(x, \varphi(x)).$$

The *Bouligand tangent cones to S at x* is defined as

$$\mathrm{T}_S^B(x) := \left\{ v \in H \mid \exists\, x_k \in S, \exists\, t_k \to 0, \text{ st } t_k^{-1}(x_k - x) \to v \text{ as } k \to +\infty \right\}.$$

Next we remind some basic concepts and properties of a maximal monotone operator. A multivalued operator $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, the *domain* and the *graph* of A are given, respectively, by

$$\mathrm{dom}A := \{x \in \mathbb{R}^n \mid A(x) \neq \emptyset\}, \ \mathrm{Gr}(A) := \{(x, y) \mid y \in A(x)\};$$

to simplify, we may identify A to $\mathrm{Gr}(A)$. The operator A is said to be monotone if

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0 \text{ for all } (x_i, y_i) \in \mathrm{Gr}(A), i = 1, 2.$$

If in addition A is not properly included in any other monotone operator then A is said that *maximal monotone*. In this case, for any $x \in \mathrm{dom}A$, then $A(x)$ is closed, convex, hence $(A(x))^\circ$ is singleton. By maximal property, if a sequence $(x_n, y_n)_n \subset A$ such that $(x_n, y_n) \to (x, y)$ as $n \to \infty$ then $(x, y) \in A$.

Concerning to evolution equations associated with maximal monotone operator. Let $T > 0$ and let $f : [0, T] \to \mathbb{R}^n$ be a function such that $f \in L^1(0, T; \mathbb{R}^n)$. The differential inclusion

$$\dot{x}(t) \in f(t) - A(x(t)) \text{ a.e. } t \in [0, T], x(0) = x_0 \in \overline{\mathrm{dom}A}$$

always has a unique solution $x(\cdot)$ (see [21]). Moreover, for almost $t \in [0, T]$, one has

$$\frac{d^+ x(t)}{dt} := \lim_{t' \downarrow t} \frac{x(t') - x(t)}{t' - t} = f(t^+) - \Pi_{A(x(t))}(f(t + 0)),$$

where $f(t^+) := \lim_{h \to 0, h \neq 0} \frac{1}{h} \int_t^{t+h} f(\tau) d\tau$.

Finally, we remind Gronwall's Lemma

Lemma 5.1. *(Gronwall's Lemma [5]) Let $T > 0$ and $a, b \in L^1(t_0, t_0 + T; \mathbb{R})$ such*

that $b(t) \geq 0$ a.e. $t \in [t_0, t_0 + T]$. If, for some $0 \leq \alpha < 1$, an absolutely continuous function $w : [t_0, t_0 + T] \to \mathbb{R}_+$ satisfies

$$(1 - \alpha)w'(t) \leq a(t)w(t) + b(t)w^\alpha(t) \quad a.e. \ t \in [t_0, t_0 + T],$$

then

$$w^{1-\alpha}(t) \leq w^{1-\alpha}(t_0)e^{\int_{t_0}^t a(\tau)d\tau} + \int_{t_0}^t e^{\int_s^t a(\tau)d\tau}b(s)ds, \ \forall t \in [t_0, t_0 + T].$$

5.3   Solutions of the system

In this section, we investigate and review some properties of the solution of differential inclusion (5.1), that is given by

$$\dot{x}(t) \in F(x(t)) - A(x(t)), \quad \text{a.e. } t \geq 0, \quad x(0) = x_0 \in \overline{\mathrm{dom}A},$$

where $A : H \rightrightarrows H$ is a maximal monotone operator and F is an L-Lipschitz Cusco mapping.

Definition 5.2. A continuous function $x : [0, \infty) \to \mathbb{R}^n$ is said to be a solution of (5.1) if it is absolutely continuous on every compact subset of $(0, +\infty)$ and satisfies

$$\dot{x}(t) \in F(x(t)) - A(x(t)) \quad \text{a.e. } t \geq 0, \ x(0) = x_0 \in \overline{\mathrm{dom}A}.$$

The following characterization will be useful in the sequel.

Proposition 5.3. *A continuous function $x : [0, \infty) \to \mathbb{R}^n$ is a solution of (5.1) iff $x(\cdot)$ is absolutely continuous on every compact subset of $(0, +\infty)$, and for every $T > 0$ there exists a function $f \in L^\infty(0, T; \mathbb{R}^n)$ with $f(t) \in F(x(t))$ a.e. $t \in [0, T]$, such that*

$$\dot{x}(t) \in f(t) - A(x(t)) \ a.e. \ t \in [0, T], \ x(0) = x_0 \in \overline{\mathrm{dom}A}. \tag{5.8}$$

Proof. The sufficient condition is clear and, so, we only need to justify the necessary part. Suppose that $x(\cdot)$ is any solution of (5.1) and fix $T > 0$. Since F is Lipschitz and $x(\cdot)$ is continuous, there exists $m > 0$ such that $F(x(t)) \subset \mathrm{B}(\theta, m)$ for all $t \in [0, T]$. We define the set-valued mapping $G : [0, T] \rightrightarrows \mathbb{R}^n$ as

$$G(t) := [\dot{x}(t) + A(x(t))] \cap F(x(t)) = \Big[[\dot{x}(t) + A(x(t))] \cap \mathrm{B}(\theta, m) \Big] \cap F(x(t)).$$

We are going to check that G is measurable. Since operator A is maximal monotone, the mappings

$$x \mapsto A_n(x) := A(x) \cap \mathrm{B}(\theta, n), \ n \geq 1,$$

are upper semi-continuous, and so are the mappings

$$t \mapsto A_n(x(t)) := A(x(t)) \cap \mathrm{B}(\theta, n), \ n \geq 1,$$

due to the continuity of the solution $x(\cdot)$. Then, due to the relation $A(x(t)) = \cup_{n \in \mathbb{N}} A_n(x(t))$, we deduce that the multifunction $t \longmapsto A(x(t))$ is measurable. Since $\dot{x}(t) = \lim_{n \to +\infty} n(x(t + \frac{1}{n}) - x(t))$ for ae $t \in [0, T]$, $\dot{x}(\cdot)$ is measurable, and we deduce that the multifunction $t \longmapsto [\dot{x}(t) + A(x(t))] \cap \mathrm{B}(\theta, m)$ is measurable. Similarly, the multifunction $t \longmapsto F(x(t))$ is measurable too. Consequently, according to [27, Proposition III.4], the mapping G is measurable, and we conclude from [27, Theorem III.6] that G admits a measurable selection; i.e., a measurable function $f : [0, T] \to \mathbb{R}^n$ such that

$$f(t) \in G(t) = [\dot{x}(t) + A(x(t))] \cap \mathrm{B}(\theta, m) \cap F(x(t)) \subset F(x(t)) \quad \text{a.e.} \quad t \in [0, T].$$

Hence, $\dot{x}(t) \in f(t) - A(x(t))$ and $\|f(t)\| \leq \|F(x(t))\| \leq m$, so that $f \in L^\infty(0, T; \mathbb{R}^n)$. $\qquad \square$

The next theorem shows that differential inclusion (5.1) has at least one solution whenever $x_0 \in \overline{\mathrm{dom} A}$. We use the following lemma, which is a particular case of [10, Theorem A].

Lemma 5.4. *Let $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a Lipschitz multifunction with nonempty, convex and compact values, and let $x \in \mathbb{R}^n, v \in G(x)$. Then there exists a Lipschitz selection f of G such that $f(x) = v$.*

Theorem 5.5. *Differential inclusion (5.1) has at least one solution.*

Proof. Fix $x_0 \in \overline{\mathrm{dom} A}$ and, according to Lemma 5.4, let f be a Lipschitz selection of F. Then the differential inclusion

$$\dot{x}(t) \in f(x(t)) - A(x(t)) \ \text{a.e.} \ t \geq 0, x(0) = x_0,$$

admits a unique solution $x(\cdot)$, which is absolutely continuous on every compact subset of $(0, +\infty)$ (see e.g. [14, 21]). It follows that $x(\cdot)$ is also a solution of

differential inclusion (5.1). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We also need to give some further properties of the solutions of differential inclusion (5.1), which will be used in the sequel.

Given a set $S \subset H$ and $x \in \mathrm{dom}A$ we denote

$$(S - A(x))^\circ := \bigcup_{s \in S}(s - A(x))^\circ = \{s - \Pi_{Ax}(s) \mid s \in S\}\,.$$

Proposition 5.6. *Fix $x_0 \in \overline{\mathrm{dom}A}$ and let $x(\cdot) := x(\cdot; x_0)$ be any solution of (5.1). Then the following assertions hold:*

(i) *$x(t) \in \mathrm{dom}A$, for every $t > 0$, and for a.e. $t \geq 0$,*

$$\frac{d^+x(t)}{dt} := \lim_{h \downarrow 0}\frac{x(t+h) - x(t)}{h} \in (F(x(t)) - A(x(t)))^\circ\,.$$

Conversely, if $x_0 \in \mathrm{dom}A$, then for any $v \in (F(x_0) - A(x_0))^\circ$, there exists a solution $y(\cdot)$ of (5.1) such that

$$y(0) = x_0,\ \frac{d^+y(0)}{dt} = v.$$

(ii) *There exists a real number $c > 0$ such that for any $x_0 \in \mathrm{dom}A$ and any solutions $x(\cdot) := x(\cdot; x_0)$ and $y(\cdot) := y(\cdot; x_0)$ of (5.1), one has for all $t \geq 0$*

$$\|x(t) - x_0\| \leq 3(\|F(x_0)\| + \|A^\circ(x_0)\|)te^{ct},$$

$$\|x(t) - y(t)\| \leq 4(\|F(x_0)\| + \|A^\circ(x_0)\|)te^{ct},$$

Consequently, for every $T > 0$ there exists $\rho > 0$ such that

$$x(t) \in \mathrm{B}(x_0, \rho)\ \forall t \in [0, T].$$

Proof. (*i*) According to Proposition 5.3, for each $T > 0$ there exists some $f \in L^\infty(0, T; \mathbb{R}^n)$ with $f(t) \in F(x(t))$ a.e. $t \in [0, T]$, such that $x(\cdot)$ is the unique solution of (5.8); hence, by [21] $x(\cdot)$ satisfies $x(t) \in \mathrm{dom}A$ for all $t \in (0, T)$ and

$$\frac{d^+x(t)}{dt} = \left(f(t^+) - A(x(t))\right)^\circ \quad \text{a.e. } t \in (0, T)\,, \qquad (5.9)$$

where $f(t^+) := \lim_{h \to 0} h^{-1}\int_0^h f(t + \tau)d\tau$. Moreover, given $\varepsilon > 0$ there exists some

$h > 0$ such that for a.e. $\tau \in (0, h)$, we have

$$f(t + \tau) \in F(x(t + \tau)) \subset F(x(t)) + L \|x(t + \tau) - x(t)\| \mathbb{B} \subset F(x(t)) + \varepsilon L \mathbb{B},$$

and so $\lim\limits_{h \to 0^+} \frac{1}{h} \int_0^h f(t + \tau) d\tau \in F(x(t)) + \varepsilon L \mathbb{B}$ (this last set is convex and closed). Hence, as ε goes to 0 we get $f(t^+) \in F(x(t))$, and (i) follows from (5.9).

Conversely, we assume that $x_0 \in \mathrm{dom} A$ and take $v \in [F(x_0) - A(x_0)]^\circ$. We choose $w \in F(x_0)$ such that $v = w - \Pi_{A(x_0)}(w)$. According to Lemma 5.4, there exists a Lipschitz selection f of F such that $f(x_0) = w$. Then the unique solution $y(\cdot)$ of the following differential inclusion

$$\dot{y}(t) \in f(y(t)) - A(y(t)), \quad y(0) = x_0,$$

satisfies
$$\frac{d^+ y(0)}{dt} = f(x_0) - \Pi_{A(x_0)}(f(x_0)) = w - \Pi_{A(x_0)}(w),$$

and the proof of (i) is complete.

(ii) Let $x(\cdot)$ be a solution of differential inclusion (5.1), with $x(0) = x_0$, and fix $T > 0$. Then by Proposition 5.3 there exist functions $k, g \in L^1(0, T; \mathbb{R}^n)$ such that $k(t) \in F(x(t))$, $g(t) \in A(x(t))$, and

$$\dot{x}(t) = k(t) - g(t) \quad \text{a.e } t \in [0, T].$$

We also choose by Lemma 5.4 a Lipschitz mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, with Lipschitz constant c $(c \geq L)$, and consider the unique solution $z(\cdot)$ of the differential inclusion

$$\dot{z}(t) \in f(z(t)) - A(z(t)) \text{ a.e. } t \geq 0, z(0) = x_0.$$

So, for any $t \geq 0$ one has $\left\| \frac{d^+ z(t)}{dt} \right\| \leq \left\| \frac{d^+ z(0)}{dt} \right\|$ and

$$\left\| \frac{d^+ z(0)}{dt} \right\| = \|(f(x_0) - A(x_0))^\circ\| \leq \|F(x_0)\| + \|A^\circ(x_0)\|,$$

so

$$
\begin{aligned}
\|z(t) - x_0\| &\leq \int_0^t e^{c\tau} \left\| \frac{d^+ z(0)}{dt} \right\| d\tau = \frac{e^{ct} - 1}{c} \left\| \frac{d^+ z(0)}{dt} \right\| \\
&\leq \frac{e^{ct} - 1}{c} (\|F(x_0)\| + \|A^\circ(x_0)\|) \quad (5.10) \\
&\leq t e^{ct} (\|F(x_0)\| + \|A^\circ(x_0)\|) \quad (5.11)
\end{aligned}
$$

By the Lipschitzianity of F we choose a function $w(\cdot) : [0, T] \to \mathbb{R}^n$ such that

$$w(t) \in F(z(t)), \quad \|k(t) - w(t)\| \leq L \|x(t) - z(t)\| \quad \forall t \in [0, T]. \qquad (5.12)$$

Then we obtain

$$
\begin{aligned}
\langle \dot{x}(t) - \dot{z}(t), x(t) - z(t) \rangle &= \big\langle k(t) - g(t) - f(z(t)) + \Pi_{A(z(t))}(f(z(t))), x(t) - z(t) \big\rangle \\
&= \langle k(t) - f(z(t)), x(t) - z(t) \rangle \\
&\quad + \underbrace{\big\langle -g(t) + \Pi_{A(z(t))}(f(z(t))), x(t) - z(t) \big\rangle}_{\leq 0, \text{ by the monotonicity of } A} \\
&\leq \langle k(t) - w(t), x(t) - z(t) \rangle + \langle w(t) - f(z(t)), x(t) - z(t) \rangle \\
&\leq L \|x(t) - z(t)\|^2 + 2 \|F(z(t))\| \, \|x(t) - z(t)\| \quad \text{(by (5.12))} \\
&\leq L \|x(t) - z(t)\|^2 + 2\big(\|F(x_0)\| + L \|z(t) - x_0\| \big) \|x(t) - z(t)\| \\
&\leq L \|x(t) - z(t)\|^2 + \\
&\quad 2\Big(\|F(x_0)\| + (e^{ct} - 1)(\|F(x_0)\| + \|A^\circ(x_0)\|) \Big) \|x(t) - z(t)\| \\
&\leq L \|x(t) - z(t)\|^2 + 2(\|F(x_0)\| + \|A^\circ(x_0)\|) e^{ct} \|x(t) - z(t)\|.
\end{aligned}
$$

Consequential, from Gronwall Lemma we get, for every $t \geq 0$,

$$\|x(t) - z(t)\| \leq 2(\|F(x_0)\| + \|A^\circ(x_0)\|) t e^{ct},$$

which together with (5.11) give us

$$\|x(t) - x_0\| \leq 3(\|F(x_0)\| + \|A^\circ(x_0)\|) t e^{ct},$$

and, for every other solution $y = y(\cdot; x_0)$,

$$\|x(t) - y(t)\| \leq \|x(t) - z(t)\| + \|y(t) - z(t)\| \leq 4(\|F(x_0)\| + \|A^\circ(x_0)\|) t e^{ct};$$

that is the conclusion of (ii) follows. $\qquad\qquad\square$

5.4 Strong and weak invariant sets

In this section, we give explicit characterizations for a closed set $S \subset \mathbb{R}^n$ to be strong or weak invariant for differential inclusion (5.1),

$$\dot{x}(t) \in F(x(t)) - A(x(t)) \quad \text{a.e. } t \geq 0, \quad x(0) = x_0 \in \overline{\mathrm{dom} A},$$

where $A : H \rightrightarrows H$ is a maximal monotone operator and F is an L-Lipschitz Cusco mapping. Invariance criteria are written exclusively by means of the data; that is, multifunction F and operator A, and involve the geometry of the set S, using the associated proximal and Fréhet normal cones.

Definition 5.7. Let S be a closed subset of $\mathbb{R}^n$.

(i) S is said to be strong invariant if for any $x_0 \in S \cap \overline{\operatorname{dom} A}$ and any solution $x(\cdot; x_0)$ of (5.1), we have

$$x(t; x_0) \in S \ \ \forall t \geq 0.$$

(ii) S is said to be weak invariant if for any $x_0 \in S \cap \overline{\operatorname{dom} A}$, there exists at least one solution $x(\cdot; x_0)$ of (5.1) such that

$$x(t; x_0) \in S \ \ \forall t \geq 0.$$

Since any solution of differential inclusion (5.1) lives in $\overline{\operatorname{dom} A}$ (Proposition 5.6), we may assume without loss of generality that S is a closed subset of $\overline{\operatorname{dom} A}$. We shall need the following two lemmas.

Lemma 5.8. *(e.g. [7, Lemma A.1])Let $S \subset \mathbb{R}^n$ be closed. Then for every $x \in \mathbb{R}^n \setminus S$ we have*

$$\partial_L d_S(\cdot)(x) \in \left\{ \frac{x - \Pi_S(x)}{d_S(x)} \right\} \text{ and } \partial_C d_S(\cdot)(x) \in \overline{\operatorname{co}}(\{ \frac{x - \Pi_S(x)}{d_S(x)} \}).$$

Lemma 5.9. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be an l-Lipschitz function. Then for every $x \in \mathbb{R}^n$ we have*
$$\varphi(x + v) \leq \varphi(x) + \varphi^0(x; v) + o(\|v\|), \ v \in \mathbb{R}^n.$$

Proof. We proceed by contradiction and suppose that for some $\alpha > 0$ and sequence $(v_n)_n \subset \mathbb{R}^n \setminus \{\theta\}$ converging to θ it holds

$$\varphi(x + v_n) - \varphi(x) > \varphi^0(x; v_n) + \alpha \|v_n\| \quad \text{for all } n \geq 1. \tag{5.13}$$

Without loss of generality, we can assume that $\frac{v_n}{\|v_n\|} \to v \neq \theta$. Then

$$\varphi(x + v_n) - \varphi(x) = \varphi(x + v_n - \|v_n\| v + \|v_n\| v) - \varphi(x + v_n - \|v_n\| v)$$
$$+ \varphi(x + v_n - \|v_n\| v) - \varphi(x)$$
$$\leq \varphi(x + v_n - \|v_n\| v + \|v_n\| v) - \varphi(x + v_n - \|v_n\| v)$$
$$+ l \|(v_n - \|v_n\| v)\|.$$

Hence, from the inequality (5.13) one gets

$$\frac{\varphi(x + v_n - \|v_n\| v + \|v_n\| v) - \varphi(x + v_n - \|v_n\| v)}{\|v_n\|} + l \left\| \frac{v_n}{\|v_n\|} - v \right\| \geq \varphi^0(x; \frac{v_n}{\|v_n\|}) + \alpha,$$

which as $n \to \infty$ leads us to the contradiction $\varphi^0(x; v) \geq \varphi^0(x; v) + \alpha > \varphi^0(x; v)$.

$\square$

Before we state the main strong invariance result we give the following result:

Proposition 5.10. *Let $S \subset \overline{\mathrm{dom}A}$ satisfy condition (5.6), and take $x_0 \in S$. If there is some $\rho > 0$ such that for any $x \in \mathrm{B}(x_0, \rho) \cap S \cap \mathrm{dom}A$,*

$$\sup_{\xi \in \mathrm{N}_S^P(x)} \sup_{v \in F(x)} \inf_{x^* \in A(x)} \langle \xi, v - x^* \rangle \leq 0, \tag{5.14}$$

then given any solution $x(\cdot; x_0)$ of (5.1), there exists $T > 0$ such that $x(t; x_0) \in S$ for every $t \in [0, T]$.

Proof. Let $x(\cdot) := x(\cdot; x_0)$ be any solution of differential inclusion (5.1), so that for some $T_1 > 0$ we have

$$x(t) \in \mathrm{B}(x_0, \frac{\rho}{3}) \cap \mathrm{dom}A, \text{ a.e. } t \in [0, T_1], \tag{5.15}$$

where $\rho > 0$ is as in the current assumption, and so (by condition (5.6))

$$\Pi_S(x(t)) \subset \mathrm{B}(x_0, \frac{2}{3}\rho) \cap S \cap \mathrm{dom}A \subset \mathrm{B}(x_0, \rho) \cap S \cap \mathrm{dom}A \text{ for a.e. } t \in (0, T_1]. \tag{5.16}$$

We denote the function $\eta : [0, T_1] \to \mathbb{R}$ as

$$\eta(t) := d_S^2(x(t)).$$

Fix $\varepsilon > 0$. Since the function $d_S^2(\cdot)$ is Lipschitz on each bounded set and $x(\cdot)$ is absolutely continuous on $[\varepsilon, T_1]$, function η is also absolutely continuous on $[\varepsilon, T_1]$;

hence, differentiable on a set $T_0 \subset [\varepsilon, T_1]$ of full measure (we may also suppose that (5.16) holds for all $t \in T_0$). We pick $t \in T_0$ so that, according to Lemma 5.9, for all $s > 0$

$$
\begin{aligned}
d_S^2(x(t+s)) &= d_S^2(x(t) + \dot{x}(t)s + o(s)) \\
&\leq d_S^2(x(t) + \dot{x}(t)s) + o(s) \\
&\leq \left(d_S(x(t)) + s d_S^0(x(t); \dot{x}(t)) + o(s) \right)^2 + o(s) \\
&\leq d_S^2(x(t)) + 2 d_S(x(t)) d_S^0(x(t); \dot{x}(t)s) + o(s),
\end{aligned}
\tag{5.17}
$$

While by Lemma 5.8 we have

$$
\begin{aligned}
d_S(x(t)) d_S^0(x(t); \dot{x}(t)) &= d_S(x(t)) \max_{\xi \in \partial_C d(x(t))} \langle \xi, \dot{x}(t) \rangle \\
&\leq \max_{u \in \Pi_S(x(t))} \langle x(t) - u, \dot{x}(t) \rangle.
\end{aligned}
\tag{5.18}
$$

Let us write $\dot{x}(t)$ as $\dot{x}(t) = v - w$ for some $v \in F(x(t))$ and $w \in A(x(t))$, and fix $u \in \Pi_S(x(t))$ ($\subset \mathrm{B}(x_0, \rho) \cap S \cap \mathrm{dom} A$ by (5.16)). By the Lipschitzianity of F we choose some $v' \in F(u)$ such that

$$
\|v - v'\| \leq L \|x(t) - u\| = L d_S(x(t)).
$$

Since $x(t) - u \in \mathrm{N}_S^P(u)$, by the current hypothesis of the theorem there exist $w' \in A(u)$ such that

$$
\langle x(t) - u, v' - w' \rangle \leq 0,
$$

which in turn yields, due to the monotonicity of A,

$$
\begin{aligned}
\langle x(t) - u, \dot{x}(t) \rangle &= \langle x(t) - u, v - w \rangle \\
&= \langle x(t) - u, v - v' \rangle + \langle x(t) - u, v' - w' \rangle \\
&\quad + \langle x(t) - u, w' - w \rangle \\
&\leq L \|x(t) - u\|^2 = L d_S^2(x(t)).
\end{aligned}
$$

Thus, continuing with (5.17) and (5.18) we arrive at

$$
\eta(t+s) \leq \eta(t) + 2L\eta(t)s + o(\|s\|),
$$

which implies that $\dot{\eta}(t) \leq 2L\eta(t)$. Hence, by Gronwall Lemma, we obtain that $\eta(t) \leq \eta(\varepsilon)e^{2L(t-\varepsilon)}$ for all $t \in T_0$, or, equivalently, $\eta(t) \leq \eta(\varepsilon)e^{2L(t-\varepsilon)}$ for all $t \in [\varepsilon, T_1]$. Then, as ε goes to 0 we conclude that $\eta(t) = 0$ for all $t \in [0, T_1]$, which

proves that $x(t) \in S$ for all $t \in [0, T_1]$. $\qquad\square$

We give the required characterization of strong invariant closed sets with respect to differential inclusion (5.1).

Theorem 5.11. *Let S be a closed subset of $\overline{\mathrm{dom}A}$ satisfying relation (5.6). Then the following statements are equivalent, provided that $\mathrm{N}_S = \mathrm{N}_S^P$ or N_S^F and $\mathrm{T}_S = \mathrm{T}_S^B$, or $\mathrm{T}_S = \overline{\mathrm{co}}\,\mathrm{T}_S^B$,*

(i) S is strong invariant for differential inclusion (5.1).

(ii) For every $x \in S \cap \mathrm{dom}A$, one has

$$v - \Pi_{A(x)}(v) \in \mathrm{T}_S(x) \ \ \forall v \in F(x). \tag{5.19}$$

(iii) For every $x \in S \cap \mathrm{dom}A$, one has

$$[v - A(x)] \cap \mathrm{T}_S(x) \neq \emptyset \ \ \forall v \in F(x). \tag{5.20}$$

(iv) For every $x \in S \cap \mathrm{dom}A$, one has

$$\sup_{\xi \in \mathrm{N}_S(x)} \ \sup_{v \in F(x)} \langle \xi, v - \Pi_{A(x)}(v) \rangle \leq 0. \tag{5.21}$$

(v) For every $x \in S \cap \mathrm{dom}A$, one has

$$\sup_{\xi \in \mathrm{N}_S(x)} \ \sup_{v \in F(x)} \ \inf_{x^* \in A(x)} \langle \xi, v - x^* \rangle \leq 0. \tag{5.22}$$

(vi) For every $x \in S \cap \mathrm{dom}A$, one has

$$\sup_{\xi \in \mathrm{N}_S(x)} \ \sup_{v \in F(x)} \ \inf_{x^* \in A(x) \cap \mathrm{B}(\theta, \|F(x)\| + \|A^\circ(x)\|)} \langle \xi, v - x^* \rangle \leq 0. \tag{5.23}$$

Proof. The implication $(ii) \Rightarrow (iii)$ and $(vi) \Rightarrow (v)$ are trivial, while the implications $(ii) \Rightarrow (iv)$ and $(iii) \Rightarrow (v)$ come from the relation $\mathrm{T}_S(x) \subset (\mathrm{N}_S^F(x))^*$ for all $x \in S$. The implications (v) (with $\mathrm{N}_S = \mathrm{N}_S^P$) $\Rightarrow (i)$ is a direct consequence of Proposition 5.10.

$(i) \Rightarrow (ii)$. To prove this implication we suppose that S is strong invariant and take $x_0 \in S \cap \mathrm{dom}A$ and $v \in F(x_0)$. According to Lemma 5.4, there exists a Lipschitz selection f of F such that $f(x_0) = v$, and so there is a unique solution $x(\cdot)$ of the following differential inclusion,

$$\dot{x}(t) \in f(x(t)) - A(x(t)), \ \text{a.e. } t \geq 0, \ x(0) = x_0.$$

It follows that $x(\cdot)$ is also a solution of differential inclusion (5.1), so that $x(t) \in S$ for any $t \geq 0$. Then we get

$$v - \Pi_{A(x_0)}(v) = (f(x_0) - A(x_0))^\circ = \frac{d^+ x(0)}{dt} = \lim_{t \downarrow 0} \frac{x(t) - x_0}{t} \in \mathrm{T}_S^B(x_0) \subset \mathrm{T}_S(x_0).$$

$(iv) \Rightarrow (vi)$. This implication holds since for any $x \in \mathrm{dom}A$ and $v \in F(x)$ we have that

$$\begin{aligned}
\left\| \Pi_{A(x)}(v) \right\| &\leq \left\| \Pi_{A(x)}(v) - A^\circ(x) \right\| + \| A^\circ(x) \| \\
&= \left\| \Pi_{A(x)}(v) - \Pi_{A(x)}(\theta) \right\| + \| A^\circ(x) \| \\
&\leq \| v \| + \| A^\circ(x) \| \leq \| F(x) \| + \| A^\circ(x) \|.
\end{aligned}$$

The proof of the theorem is complete. $\qquad\square$

The following proposition, which provides the counterpart of Proposition 5.10 for the weak invariance, is essentially given in [39, Theorem 1]. The specification of the interval on which the solution remains in S also comes from the proof given in that paper.

Proposition 5.12. *Let $S \subset \mathrm{dom}A$ be closed and take $x_0 \in S$ such that, for some $r, m > 0$,*

$$\| A^\circ(x) \| \leq m \ \ \forall x \in S \cap \mathrm{B}(x_0, r). \tag{5.24}$$

Assume that for all $x \in S \cap \mathrm{B}(x_0, r)$,

$$\sup_{\xi \in \mathrm{N}_S^P(x)} \inf_{v \in F(x)} \inf_{x^* \in A(x) \cap \mathrm{B}_{m + \| F(x) \|}} \langle \xi, v - x^* \rangle \leq 0. \tag{5.25}$$

Then there exists a solution $x(\cdot; x_0)$ of (5.1) such that $x(t; x_0) \in S$ for every $t \in [0, T]$ with $T = \frac{r}{3} \Big(m + \sup\limits_{x \in \mathrm{B}(x_0, r) \cap S} \| F(x) \| \Big)^{-1}$.

Consequently, we obtain the desired characterization of weak invariant sets with respect to differential inclusion (5.1). Recall that A° is said to be locally bounded on S if for every $x \in S$ we have

$$\mathfrak{m}(x) := \limsup_{y \to x, y \in S} \| A^\circ(y) \| < +\infty. \tag{5.26}$$

Theorem 5.13. *Let $S \subset \mathrm{dom}A$ be a closed set such that A° is locally bounded on S. Then the following statements are equivalent provided that T_S and N_S are the same as the ones in Theorem 5.11:*

(i) *S is weak invariant for differential inclusion (5.1).*

(ii) *For every $x \in S$, one has*

$$\cup_{v \in F(x)} \left[v - A(x) \cap \mathrm{B}_{\mathfrak{m}(x) + \|F(x)\|} \right] \cap \mathrm{T}_S(x) \neq \emptyset. \tag{5.27}$$

(iii) *For every $x \in S$, one has*

$$\sup_{\xi \in \mathrm{N}_S(x)} \inf_{v \in F(x)} \inf_{x^* \in A(x) \cap \mathrm{B}_{\mathfrak{m}(x) + \|F(x)\|}} \langle \xi, v - x^* \rangle \leq 0. \tag{5.28}$$

Proof. $(i) \Rightarrow (ii)$. Given an $x_0 \in S$ we choose a solution $x(\cdot) := x(\cdot; x_0)$ of (5.1) that belongs to S. Fix $\varepsilon > 0$. By (5.26) and the current assumption we also choose $\rho > 0$ such that

$$\|A^\circ(x)\| \leq \mathfrak{m}(x_0) + \varepsilon \ \text{ for all } \ x \in \mathrm{B}(x_0, \rho) \cap S.$$

Then for any $x \in \mathrm{B}(x_0, \rho) \cap S$ and any $v \in F(x)$ we get

$$\left\|\Pi_{A(x)}(v)\right\| \leq \left\|\Pi_{A(x)}(v) - A^\circ(x)\right\| + \|A^\circ(x)\| \leq \|v\| + \|A^\circ(x)\| \leq \|F(x)\| + \mathfrak{m}(x_0) + \varepsilon;$$

Let $T > 0$ be such that $x(t) \in \mathrm{B}(x_0, \rho) \cap S$ for all $t \in [0, T]$, so that for all $v \in F(x(t))$ and $t \in [0, T]$ we have

$$\left\|\Pi_{A(x(t))}(v)\right\| \leq \|F(x(t))\| + \mathfrak{m}(x_0) + \varepsilon;$$

hence, by Proposition 5.6(i),

$$\dot{x}(t) \in F(x(t)) - A(x(t)) \cap \mathrm{B}_{\|F(x(t))\| + \mathfrak{m}(x_0) + \varepsilon} \ \ a.e. \ \ t \in [0, T], \tag{5.29}$$

and $x(\cdot)$ is Lipschitz on $[0, T]$ (observing that $\mathrm{B}_{\|F(x(t))\| + \mathfrak{m}(x_0) + \varepsilon} \subset \mathrm{B}_{\|F(x_0)\| + L\rho + \mathfrak{m}(x_0) + \varepsilon}$). Take $w \in \mathrm{Limsup}_{t \downarrow 0} t^{-1}(x(t) - x_0)$ (this Painleve-Kuratowski upper limit is nonempty, due to the Lipschitzianity of $x(\cdot)$). Then, since the mappings $x \mapsto A(x) \cap \mathrm{B}(\theta, \|F(x)\| + \mathfrak{m}(x_0) + \varepsilon)$ and $x \mapsto F(x)$ are upper

semi-continuous, by using (5.29) we get

$$
\begin{aligned}
w \;\in\; & \mathrm{Limsup}_{t\downarrow 0}\frac{1}{t}\int_0^t \dot{x}(\tau)d\tau \\
\subset\; & \mathrm{Limsup}_{t\downarrow 0}\left(\overline{\mathrm{co}}\left(\bigcup_{\tau\in[0,t]} F(x(\tau)) - A(x(\tau))\cap \mathrm{B}_{\|F(x(\tau))\|+\mathfrak{m}(x_0)+\varepsilon}\right)\right) \\
\subset\; & F(x_0) - A(x_0)\cap \mathrm{B}_{\|F(x_0)\|+\mathfrak{m}(x_0)+\varepsilon},
\end{aligned}
\tag{5.30}
$$

and we conclude that, as ε goes to 0 (observe that v is independent of ε),

$$
w \in F(x_0) - A(x_0)\cap \mathrm{B}_{\|F(x_0)\|+\mathfrak{m}(x_0)}.
$$

Thus, (ii) follows, due to the obvious fact that $\mathrm{Limsup}_{t\downarrow 0} t^{-1}(x(t)-x_0) \subset \mathrm{T}_S(x_0)$.

$(iii) \Rightarrow (i)$. Fix $x_0 \in S$. By (5.26) we choose $r, m > 0$ such that $\mathfrak{m}(x) \le m$ for every $x \in S \cap \mathrm{B}(x_0, r)$. It suffices to prove that

$$
\bar{T} := \sup\{T : \exists\, x(\cdot\,; x_0) \text{ a solution of } (5.1) \text{ such that } x(t; x_0) \in S \;\; \forall t \in [0, T]\} = +\infty.
$$

According to Proposition 5.12, there exist some $T_1 > 0$ and a solution $x_1(\cdot\,; x_0)$ of differential inclusion (5.1) such that $x_1(t; x_0) \in S$ for all $t \in [0, T_1]$; hence, $\bar{T} \ge T_1 > 0$.

We proceed by contradiction and assume that $\bar{T} < +\infty$. By Proposition 5.6, we let $r_1 > 0$ be such that for every solution $x(\cdot\,; x_0)$ of (5.1) we have

$$
x(t; x_0) \in \mathrm{B}(x_0, r_1) \;\; \forall t \in [0, \bar{T}].
$$

We set

$$
k := \sup_{x\in \mathrm{B}(x_0, r_1+1)} \|F(x)\| + \sup_{x\in \mathrm{B}(x_0, r_1+1)\cap S} \|A^{\circ}(x)\|,
$$

so that $k < +\infty$, due to (5.26) and the compactness of the set $\mathrm{B}(x_0, r_1+1)\cap S$. By definition of $\bar{T}$, for $0 < \varepsilon < \min\left\{\frac{1}{3k}, \bar{T}\right\}$ we choose a solution $x_\varepsilon(\cdot\,; x_0)$ of (5.1) such that $x_\varepsilon(t; x_0) \in S$ for all $t \in [0, \bar{T}-\varepsilon]$. We put $y_0 := x_\varepsilon(\bar{T}-\varepsilon; x_0) \in \mathrm{B}(x_0, r_1)\cap S$, so that $\mathrm{B}(y_0, 1) \subset \mathrm{B}(x_0, r_1+1)$ and the following relations follows easily

$$
\|A^{\circ}(y)\| \le \sup_{u\in \mathrm{B}(x_0, r_1+1)\cap S} \|A^{\circ}(u)\| =: m_1 \;\; \forall y \in S \cap \mathrm{B}(y_0, 1),
$$

$$
\sup_{\xi\in \mathrm{N}_S(y)} \;\inf_{v\in F(y)}\; \inf_{x^*\in A(y)\cap \mathrm{B}_{m_1+\|F(y)\|}} \langle \xi, v - x^*\rangle \le 0 \text{ for all } y \in S \cap \mathrm{B}(y_0, 1).
$$

Then, according to Proposition 5.12, there exists a solution $x_2(\cdot; y_0)$ of (5.1) such that $x_2(t; y_0) \in S$ for all $t \in [0, \frac{1}{3k}]$. Consequently, the function $z(\cdot; x_0)$ defined as

$$z(t; x_0) := \begin{cases} x_\varepsilon(t; x_0) & \text{if } s \in [0, \bar{T} - \varepsilon] \\ x_2(t - \bar{T} + \varepsilon; y_0) & \text{if } s \in [\bar{T} - \varepsilon, +\infty[, \end{cases}$$

is a solution of (5.1) and satisfies $z(t; x_0) \in S$ for all $t \in [0, \tilde{T}]$ with $\tilde{T} := \bar{T} + \frac{1}{3k} - \varepsilon > \bar{T}$, which contradicts the definition of $\bar{T}$. Hence $\bar{T} = \infty$, and S is weak invariant. $\qquad\square$

5.5 Strong a-Lyapunov pairs

In this section, we use the invariance results of the previous section to characterize strong a-Lyapunov pairs with respect to differential inclusion (5.1),

$$\dot{x}(t) \in F(x(t)) - A(x(t)), \quad \text{a.e. } t \geq 0, \quad x(0) = x_0 \in \overline{\text{dom}A},$$

where $A : H \rightrightarrows H$ is a maximal monotone operator and F is an L-Lipschitz Cusco mapping.

Definition 5.14. Let $V, W : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lower semi-continuous functions such that $W \geq 0$ and let $a \geq 0$. We say that (V, W) is a strong a-Lyapunov pair for (5.1) if for any $x_0 \in \overline{\text{dom}A}$ we have

$$e^{at}V(x(t; x_0)) + \int_0^t W(x(\tau; x_0))d\tau \leq V(x_0) \qquad \forall t \geq 0, \qquad (5.31)$$

for every solution $x(\cdot; x_0)$ of (5.1).

The following lemma shows that the non-regularity of the functions V, W candidates to form a-Lyapunov pairs is mainly carried by the function V. For $k \geq 1$ we denote

$$W_k(x) := \inf_{z \in \mathbb{R}^n} \{W(z) + k\|x - z\|\}. \qquad (5.32)$$

Lemma 5.15. *Given a function $W : \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$, W_k defined in (5.32) is k-Lipschitz, and we have $W_k(x) \nearrow W(x)$ for all $x \in \mathbb{R}^n$. Moreover, the following assertions hold true for every $T > 0$ and $x_0 \in \text{dom}V$:*

(i) If $x(\cdot; x_0)$ is a solution of differential inclusion (5.1), then W satisfies inequality (5.31) iff W_k does for all $k \geq 1$.

(ii) *If there are solutions* $x_k(\cdot; x_0)$, $k \geq 1$, *of* (5.1) *such that* $\sup_{k \geq 1} \operatorname{ess\,sup}_T \|\dot{x}_k(\cdot; x_0)\| < +\infty$ *and*

$$e^{at}V(x_k(t; x_0)) + \int_0^t W_k(x_k(\tau; x_0))d\tau \leq V(x_0) \ \forall t \in [0, T] \ and \ \forall k \geq 1,$$

then there exists a solution $x(\cdot; x_0)$ *of* (5.1) *such that*

$$e^{at}V(x(t; x_0)) + \int_0^t W(x(\tau; x_0))d\tau \leq V(x_0) \ \forall t \in [0, T].$$

Proof. The first statement of the lemma is known (see, e.g., [30]). Statement (i) follows easily from Fatou's lemma. To prove statement (ii) we observe that the current assumption, that $\sup_{k \geq 1} \operatorname{ess\,sup}_T \|\dot{x}_k(\cdot; x_0)\| \leq m$, yields the existence of some $r > 0$ such that $x_k(t; x_0) \in \mathrm{B}(x_0; r)$ for all $t \in [0, T]$ and all $k \geq 1$. Hence, there exists $M > 2m$ such that $\sup_{k \geq 1} \sup_{t \in [0,T]} \|F(x_k(t; x_0))\| \leq \frac{M}{2}$; in other words, for all $k \geq 1$ it holds

$$\dot{x}_k(t; x_0) \in F(x_k(t; x_0)) - A(x_k(t; x_0)) \cap \mathrm{B}_M \ \text{a.e.} \ t \in [0, T], \ x_k(0; x_0) = x_0. \quad (5.33)$$

We may assume without loss of generality that $\dot{x}_k(\cdot; x_0)$ converges weakly in $L^2(0, T; \mathbb{R}^n)$ to some $v(\cdot) \in L^2(0, T; \mathbb{R}^n)$, and, due to Arzela-Ascoli's Theorem, that $x_k(\cdot; x_0)$ converges uniformly to a continuous function $x(\cdot)$ on $[0, T]$. It follows that $\dot{x}(t) = v(t)$ a.e. $t \in [0, T]$, and so, since the mapping $F - A \cap \mathrm{B}_M$ has closed graph, as $k \to +\infty$ in (5.33) we get

$$\dot{x}(t) \in F(x(t)) - A(x(t)) \cap \mathrm{B}_M \ \text{a.e.} \ t \in [0, T]; \quad (5.34)$$

that is, $x(\cdot)$ is a solution of differential inclusion (5.1). Finally, by taking the limits as $k, p \to +\infty$ for $k \geq p$ in the inequalities for all $t \in [0, T]$

$$e^{at}V(x_k(t; x_0)) + \int_0^t W_p(x_k(\tau; x_0))d\tau \leq e^{at}V(x_k(t; x_0)) + \int_0^t W_k(x_k(\tau; x_0))d\tau \leq V(x_0),$$

we obtain that

$$e^{at}V(x(t)) + \int_0^t W(x(\tau))d\tau \leq V(x_0) \ \forall t \in [0, T],$$

as we wanted to prove. $\qquad \square$

Lemma 5.16. *Consider the operator* $\hat{A} : \mathbb{R}^n \times \mathbb{R}^3 \to \mathbb{R}^{n+3}$ *and the function*

$\tilde{V} : \mathbb{R}^{n+1} \times \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ *defined as*

$$\hat{A}(x, \alpha, \beta, \gamma) := (A(x), \theta_{\mathbb{R}^3}), \quad \tilde{V}(x, \alpha, \beta) := e^{\alpha\beta}V(x) + \alpha, \qquad (5.35)$$

together with the mappings $\hat{F}_k : \mathbb{R}^{n+3} \to \mathbb{R}^{n+3}$, $k \geq 1$, *given by (recall (5.32))*

$$\hat{F}_k(x, \alpha, \beta, \gamma) := (F(x), W_k(x), 1, 0).$$

Then $\hat{A}$ *is maximal monotone with* $\mathrm{dom}\hat{A} = \mathrm{dom}A \times \mathbb{R}^3$, $\hat{F}_k$ *is Lipschitz with constant* $(L^2 + k^2)^{\frac{1}{2}}$, *and consequently, the following differential inclusion possesses solutions,*

$$\dot{z}(t) \in \hat{F}_k(z(t)) - \hat{A}(z(t)) \ \ a.e. \ \ t \geq 0, \ \ z(0) = z_0 = (x_0, y_0, z_0, w_0) \in \overline{\mathrm{dom}A} \times \mathbb{R}^3, \tag{5.36}$$

and every solutions is written as

$$z(t; z_0) = \left(x(t; x_0), y_0 + \int_0^t W_k(x(\tau; x_0))d\tau, z_0 + t, w_0\right),$$

for a solution $x(\cdot; x_0)$ *of* (5.1).

We need the following result which provides us with a local criterion for strong a-Lyapunov pairs.

Proposition 5.17. *Let* $V, W : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be two proper lower semi-continuous functions such that* $\mathrm{dom}V \subset \mathrm{dom}A, W \geq 0$ *and let* $a \geq 0$. *Fix* $x_0 \in \mathrm{dom}V$ *and assume that for some* $\rho > 0$ *we have, for all* $x \in \mathrm{B}(x_0, \rho)$,

$$\sup_{\xi \in \partial_P V(x)} \sup_{v \in F(x)} \inf_{x^* \in A(x)} \langle \xi, v - x^* \rangle + aV(x) + W(x) \leq 0, \tag{5.37}$$

$$\sup_{\xi \in \partial_{P,\infty} V(x)} \sup_{v \in F(x)} \inf_{x^* \in A(x)} \langle \xi, v - x^* \rangle \leq 0. \tag{5.38}$$

Then there exists some $T > 0$ *such that for every solution* $x(\cdot; x_0)$ *of differential inclusion* (5.1) *one has*

$$e^{at}V(x(t; x_0)) + \int_0^t W(x(\tau; x_0))d\tau \leq V(x_0) \ \forall t \in [0, T].$$

Proof. First, by Proposition 5.6(ii) we let $c > 0$ be such that for any solutions

$x(\cdot) := x(\cdot; x_0)$ of (5.1) it holds

$$\|x(t) - x_0\| \le 3(\|F(x_0)\| + \|A^\circ(x_0)\|)te^{ct} \text{ for all } t \ge 0,$$

and choose $T > 0$ such that

$$3(\|F(x_0)\| + \|A^\circ(x_0)\|)Te^{cT} \le \rho. \tag{5.39}$$

As in Lemma 5.16, we define the proper and lower semi-continuous function $\tilde{V}$: $\mathbb{R}^{n+1} \times \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ as $\tilde{V}(x, \alpha, \beta) := e^{a\beta}V(x) + \alpha$, so that epi$\tilde{V}$ is closed and satisfies

$$\text{epi}\tilde{V} \subset \text{dom}V \times \mathbb{R}^3 \subset \text{dom}A \times \mathbb{R}^3 = \text{dom}\hat{A},$$

where $\hat{A}$ is also defined as in Lemma 5.16; hence, condition (5.6) is obviously satisfied for epi$\tilde{V}$.

Claim. We claim that for any given $\tilde{z} := (x_1, y_1, z_1, w_1) \in \text{epi}\tilde{V}$ with $\|x_1 - x_0\| < \rho$, there exists small enough $\varepsilon > 0$ such that for each $(x, y, z, w) \in \text{B}(\tilde{z}, \varepsilon) \cap \text{epi}\tilde{V}$, $(\tilde{\xi}, -\kappa) \in \text{N}^P_{\text{epi}\tilde{V}}(x, y, z, w)$, and $(v, W_k(x), 1, 0) \in \hat{F}_k(x, y, z, w)$ there exists $x^* \in A(x)$ such that

$$\langle (\tilde{\xi}, -\kappa), (v - x^*, W_k(x), 1, 0) \rangle \le 0. \tag{5.40}$$

Indeed, with $\tilde{z}$ as in the claim let us choose $\varepsilon > 0$ such that

$$(x, y, z, w) \in \text{B}(\tilde{z}, \varepsilon) \cap \text{epi}\tilde{V} \Rightarrow x \in \text{B}(x_0, \rho).$$

Let (x, y, z, w), $(\tilde{\xi}, -\kappa)$, and $(v, W_k(x), 1, 0)$ be as in the claim, so that $x \in \text{B}(x_0, \rho) \cap \text{dom}V$ and $v \in F(x)$, as well as $\kappa \ge 0$ (see [30, Exercise 2.1]). We may distinguish two cases:

(i) If $\kappa > 0$, then $w = \tilde{V}(x, y, z)$ and, without loss of generality, we may suppose that $\kappa = 1$. Hence, $\tilde{\xi} = (e^{az}\xi, 1, ae^{az}V(x)) \in \partial_P\tilde{V}(x, y, z)$ for some $\xi \in \partial_P V(x)$. Consequently, by the current hypothesis there exists $x^* \in A(x)$ such that

$$\langle \xi, v - x^* \rangle + aV(x) + W_k(x) \le \langle \xi, v - x^* \rangle + aV(x) + W(x) \le 0.$$

In other words, we have $(v - x^*, W_k(x), 1, 0) \in \hat{F}_k(x, y, z, w) - \hat{A}(x, y, z, w)$ and

$$
\begin{aligned}
\langle(\tilde{\xi}, -1), (v - x^*, W_k(x), 1, 0)\rangle &= \langle(e^{az}\xi, 1, ae^{az}V(x), -1), (v - x^*, W_k(x), 1, 0)\rangle \\
&= e^{az}\langle\xi, v - x^*\rangle + W_k(x) + ae^{az}V(x) \\
&= e^{az}(\langle\xi, v - x^*\rangle + aV(x) + W_k(x)) \\
&\quad + (1 - e^{az})W_k(x) \leq 0,
\end{aligned}
\tag{5.41}
$$

and (5.40) follows.

(ii) If $\kappa = 0$, then $\tilde{\xi} \in \partial_{P,\infty}\tilde{V}(x, y, z)$ and, so, $(\tilde{\xi}, -\kappa) = (\xi, \theta_{\mathbb{R}^3})$ for some $\xi \in \partial_{P,\infty}V(x)$. Then, by arguing as in the paragraph above, the current hypothesis yields some $x^* \in A(x)$ such that $\langle\xi, v - x^*\rangle \leq 0$. Hence, $(v - x^*, W_k(x), 1, 0) \in \hat{F}_k(x, y, z, w) - \hat{A}(x, y, z, w)$ and

$$
\langle(\tilde{\xi}, 0), (v - x^*, W_k(x), 1, 0)\rangle = \langle\xi, v - x^*\rangle \leq 0;
\tag{5.42}
$$

that is, (5.40) follows in this case too. The claim is proved.

Now, we take a solution $x(\cdot; x_0)$ of (5.1), so that $z(\cdot; z_0) := (x(\cdot; x_0), \int_0^\cdot W_k(x(\tau; x_0))d\tau, \cdot, V(x_0))$, with $z_0 := (x_0, 0, 0, V(x_0))$, becomes a solution of (5.36). Then, from the claim (with $\tilde{z} := z_0$) above and Proposition 5.10, there exists some $\bar{t} > 0$ such that

$$
z(t; z_0) \in \text{epi}\tilde{V} \quad \forall t \in [0, \bar{t}];
\tag{5.43}
$$

that is,

$$
\bar{T} := \sup\{t \geq 0 : \text{such that } z(s; z_0) \in \text{epi}\tilde{V} \; \forall s \in [0, t]\} > 0.
\tag{5.44}
$$

Let us show that $\bar{T} \geq T$, where T is defined in (5.39). We proceed by contradiction and assume that $\bar{T} < T$. Then, because (by Proposition 5.6(ii))

$$
\|x(\bar{T}; x_0) - x_0\| \leq 3(\|F(x_0)\| + \|A^\circ(x_0)\|)\bar{T}e^{c\bar{T}} < \rho,
$$

and $z(\bar{T}; z_0) = (x(\bar{T}; x_0), \int_0^{\bar{T}} W_k(x(\tau; x_0))d\tau, \bar{T}, V(x_0)) \in \text{epi}\tilde{V}$, from the claim above (with $\tilde{z} := z(\bar{T}; z_0)$) and Proposition 5.10, there exists some $t_1 > 0$ such that $z(t; z(\bar{T}; z_0)) \in \text{epi}\tilde{V}$ for all $t \in [0, t_1]$. Thus, $z(t + \bar{T}; z_0) = z(t; z(\bar{T}; z_0)) \in \text{epi}\tilde{V}$ for every $t \in [0, t_1]$, and we get a contradiction to the definition of $\bar{T}$.

Finally, from (5.44) we get

$$
e^{at}V(x(t; x_0)) + \int_0^t W_k(x(\tau; x_0))d\tau \leq V(x_0) \; \forall t \in [0, T].
$$

Moreover, because T is independent of k, by taking the limit as $k \to \infty$ we arrive at (as $W_k(x) \nearrow W(x)$, by Lemma 5.15)

$$e^{at}V(x(t; x_0)) + \int_0^t W(x(\tau; x_0))d\tau \leq V(x_0) \; \forall t \in [0, T],$$

which is the desired inequality. □

We give now the desired characterization of strong a-Lyapunov pairs.

Theorem 5.18. *Let V, W, and a be as in Proposition 5.17, and let ∂ stand for either ∂_P or ∂_F. Then the pair (V, W) is a strong a-Lyapunov pair for (5.1) iff for all $x \in \mathrm{dom}V$*

$$\sup_{\xi \in \partial V(x)} \sup_{v \in F(x)} \inf_{x^* \in A(x)} \langle \xi, v - x^* \rangle + aV(x) + W(x) \leq 0, \tag{5.45}$$

$$\sup_{\xi \in \partial_{P,\infty} V(x)} \sup_{v \in F(x)} \inf_{x^* \in A(x)} \langle \xi, v - x^* \rangle \leq 0. \tag{5.46}$$

Proof. To prove the sufficiency part, we take $x_0 \in \mathrm{dom}V$ and a solution $x(\cdot; x_0)$ of differential inclusion (5.1). By Proposition 5.17 there exists some $T > 0$ such that

$$e^{at}V(x(t; x_0)) + \int_0^t W(x(\tau; x_0))d\tau \leq V(x_0) \; \forall t \in [0, T]. \tag{5.47}$$

It suffices to prove that the following quantity is $+\infty$,

$$T := \sup\{s \geq 0 : (5.47) \text{ holds } \forall t \in [0, s]\}.$$

Otherwise, if T is finite, then $x(T; x_0) \in \mathrm{dom}V$ (because V is lower semi-continuous), and again from Proposition 5.17 we find $\eta > 0$ such that for all $t \in [0, \eta]$, using the semi-group property of $x(\cdot; x_0)$,

$$e^{a(t+T)}V(x(t + T; x_0)) + \int_0^{t+T} W(x(\tau; x_0))d\tau$$

$$\leq e^{aT}\left(e^{at}V(x(t + T; x_0)) + \int_T^{t+T} W(x(\tau; x_0))d\tau\right) + \int_0^T W(x(\tau; x_0))d\tau$$

$$\leq e^{aT}V(x(T; x_0)) + \int_0^T W(x(\tau; x_0))d\tau \leq V(x_0),$$

and we get the contradiction $T \geq T + \eta$. Hence, $T = +\infty$ and (5.47) holds for all $t \geq 0$, showing that (V, W) forms a strong Lyapunov pair for differential inclusion (5.1).

To prove the necessity of the current conditions, we start by verifying (5.45) with $\partial = \partial_F$. We fix $x_0 \in \mathrm{dom}V$ ($\subset \mathrm{dom}A$) and $v \in F(x_0)$, and, according to Proposition 5.6, we choose a solution $x(\cdot; x_0)$ of differential inclusion (5.1) such that $\frac{d^+x(0;x_0)}{dt} = v - \Pi_{A(x_0)}(v)$. Thus, since (V, W) is assumed to be a strong a-Lyapunov pair for (5.1), we obtain for every $t > 0$

$$\frac{V(x(t; x_0)) - V(x_0)}{t} + \frac{e^{at} - 1}{t} V(x(t; x_0)) + \frac{1}{t} \int_0^t W(x(\tau; x_0)) d\tau \leq 0,$$

which give us, as $t \downarrow 0$,

$$\begin{aligned} \sigma_{\partial_F V(x_0)}(v - \Pi_{A(x_0)}(v)) &\leq V'(x_0; v - \Pi_{A(x_0)}(v)) \\ &\leq \liminf_{t \downarrow 0} \frac{V(x(t; x_0)) - V(x_0)}{t} \leq -aV(x_0) - W(x_0). \end{aligned}$$

Hence, (5.45) follows with either $\partial = \partial_F$ or $\partial = \partial_P$. To verify (5.46) we fix $x_0 \in \mathrm{dom}V$, $v \in F(x_0)$ and $\xi \in \partial_{P,\infty} V(x_0)$; that is, $(\xi, 0) \in \mathrm{N}^P_{\mathrm{epi}V}(x_0, V(x_0))$. According to Proposition 5.6, we choose a solution $x(\cdot; x_0)$ of differential inclusion (5.1) such that $\frac{d^+x(0;x_0)}{dt} = v - \Pi_{A(x_0)}(v)$. Since (V, W) is strong a-Lyapunov for differential inclusion (5.1), one has that $(x(t; x_0), e^{-at}V(x_0)) \in \mathrm{epi}V$ for all $t \geq 0$. Then, by the definition of the proximal normal cone, there exists $\eta > 0$ such that for all small $t \geq 0$

$$\langle (\xi, 0), (x(t; x_0), e^{-at}V(x_0)) - (x_0, V(x_0)) \rangle \leq \eta \left(\|x(t; x_0) - x_0\|^2 + |e^{-at}V(x_0) - V(x_0)| \right)^2,$$

and so

$$\langle \xi, x(t; x_0) - x_0 \rangle \leq \eta \left(\|x(t; x_0) - x_0\|^2 + (e^{-at} - 1)^2 |V(x_0)|^2 \right).$$

Hence, by dividing on $t > 0$ and taking limits as $t \downarrow 0$, we obtain that $\langle \xi, v - \Pi_{A(x_0)}(v) \rangle \leq 0$, as we wanted to prove. $\qquad \square$

We give in the following corollary other criteria for strong a-Lyapunov pairs for (5.1), Recall that A° is said to be locally bounded on $\mathrm{dom}V$ if condition (5.26) holds for all $x \in \mathrm{dom}V$; that is, for every $x \in \mathrm{dom}V$ we have

$$\mathfrak{m}(x) = \limsup_{y \to x, y \in \mathrm{dom}V} \|A^\circ(y)\| < +\infty.$$

We also observe that the function $\mathfrak{m}$ is upper semi-continuous at every $x \in \mathbb{R}^n$

such that $\mathfrak{m}(x) < +\infty$; that is,

$$\limsup_{y \to x, y \in \mathrm{dom}V} \mathfrak{m}(y) = \mathfrak{m}(x). \tag{5.48}$$

Corollary 5.19. *Let V, W, and a be as in Proposition 5.17, and let ∂ stand for either ∂_P, ∂_F, or ∂_L. If A° is locally bounded on $\mathrm{dom}V$, then (V, W) is a strong a-Lyapunov pair for (5.1) iff one of the following statements holds.*

(i) For any $x \in \mathrm{dom}V$,

$$\sup_{\xi \in \partial V(x)} \sup_{v \in F(x)} \inf_{x^* \in A(x) \cap \mathrm{B}_{\|F(x)\| + \mathfrak{m}(x)}} \langle \xi, v - x^* \rangle + aV(x) + W(x) \leq 0.$$

(ii) For any $x \in \mathrm{dom}V$,

$$\sup_{v \in F(x)} V'(x; v - \Pi_{A(x)}(v)) + aV(x) + W(x) \leq 0.$$

(iii) For any $x \in \mathrm{dom}V$,

$$\sup_{v \in F(x)} \inf_{x^* \in A(x) \cap \mathrm{B}_{\|F(x)\| + \mathfrak{m}(x)}} V'(x; v - x^*) + aV(x) + W(x) \leq 0.$$

Proof. $(ii) \Rightarrow (iii)$. This implication follows since that for any $x \in \mathrm{dom}V$ ($\subset \mathrm{dom}A$) any $v \in F(x)$

$$\|\Pi_{A(x)}(v)\| \leq \|A^\circ(x)\| + \|\Pi_{A(x)}(v) - A^\circ(x)\| \leq \|A^\circ(x)\| + \|v\| \leq \mathfrak{m}(x) + \|F(x)\|.$$

$(iii) \Rightarrow (i)$. When ∂ stands for either ∂_P or ∂_F this implication follows from the relation $\sigma_{\partial_P V(x)}(\cdot) \leq \sigma_{\partial_F V(x)}(\cdot) \leq V'(x; \cdot)$. If $\partial = \partial_L$, we take $\xi \in \partial_L V(x)$ and $v \in F(x)$, and choose sequences (x_i) and (ξ_i) such that

$$x_i \xrightarrow{V} x, \ \xi_i \in \partial_P V(x_i), \ \xi_i \to \xi \ \text{as } i \to \infty;$$

moreover, due to the upper semi-continuity of $\mathfrak{m}$ at x and $\mathfrak{m}(x) < +\infty$, by assumption, we may assume up to a subsequence that

$$\mathfrak{m}(x_i) \leq \mathfrak{m}(x) + \frac{1}{i} \ \forall i \in \mathbb{N}. \tag{5.49}$$

By the Lipschitzianity of F we also choose a sequence $(v_i)_{i \geq 1}$ such that $v_i \in F(x_i)$ and $v_i \to v$. Since (i) holds with $\partial = \partial_P$, for each i there exists $x_i^* \in A(x_i) \cap$

$B_{\|F(x_i)\|+\mathfrak{m}(x_i)}$ such that

$$\langle \xi_i, v_i - x_i^* \rangle + aV(x_i) + W(x_i) \le 0. \tag{5.50}$$

Then, since the maximal monotone operator A has a closed graph, and $(x_i^*)_i$ is bounded, we assume w.l.o.g. that

$$x_i^* \to x^* \in A(x) \cap B(\theta, \mathfrak{m}(x)) \text{ as } i \to \infty.$$

So, by passing to the limit in (5.50) as $i \to \infty$, and using the lower semicontinuity of W, we obtain that

$$\langle \xi, v - x^* \rangle + aV(x) + W(x) \le 0, \tag{5.51}$$

which shows that (i) holds when $\partial = \partial_L$.

$(i) \Rightarrow (V, W)$ is a strong a-Lyapunov pair for (5.1). According to Theorem 5.18 we only need to show that (5.46) holds. We fix $x \in \text{dom} V$, $\xi \in \partial_{P,\infty} V(x)$ and $v \in F(x)$. There exist sequences $(x_i)_i$, $(\xi_i)_i$, and $(\alpha_i)_i$ such that

$$x_i \overset{V}{\to} x, \ \xi_i \in \partial_P V(x_i), \ \alpha_i \downarrow 0, \ \alpha_i \xi_i \to \xi \text{ as } i \to \infty.$$

By arguing as in the last paragraph above there also exists a sequence $(v_i)_i$ such that $v_i \in F(x_i)$ and $v_i \to v$ as $i \to \infty$. Moreover, using the current assumption on A°, there exists $m > 0$ such that $\sup_i \mathfrak{m}(x_i) \le m$. Now, by assumption (ii), for each $i \in \mathbb{N}$ there exists a sequences $x_i^* \in A(x_i) \cap B_{\|F(x_i)\|+\mathfrak{m}(x_i)} \subset A(x_i) \cap B_{\|F(x_i)\|+m}$ and

$$\langle \xi_i, v_i - x_i^* \rangle + aV(x_i) + W(x_i) \le 0. \tag{5.52}$$

By using again that A has a closed graph, and that $x_i^* \to x^* \in A(x)$, By multiplying the last inequality above (5.52) by α_i and next taking limits as $i \to \infty$, we arrive at (5.46). The proof of the corollary is finished since (ii) is a necessary condition for strong a-Lyapunov pairs, as we have shown in the proof of Theorem 5.18. $\qquad \square$

5.6 Weak a-Lyapunov pairs

In this section, we characterize weak a-Lyapunov pairs with respect to differential inclusion (5.1),

$$\dot{x}(t) \in F(x(t)) - A(x(t)), \quad \text{a.e. } t \geq 0, \quad x(0) = x_0 \in \overline{\mathrm{dom}A},$$

where $A : H \rightrightarrows H$ is a maximal monotone operator and F is an L-Lipschitz Cusco mapping.

Definition 5.20. Let $V, W : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lower semi-continuous functions such that $W \geq 0$ and let $a \geq 0$. We say that (V, W) is a weak a-Lyapunov pair for (5.1) if for any $x_0 \in \overline{\mathrm{dom}A}$, there exists at least one solution $x(\cdot; x_0)$ of (5.1) such that

$$e^{at}V(x(t; x_0)) + \int_0^t W(x(\tau; x_0))d\tau \leq V(x_0) \qquad \forall t \geq 0.$$

Definition 5.21. Let $c > 0$ be as in Proposition 5.6(ii), and take $x_0 \in \mathrm{dom}A$ and $\rho > 0$. We denote by $T_{c,\rho}(x_0)$ the positive number that satisfies the following equation in t,

$$3(\|F(x_0)\| + \|A^\circ(x_0)\|)te^{ct} = \frac{\rho}{2}.$$

Proposition 5.22. *Let $V, W : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be two proper lower semi-continuous functions such that $\mathrm{dom}V \subset \mathrm{dom}A, W \geq 0$ and let $a \geq 0$. Fix $x_0 \in \mathrm{dom}V$ and assume that for some $m, \rho > 0$ we have, for all $x \in \mathrm{B}(x_0, \rho) \cap \mathrm{dom}V$*

$$\sup_{\xi \in \partial_P V(x)} \inf_{v \in F(x)} \inf_{x^* \in A(x) \cap \mathrm{B}_{m+\|F(x)\|}} \langle \xi, v - x^* \rangle + aV(x) + W(x) \leq 0.$$

Then there is a solution $x(\cdot; x_0)$ of differential inclusion (5.1) such that

$$e^{at}V(x(t; x_0)) + \int_0^t W(x(\tau; x_0))d\tau \leq V(x_0) \ \forall t \in [0, T_{c,\rho}(x_0)],$$

where $T_{c,\rho}(x_0) > 0$ is given in Definition 5.21.

Proof. We fix $k \in \mathbb{N}$, and let $\hat{A}$, $\tilde{V}$, and F_k be as in Lemma 5.16. First, since $\{x \in \mathbb{R}^n \mid \partial_P V(x) \neq \emptyset\}$ is dense in $\mathrm{dom}V$ (see, e.g., [30, Theorem 1.3.1]), and the maximal monotone operator has closed graph, from the current assumption and the Cusco property of F, there is some $m_1 > 0$ that, for all $(x, y, z, w) \in \mathrm{epi}\tilde{V}$ such

that $\|x - x_0\| < \rho$,

$$\left\| \hat{A}^\circ(x, y, z, w) \right\| \le m + \|F(x)\| \le m_1. \tag{5.53}$$

We proceed by steps:

Step 1). We prove that for every $(x, y, z, w) \in \text{epi}\tilde{V}$ such that $\|x - x_0\| < \rho$ and $(\tilde{\xi}, -\kappa) \in \mathrm{N}^P_{\text{epi}\tilde{V}}(x, y, z, w)$, there exist $v \in F(x)$ and $x^* \in A(x) \cap \mathrm{B}_{m+\|F(x)\|}$ such that

$$(v, W_k(x), 1, 0) \in \hat{F}_k(x, y, z, w), \quad (x^*, \theta_{\mathbb{R}^3}) \in \hat{A}(x, y, z, w) \cap \mathrm{B}_{m+\|F(x)\|} \quad \text{and} \tag{5.54}$$

$$\langle (\tilde{\xi}, -\kappa), (v - x^*, W_k(x), 1, 0) \rangle \le 0. \tag{5.55}$$

Indeed, let $(x, y, z, w) \in \text{epi}\tilde{V}$ and $(\tilde{\xi}, -\kappa)$ be as in the claim, so that $\kappa \ge .$ If $\kappa > 0$, say $\kappa = 1$, we get $\tilde{\xi} = (e^{az}\xi, 1, ae^{az}V(x)) \in \partial_P\tilde{V}(x, y, z)$ for some $\xi \in \partial_P V(x)$. Then, by the current hypothesis, there exist $v \in F(x)$ and $x^* \in A(x) \cap \mathrm{B}_{m+\|F(x)\|}$, that satisfy (5.54), such that

$$\langle \xi, v - x^* \rangle + aV(x) + W_k(x) \le \langle \xi, v - x^* \rangle + aV(x) + W(x) \le 0.$$

Hence,

$$\begin{aligned}
\langle (\tilde{\xi}, -1), (v - x^*, W_k(x), 1, 0) \rangle &= e^{az}\langle \xi, v - x^* \rangle + W_k(x) + ae^{az}V(x) \\
&= e^{az}(\langle \xi, v - x^* \rangle + aV(x) + W_k(x)) \\
&\quad + (1 - e^{az})W_k(x) \le 0,
\end{aligned} \tag{5.56}$$

and (5.55) follows. If $\kappa = 0$, then $\tilde{\xi} = (\xi, 0, 0) \in \partial_{P,\infty}\tilde{V}(x, y, z)$ for some $\xi \in \partial_{P,\infty}V(x)$. Then, taking into account that $\|x - x_0\| < \rho$, there are sequences $(x_i) \subset \mathrm{B}(x_0, \rho) \cap \text{dom}V$, $(\xi_i) \subset \mathbb{R}^n$, and $(\alpha_i) \subset \mathbb{R}$ such that $x_i \xrightarrow{V} x$, $\xi_i \in \partial_P V(x_i)$, $\alpha_i \downarrow 0$, and $\alpha_i\xi_i \to \xi$ as $i \to \infty$. Hence, for each $i \in \mathbb{N}$, by the current hypothesis, there exist $v_i \in F(x_i)$ and $x_i^* \in A(x_i) \cap \mathrm{B}_{m+\|F(x_i)\|}$ such that

$$\langle \xi_i, v_i - x_i^* \rangle + aV(x_i) + W_k(x_i) \le 0. \tag{5.57}$$

Because F is of Cusco and A is maximal monotone, we may suppose w.l.o.g. that $v_i \to v \in F(x)$ and $x_i^* \to x^* \in A(x) \cap \mathrm{B}_{m+\|F(x)\|}$ as $i \to \infty$. Then, by multiplying both sides of the inequality above by α_i and taking limits as $i \to \infty$, we obtain

$\langle \xi, v - x^* \rangle \leq 0$; that is,

$$\langle (\tilde{\xi}, 0), (v - x^*, W_k(x), 1, 0) \rangle = \langle \xi, v - x^* \rangle \leq 0,$$

and we get (5.55).

Step 2). Given $\tilde{z} := (\tilde{x}, \tilde{y}, \tilde{u}, \tilde{w}) \in \text{epi}\tilde{V}$ such that $\|\tilde{x} - x_0\| < \rho$, we prove that there exists $z(\cdot; \tilde{z})$ solution of differential inclusion (5.36) such that

$$z(t; \tilde{z}) \in \text{epi}\tilde{V} \ \forall t \in \left[0, \frac{\rho - \|\tilde{x} - x_0\|}{6(m_1 + \beta_k)} \right],$$

where

$$\beta_k := \left((\|F(x_0)\| + L\rho)^2 + (\|W(x_0)\| + k\rho)^2 + 1 \right)^{\frac{1}{2}}. \tag{5.58a}$$

Indeed, let $\tilde{z}$ be as in the claim. Then for every $z := (x, y, u, w) \in \text{epi}\tilde{V} \cap B(\tilde{z}, \frac{1}{2}(\rho - \|\tilde{x} - x_0\|))$ we have

$$\|x - x_0\| \leq \|\tilde{x} - x_0\| + \|x - \tilde{x}\| \leq \|\tilde{x} - x_0\| + \frac{1}{2}(\rho - \|\tilde{x} - x_0\|) = \frac{1}{2}(\rho + \|\tilde{x} - x_0\|) < \rho, \tag{5.59}$$

and, so, $\left\| \hat{A}^\circ(z) \right\| \leq m + \|F(x)\| \leq m_1$ (recall (5.53)). In other words, according to the first step, for every $z \in \text{epi}\tilde{V} \cap B(\tilde{z}, \frac{1}{2}(\rho - \|\tilde{x} - x_0\|))$ and $(\tilde{\xi}, -\kappa) \in N^P_{\text{epi}\tilde{V}}(z)$, there exist $(v, W_k(x), 1, 0) \in \hat{F}_k(z)$ and $(x^*, \theta_{\mathbb{R}^3}) \in \hat{A}(z) \cap B_{m + \|F(x)\|} \subset \hat{A}(z) \cap B_{m_1 + \|F_k(z)\|}$ such that (5.55) holds. Consequently, taking into account that (using (5.59))

$$\sup_{p \in \text{epi}\tilde{V} \cap B(\tilde{z}, \frac{1}{2}(\rho - \|\tilde{x} - x_0\|))} \left\| \hat{F}_k(p) \right\| = \sup_{x \in B(x_0, \rho)} \|(F(x), W_k(x), 1, 0)\| \leq \beta_k, \tag{5.60}$$

by Proposition 5.12 there exists a solution of differential inclusion (5.36) as required.

Step 3). We put $z_0 := (x_0, 0, 0, V(x_0)) \in \text{epi}\tilde{V}$. Then

$$\bar{T} := \sup\{t \geq 0 : \text{there exists a solution } z_k(\cdot; z_0) \text{ of (5.36) st. } z_k(s; z_0) \in \text{epi}\tilde{V} \ \forall s \in [0, t]\}$$
$$\geq T_{c,\rho}(x_0), \tag{5.61}$$

where $T_{c,\rho}(x_0)$ is given in Definition 5.21; hence, $T_{c,\rho}(x_0)$ satisfies

$$3(\|F(x_0)\| + \|A^\circ(x_0)\|)T_{c,\rho}(x_0)e^{cT_{c,\rho}(x_0)} < \rho, \tag{5.62}$$

with $c > 0$ being such that (see Proposition 5.6(ii)) for any solutions $x(\cdot) := x(\cdot; x_0)$ of (5.1) it holds

$$\|x(t) - x_0\| \leq 3(\|F(x_0)\| + \|A^\circ(x_0)\|)te^{ct} \quad \text{for all } t \geq 0. \tag{5.63}$$

Indeed, from Step 2 we have that $\bar{T} \geq \frac{\rho}{6(m_1 + \beta_k)}$, where β_k is defined in (5.58a).

To prove (5.61) we proceed by contradiction and assume that $\bar{T} < T_{c,\rho}(x_0)$. By definition of $\bar{T}$, for every $0 < \varepsilon < \min\{\bar{T}, \frac{\rho - 3(\|F(x_0)\| + \|A^\circ(x_0)\|)T_{c,\rho}(x_0)e^{cT_{c,\rho}(x_0)}}{6(m_1 + \beta_k)}\}$ there exists a solution $z_k(\cdot; z_0)$ of (5.36), and a solution $x_k(\cdot; x_0)$ of (5.1), such that

$$z_k(t; z_0) = (x_k(t; x_0), \int_0^t W_k(x_k(\tau; x_0))d\tau, t, V(x_0)) \in \text{epi}\tilde{V} \ \forall t \in [0, \bar{T} - \varepsilon].$$

But by (5.63) and (5.62) we have that

$$\begin{aligned}
\left\| x_k(\bar{T} - \varepsilon; x_0) - x_0 \right\| &\leq 3(\|F(x_0)\| + \|A^\circ(x_0)\|)(\bar{T} - \varepsilon)e^{c(\bar{T} - \varepsilon)} \\
&\leq 3(\|F(x_0)\| + \|A^\circ(x_0)\|)T_{c,\rho}(x_0)e^{cT_{c,\rho}(x_0)} < \rho, \tag{5.64}
\end{aligned}$$

and, so, by Step 2 there exists $\bar{z}_k(\cdot; z_k(\bar{T} - \varepsilon; z_0))$ a solution of differential inclusion (5.36) such that

$$\bar{z}_k(t; z_k(\bar{T} - \varepsilon; z_0)) \in \text{epi}\tilde{V} \ \forall t \in \left[0, \frac{\rho - \left\| x_k(\bar{T} - \varepsilon; z_0) - x_0 \right\|}{6(m_1 + \beta_k)}\right].$$

We denote

$$\tilde{z}_k(t; z_0) := \begin{cases} z_k(t; z_0) & \text{if } t \in [0, \bar{T} - \varepsilon] \\ \bar{z}_k(t - \bar{T} + \varepsilon; z_k(\bar{T} - \varepsilon; z_0)) & \text{if } t \in [\bar{T} - \varepsilon, \infty). \end{cases}$$

Then $\tilde{z}_k(\cdot; z_0)$ is a solution of (5.36) and we have that

$$\tilde{z}_k(t; z_0) \in \text{epi}\tilde{V} \ \forall t \in \left[0, \bar{T} + \frac{\rho - \left\| x_k(\bar{T} - \varepsilon; z_0) - x_0 \right\|}{6(m_1 + \beta_k)} - \varepsilon\right].$$

Thus, since (recall (5.64))

$$\frac{\rho - \left\| x_k(\bar{T} - \varepsilon; z_0) - x_0 \right\|}{6(m_1 + \beta_k)} \geq \frac{\rho - 3(\|F(x_0)\| + \|A^\circ(x_0)\|)T_{c,\rho}(x_0)e^{cT_{c,\rho}(x_0)}}{6(m_1 + \beta_k)} > \varepsilon,$$

we get the contradiction $\bar{T} \geq \bar{T} + \frac{\rho - \left\| x_k(\bar{T} - \varepsilon; z_0) - x_0 \right\|}{6(m_1 + \beta_k)} - \varepsilon > \bar{T}$. Step 3 is now proved.

Step 4). In this last step we get the conclusion of the proposition. From Step 3 there is a solution $x_k(\cdot; x_0)$ of (5.1) such that

$$z_k(t; z_0) = \left(x_k(t; x_0), \int_0^t W_k(x_k(\tau; x_0))d\tau, t, V(x_0)\right) \in \mathrm{epi}\tilde{V} \quad \text{for all } t \in [0, T_{c,\rho}(x_0)];$$

that is,

$$e^{at}V(x_k(t; x_0)) + \int_0^t W_k(x_k(\tau; x_0))d\tau \leq V(x_0) \quad \text{for all } t \in [0, T_{c,\rho}(x_0)].$$

Moreover, since that $\|x_k(\cdot; x_0) - x_0\| \leq 3(\|F(x_0)\| + \|A^\circ(x_0)\|)T_{c,\rho}(x_0)e^{cT_{c,\rho}(x_0)} < \rho$ (by Proposition 5.6(ii)), by using the Lipschitz property of F and (5.53) from Proposition 5.6(i) we obtain that

$$\|\dot{x}_k(t; x_0)\| \leq \|F(x_k(t; x_0))\| + \|A^\circ(x_k(t; x_0))\| \leq \|F(x_0)\| + L\rho + m_1.$$

Consequently, by Lemma 5.15 there exists a solution $\hat{x}(\cdot; x_0)$ of (5.1) such that

$$e^{at}V(\hat{x}(t; x_0)) + \int_0^t W(\hat{x}(\tau; x_0))d\tau \leq V(x_0) \; \forall t \in [0, T_{c,\rho}(x_0)],$$

which yields the conclusion of the proposition. $\qquad\square$

The assumption of Proposition 5.22 easily implies that (see (5.53)) A° is locally bounded on $\mathrm{dom}V$; that is,

$$\mathfrak{m}(x) = \limsup_{y \overset{\mathrm{dom}V}{\longrightarrow} x} \|A^\circ(y)\| < +\infty \text{ for all } x \in \mathrm{dom}V. \tag{5.65}$$

The following theorem characterizes weak a-Lyapunov pairs for (5.1) under the condition above.

Theorem 5.23. *Let V, W, and $a \geq 0$ be as in Proposition 5.22, and let ∂ stand for either ∂_P, ∂_F, or ∂_L. Under the local boundedness of A° on $\mathrm{dom}V$, (V, W) is a weak a-Lyapunov pair for (5.1) iff one of the following assertions holds:*

(i) For every $x \in \mathrm{dom}V$,

$$\sup_{\xi \in \partial V(x)} \inf_{v \in F(x)} \inf_{x^* \in A(x) \cap B_{\mathfrak{m}(x) + \|F(x)\|}} \langle \xi, v - x^* \rangle + aV(x) + W(x) \leq 0.$$

(ii) For every $x \in \text{dom}V$,

$$\inf_{v \in F(x)} \inf_{x^* \in A(x) \cap B_{\|F(x)\| + \mathfrak{m}(x)}} V'(x; v - x^*) + aV(x) + W(x) \leq 0.$$

Proof. $(i) \Rightarrow ((V, W)$ is a weak a-Lyapunov pair for $(5.1))$.

As in Lemma 5.16 we let the maximal monotone operator $\hat{A} : \mathbb{R}^n \times \mathbb{R}^3 \to \mathbb{R}^{n+3}$, the proper lower semi-continuous function $\tilde{V} : \mathbb{R}^{n+1} \times \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$, and the Cusco mappings $\hat{F}_k : \mathbb{R}^{n+3} \to \mathbb{R}^{n+3}$, $k \geq 1$, defined by

$$\hat{A}(x, \alpha, \beta, \gamma) := (A(x), \theta_{\mathbb{R}^3}), \quad \tilde{V}(x, \alpha, \beta) := e^{a\beta}V(x) + \alpha, \quad \hat{F}_k(x, \alpha, \beta, \gamma) := (F(x), W_k(x), 1, 0).$$

Let us fix $z_0 = (x_0, 0, 0, V(x_0)) \in \text{epi}\tilde{V}$ and $k \in \mathbb{N}$. We set

$$T := \sup\{s : \text{ there exists a solution } z_k(\cdot; z_0)) \text{ of } (5.36) \text{ s.t. } z_k(\cdot; z_0) \in \text{epi}\tilde{V} \; \forall t \in [0, s]\}$$
$$(5.66)$$

By current hypothesis (i) and the locally boundedness of A°, there exists $m > 0$ and $\rho > 0$ such that for all $x \in B(x_0, \rho)$

$$\sup_{\xi \in \partial_P V(x)} \inf_{v \in F(x)} \inf_{x^* \in A(x) \cap B_{m + \|F(x)\|}} \langle \xi, v - x^* \rangle + aV(x) + W_k(x) \leq 0.$$

Then, according to Proposition 5.22, there exists $s > 0$ and a solution $x(\cdot; x_0)$ of (5.1) such that

$$e^{at}V(x(t; x_0)) + \int_0^t W_k(x(\tau; x_0))d\tau \leq V(x_0) \; \forall t \in [0, s];$$

that is, $z_k(\cdot; z_0) := (x(\cdot; x_0), \int_0^{\cdot} W_k(x(\tau; x_0))d\tau, \cdot, V(x_0))$ is a solution of (5.36) such that $z_k(t; z_0) \in \text{epi}\tilde{V}$ for all $t \in [0, s]$. Hence, $T > 0$. If T is finite, then by Proposition 5.6 there would exist $r > 0$ such that for any solution $z_k(\cdot; z_0)$ of (5.36) we have

$$z_k(\cdot; z_0) \in B(z_0, r) \; \forall t \in [0, T].$$

Also, since the set $B(z_0, r + 2) \cap \text{epi}\tilde{V}$ and its projection on $\mathbb{R}^n$

$$E = \left\{ x \in \mathbb{R}^n : \text{ there exist } (y, u, v) \in \mathbb{R}^3 \text{ s.t. } (x, y, u, v) \in B(z_0, r + 2) \cap \text{epi}\tilde{V} \right\},$$

are compact, by the current assumptions there exists $M > 0$ such that

$$\|(\hat{A})^\circ(z)\| \leq M \; \forall z \in B(z_0, r + 2) \cap \text{epi}\tilde{V}, \; \mathfrak{m}(x) \leq M \; \forall x \in E, \quad (5.67)$$

and so

$$\tilde{\mathfrak{m}}(z) := \limsup_{z' \xrightarrow{\text{epi}\tilde{V}} z} \|(\hat{A})^\circ(z')\| \leq M \ \forall z \in \mathrm{B}(z_0, r+1) \cap \mathrm{epi}\tilde{V}.$$

To continue we shall proceed by steps:

Step 1). We show that for any $z \in \mathrm{B}(z_0, r+1) \cap \mathrm{epi}\tilde{V}$ we have

$$\sup_{\xi \in \mathrm{N}^P_{\mathrm{epi}\tilde{V}}(z)} \inf_{v \in \hat{F}_k(z)} \inf_{z^* \in \hat{A}(z) \cap \mathrm{B}_{M+\|F_k(z)\|}} \langle \xi, v - z^* \rangle \leq 0. \tag{5.68}$$

Indeed, let $z := (x, y, u, w) \in \mathrm{B}(z_0, r+1) \cap \mathrm{epi}\tilde{V}$ and pick $(\tilde{\xi}, -\kappa) \in \mathrm{N}^P_{\mathrm{epi}\tilde{V}}(z)$; hence, $\kappa \geq 0$ and $x \in E$.

If $\kappa > 0$, say $\kappa = 1$ (w.l.o.g.), then we get $\tilde{\xi} = (e^{au}\xi, 1, ae^{au}V(x))$ for some $\xi \in \partial_P V(x)$. Thus, by the current hypothesis (i) and (5.67) there exist $v \in F(x)$ and $x^* \in A(x) \cap \mathrm{B}_{\mathfrak{m}(x)+\|F(x)\|} \subset A(x) \cap \mathrm{B}_{M+\|F(x)\|}$ such that

$$\langle \xi, v - x^* \rangle + aV(x) + W_k(x) \leq 0.$$

In other words, we have that $(x^*, 0, 0, 0) \in \hat{A}(z) \cap \mathrm{B}_{M+\|F(x)\|} \subset \hat{A}(z) \cap \mathrm{B}_{M+\|F_k(z)\|}$ and

$$\begin{aligned}
\langle (\tilde{\xi}, -1), (v, W_k(x), 1, 0) - (x^*, 0, 0, 0) \rangle &= \langle (e^{au}\xi, 1, ae^{au}V(x), -1), (v - x^*, W_k(x), 1, 0) \rangle \\
&= e^{au}(\langle \xi, v - x^* \rangle + aV(x) + W_k(x)) \\
&\quad + (1 - e^{au})W_k(x) \leq 0,
\end{aligned}$$

which entails (5.68).

f $\kappa = 0$, then $\tilde{\xi} = (\xi, 0, 0) \in \partial_{P,\infty}\tilde{V}(x, y, z)$ for some $\xi \in \partial_{P,\infty}V(x)$. Hence, there are sequences (x_i), $(\xi_i) \subset \mathbb{R}^n$, and $(\alpha_i) \subset \mathbb{R}$ such that $x_i \xrightarrow{V} x$, $\xi_i \in \partial_P V(x_i)$, $\alpha_i \downarrow 0$, and $\alpha_i \xi_i \to \xi$ as $i \to \infty$. We denote $w_i := w + e^{au}(V(x_i) - V(x))$, $i \in \mathbb{N}$, so that $w_i \to w$ and, w.l.o.g. on $i \geq 1$, $(x_i, y, u, w_i) \in \mathrm{B}(z_0, r+2)$ and

$$\tilde{V}(x_i, y, u) = e^{au}V(x_i) + y = w_i - (w - e^{au}V(x) - y) = w_i - (w - \tilde{V}(x, y, u)) \leq w_i;$$

that is, $(x_i, y, u, w_i) \in \mathrm{B}(z_0, r+2) \cap \mathrm{epi}\tilde{V}$. Now, by the current hypothesis, for each $i \in \mathbb{N}$ there exist $v_i \in F(x_i)$ and $x_i^* \in A(x_i) \cap \mathrm{B}_{m(x)+\|F(x_i)\|} \subset A(x_i) \cap \mathrm{B}_{M+\|F(x_i)\|}$ such that

$$\langle \xi_i, v_i - x_i^* \rangle + aV(x_i) + W_k(x_i) \leq 0. \tag{5.69}$$

Because F is of Cusco and A is maximal monotone, we may suppose w.l.o.g. that

$v_i \to v \in F(x)$ and $x_i^* \to x^* \in A(x) \cap B_{M+\|F(x)\|}$ as $i \to \infty$. Then, by multiplying both sides of the inequality above by α_i and taking limits as $i \to \infty$, we obtain $\langle \xi, v - x^* \rangle \leq 0$; that is,

$$\langle (\tilde{\xi}, 0), (v - x^*, W_k(x), 1, 0) \rangle = \langle \xi, v - x^* \rangle \leq 0,$$

and (5.68) also follows in this case.

Step 2). We show that $T = \infty$. From Step 1 and Proposition 5.12, for every $z \in B(z_0, r) \cap \text{epi}\tilde{V}$, there exists a solution $z_k(\cdot; z)$ such that

$$z_k(t; z) \in \text{epi}\tilde{V} \ \forall t \in [0, \bar{t}],$$

where $\bar{t} := \frac{1}{3}(M + \sup\limits_{z' \in B(z_0, r+1) \cap \text{epi}\tilde{V}} \|F_k(z')\|)^{-1}$. Let us fix $\varepsilon \in (0, \bar{t})$. From the definition of T, there exists a solution $\bar{z}_k(\cdot; z_0)$ of (5.36) such that $\bar{z}_k(t, z_0) \in \text{epi}\tilde{V}$ for all $t \in [0, T - \varepsilon]$. By the result above, it is easy to find a solution $\tilde{z}_k(\cdot; z_0)$ of (5.36) such that $\tilde{z}_k(t; z_0) \in \text{epi}\tilde{V}$ for all $t \in [0, T + \bar{t} - \varepsilon]$ which contradicts the definition of T, hence we get $T = +\infty$.

Step 3). In this step, we get (V, W) is a weak a-Lyapunov pair for (5.1). By the result of Step 2, then for every $\overline{T}$ and $k \in \mathbb{N}$, there exists a solution $x_k(\cdot; x_0)$ of (5.1) such that

$$z_k(t; z_0) := (x_k(t; x_0), \int_0^t W_k(x_k(\tau; x_0))d\tau, t, V(x_0)) \in \text{epi}\tilde{V} \ \forall \ t \in [0, \overline{T}],$$

or, equivalently,

$$e^{at}V(x_k(t; x_0)) + \int_0^t W_k(x_k(\tau; x_0))d\tau \leq V(x_0) \ \forall t \in [0, \overline{T}],$$

that is, using Proposition 5.6(ii),

$$(x_k(t; x_0))_k \subset D := [V \leq V(x_0)] \cap B\left(x_0, 3\left(\|F(x_0)\| + \|A^\circ(x_0)\|\right)\overline{T}e^{c\overline{T}}\right)$$

where $c > 0$ is defined in Proposition 5.6(ii). Thus, by the current assumption, and the lower semi-continuity of the function V, A° is bounded on the compact set D, so that by Proposition 5.6(i) we obtain some $\overline{M} > 0$ such that $\|\dot{x}_k(t; x_0)\| \leq \overline{M}$ for all $k \geq 1$. Therefore, by Lemma 5.15 there exists a solution $x(\cdot; x_0)$ of (5.1)

such that

$$e^{at}V(x(t;x_0)) + \int_0^t W(x(\tau;x_0))d\tau \le V(x_0) \ \forall t \in [0,T]. \tag{5.70}$$

As we proceeded many times we can show that

$$\tilde{T} := \sup\{s \ge 0 \mid \exists \ x(\cdot;x_0) \text{ solution of (5.1) st. (5.70) holds } \forall t \in [0,s]\} = +\infty,$$

which ensures that (V,W) is a weak a-Lyapunov pair for (5.1).

$(ii) \Rightarrow (i)$. This implication follows when $\partial = \partial_P$ or $\partial = \partial_F$, due to the relations

$$\sigma_{\partial_P V(x)}(\cdot) \le \sigma_{\partial_F V(x)}(\cdot) \le V'(x;\cdot).$$

It remains to check the case $\partial = \partial_L$. We take $\xi \in \partial_L V(x)$ and, be definition, let the sequences (x_k) and (ξ_k) converge to x and ξ, respectively, such that $\xi_k \in \partial_P V(x_k)$ for all k. Since (i) already holds for $\partial = \partial_P$, for each $k \ge 1$ there exist $v_k \in F(x_k)$ and $x_k^* \in A(x_k) \cap B_{\mathfrak{m}(x_k)+\|F(x_k)\|}$ such that

$$\langle \xi_k, v_k - x_k^* \rangle + aV(x_k) + W(x_k) \le 0.$$

We may suppose that $x_k^* \to x^* \in A(x) \cap B_{\mathfrak{m}(x)+\|F(x)\|}$ and $v_k \to v \in F(x)$ as $k \to \infty$. Hence, by taking the limit as $k \to \infty$ in the last inequality we get

$$\langle \xi, v - x^* \rangle + aV(x) + W(x) \le 0,$$

as we wanted to prove.

$((V,W)$ is a weak a-Lyapunov pair for (5.1)$) \Rightarrow (ii)$. Take $x_0 \in \text{dom}V$ and let $x(\cdot;x_0)$ be any solution of (5.1) that satisfies

$$e^{at}V(x(t;x_0)) + \int_0^t W(x(\tau;x_0))d\tau \le V(x_0) \ \ \forall t \ge 0. \tag{5.71}$$

Then, as in (5.30) for some $t_n \downarrow 0$ we have that

$$v := \lim_{t_n \downarrow 0} \frac{x(t_n;x_0) - x_0}{t_n} \in F(x_0) - (A(x_0) \cap B_{\|F(x_0)\|+\mathfrak{m}(x_0)}),$$

and so, using (5.71),

$$
\begin{aligned}
V'(x_0; v) &\leq \liminf_{n \to \infty} \frac{V(x(t_n; x_0)) - V(x_0)}{t_n} \\
&\leq \liminf_{n \to \infty} \left(- \frac{e^{at_n} - 1}{t_n} V(x(t_n; x_0)) - \frac{1}{t_n} \int_0^{t_n} W(x(\tau; x_0) d\tau \right) \\
&\leq -aV(x_0) - W(x_0),
\end{aligned}
$$

and we get (ii). $\qquad\square$

5.7 Differential inclusions with prox-regular sets

In this section, we use the previous results to characterize Lyapunov pairs associated to the following differential inclusion

$$
\begin{cases}
\dot{x}(t) \in F(x(t)) - \mathrm{N}_C(x(t)) \ \text{ a.e. } \ t \geq 0, \\
x(0) = x_0 \in C.
\end{cases}
\tag{5.72}
$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an L-Lipschitz Cusco mapping and C is an r-uniformly-prox-regular set of $\mathbb{R}^n$.

Definition 5.24. [63, 73] For positive numbers r and α, a closed set S is said to be (r, α)-prox-regular at $\overline{x} \in S$ provided that one has $x = \Pi_S(x + v)$, for all $x \in S \cap \mathrm{B}(\overline{x}, \alpha)$ and all $v \in \mathrm{N}_S^P(x)$ such that $\|v\| < r$.
The set S is r-prox-regular (resp., prox-regular) at $\overline{x}$ when it is (r, α)-prox-regular at $\overline{x}$ for some real $\alpha > 0$ (resp., for some numbers $r, \alpha > 0$). The set S is said to be r-uniformly prox-regular when $\alpha = +\infty$.

When S is r-uniformly prox-regular, the set-valued mapping defined by $x \mapsto \mathrm{N}_S^P(x) \cap \mathbb{B}$ is $\frac{1}{r}$-hypo-monotone, and for every $x \in S$ we have ([73])

$$
\mathrm{N}_S^P(x) = \mathrm{N}_S^F(x) = \mathrm{N}_S^L(x) = \mathrm{N}_S^C(x),
$$

so that in the sequel we simply write $\mathrm{N}_S(x)$ to refer to any one of theses cones. We shall use the following property of r-uniformly-prox-regular sets.

Lemma 5.25. *Given* $\kappa > 0$ *and a maximal monotone extension* $A_{C,\kappa}$ *of the mapping* $\mathrm{N}_C(\cdot) \cap \mathrm{B}_\kappa + \frac{\kappa}{r}\mathrm{Id}$, *where* Id *is the identity mapping, we have for every*

$x \in C$ and $v \in B_\kappa$

$$N_C(x) \cap B(\theta, \kappa) + \frac{\kappa}{r}x \subset A_{C,\kappa}(x) \subset N_C(x) + \frac{\kappa}{r}x, \qquad (5.73)$$

$$(v - N_C(x))^\circ = (v + \frac{\kappa}{r}x - A_{C,\kappa}(x))^\circ. \qquad (5.74)$$

Proof. We refer [5] for (5.73). To verify (5.74) we observe that (by (5.73))

$$v - N_C(x) \cap B(\theta, \kappa) \subset v + \frac{\kappa}{r}x - A_{C,\kappa}(x) \subset v - N_C(x).$$

Then, since $\|v\| \leq \kappa$ and, so, $(v - N_C(x) \cap B(\theta, \kappa))^\circ = (v - N_C(x))^\circ$, we get

$$(v - N_C(x) \cap B(\theta, \kappa))^\circ = (v + \frac{\kappa}{r}x - A_{C,\kappa}(x))^\circ = (v - N_C(x))^\circ.$$

$\square$

The following theorem reviews the main properties of (5.72) (see [5]).

Theorem 5.26. *Fix $x_0 \in C$, $w \in (F(x_0) - N_C(x_0))^\circ$, $T > 0$ and $m > e^{LT} \|F(x_0)\|$. Then:*

(i) *There exists a solution $x(\cdot; x_0)$ of (5.72) such that $\frac{d^+ x(0)}{dt} = w$, and the function $t \mapsto \frac{d^+ x(t)}{dt}$ is right-continuous.*

(ii) *For every solution $x(\cdot; x_0)$ of (5.72) we have, for ae $t \geq 0$,*

$$\|x(t) - x_0\| \leq \frac{\|F(x_0)\|}{L}(e^{Lt} - 1), \quad \|\dot{x}(t)\| \leq \|F(x_0)\| e^{Lt}. \qquad (5.75)$$

(iii) *Differential inclusion (5.72) has the same solutions set on $[0, T]$ as the differential inclusion*

$$\dot{x}(t) \in (F + \frac{m}{r}\mathrm{Id})(x(t)) - A_{C,m}(x(t)) \ a.e. \ t \in [0, T]; \ x(0) = x_0 \in C, \quad (5.76)$$

where $A_{C,m} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is any maximal monotone extension of $N_C(\cdot) \cap B_m + \frac{m}{r}\mathrm{Id}$. Consequently, every solution $x(\cdot; x_0)$ of (5.72) on $[0, T]$ satisfies

$$\dot{x}(t) \in (F(x(t)) - N_C(x(t)))^\circ \ a.e. \ t \geq 0.$$

Proof. (i) Let $v \in F(x_0)$ be such that $w = v - \Pi_{N_C(x_0)}(v)$, and, according to Lemma 5.4, let f be a Lipschitz selection of F such that $f(x_0) = v$. Then the

following differential inclusion

$$\dot{x}(t) \in f(x(t)) - \mathrm{N}_C(x(t)), \quad x(0) = x_0, \text{ ae } t \geq 0,$$

has a unique solution, which satisfies the conditions of statement (i) (see [5]).

(ii) Let $x(\cdot; x_0)$ be any solution of (5.72) and fix $T_1 > 0$. By Proposition 5.3, we choose a function $g \in L^1([0, T_1]; \mathbb{R}^n)$ such that $g(t) \in F(x(t))$ and (see [5, Lemma 9])

$$\langle g(t) - \dot{x}(t), \dot{x}(t) \rangle = 0 \text{ and } \dot{x}(t) \in g(t) - \mathrm{N}_C(x(t)) \text{ a.e. } t \in [0, T_1];$$

hence,

$$\|\dot{x}(t)\| \leq \|g(t)\| \leq \|F(x(t))\| \leq \|F(x_0)\| + L\|x(t) - x_0\|. \tag{5.77}$$

Now, we introduce the function $\eta(t) := \|x(t) - x_0\|^2$, $t \in [0, T_1]$. Then, for all t in a full measure subset of $[0, T_1]$ the function $x(\cdot; x_0)$ is differentiable at $t \in [0, T]$ and we have, using (5.77),

$$
\begin{aligned}
\dot{\eta}(t) &= 2\langle \dot{x}(t), x(t) - x_0 \rangle \\
&\leq 2\|\dot{x}(t)\| \|x(t) - x_0\| \\
&\leq 2\|F(x_0)\| \|x(t) - x_0\| + 2L\|x(t) - x_0\|^2 \\
&= 2L\eta(t) + 2\|F(x_0)\| \eta^{\frac{1}{2}}(t).
\end{aligned}
$$

So, on the one hand, by Lemma 5.1 we obtain $\eta^{\frac{1}{2}}(t) \leq \frac{\|F(x_0)\|}{L}(e^{Lt} - 1)$ for all $t \in [0, T_1]$, and on the other hand, this last inequality together with (5.77) give us for ae $t \in [0, T_1]$

$$\|\dot{x}(t)\| \leq \|F(x(t))\| \leq \|F(x_0)\| + L\|x(t) - x_0\| \leq \|F(x_0)\| e^{Lt}.$$

This proves (5.75).

(iii) If $x(\cdot) := x(\cdot; x_0)$ is a solution of of (5.72) on $[0, T]$, then by (5.75) we get for ae $t \in [0, T]$

$$\|\dot{x}(t)\| \leq m, \quad \|F(x(t))\| \leq \|F(x_0)\| + L\|x(t) - x_0\| \leq m; \tag{5.78}$$

that is, using Lemma 5.25,

$$\dot{x}(t) \in F(x(t)) - \mathrm{N}_C(x(t)) \cap \mathrm{B}_{2m} \subset F(x(t)) + \frac{2m}{r}x(t) - \hat{A}_{C,2m}(x(t)),$$

Thus, due to Proposition 5.6,

$$\dot{x}(t) \in (F(x(t)) + \frac{2m}{r}x(t) - \hat{A}_{C,2m}(x(t)))^\circ \quad \text{for ae } t \in [0, T], \tag{5.79}$$

and we get, using again Lemma 5.25 and combining with (5.78),

$$
\begin{aligned}
\dot{x}(t) \ & \in \ (F(x(t)) - \mathrm{N}_C(x(t)))^\circ \\
& = \ (F(x(t)) - \mathrm{N}_C(x(t)) \cap \mathrm{B}_m)^\circ \\
& \subset \ F(x(t)) - \mathrm{N}_C(x(t)) \cap \mathrm{B}_m \\
& \subset \ F(x(t)) + \frac{m}{r}x(t) - A_{C,m}(x(t)),
\end{aligned}
$$

and $x(\cdot)$ is a solution of (5.76).

Conversely, let $x(\cdot) := x(\cdot; x_0)$ be a solution of (5.76) on $[0, T]$. So, according to Lemma 5.25, we only need to verify that $x(t) \in C$ for all $t \in [0, T]$. For this aim, we take $\varepsilon > 0$ such that $\|F(x_0)\| e^{LT} + L\varepsilon < m$. Next, given $y \in C \cap \mathrm{B}(x_0, \frac{\|F(x_0)\|(e^{LT}-1))}{L} + \varepsilon)$ and $v \in F(y)$, we have $\|v\| < m$ and, so by Lemma 5.25 it follows that

$$(v + \frac{m}{r}y - A_{C,m}(y))^\circ = (v - \mathrm{N}_C(y))^\circ = v - \Pi_{\mathrm{N}_C(y)}(v) \in \mathrm{T}_C(y) = (\mathrm{N}_C(y))^*; \tag{5.80}$$

that is,

$$\sup_{\xi \in \mathrm{N}_C(y)} \sup_{v \in F(y) + \frac{m}{r}x} \inf_{x^* \in A_{C,m}(y)} \langle \xi, v - x^* \rangle \le 0.$$

Then, according to Theorem 5.11, there exists $\bar{t} \in (0, T]$ such that for every solution $y(\cdot) = y(\cdot; x_0)$ of (5.76) we have that $y(t) \in C$ for all $t \in [0, \bar{t}]$. Hence, by Lemma 5.25, for ae $t \in [0, \bar{t}]$

$$\dot{x}(t) \in (F(x(t)) + \frac{m}{r}x(t) - A_{C,m}(x(t)))^\circ = (F(x(t)) - \mathrm{N}_C(x(t)))^\circ. \tag{5.81}$$

In particular,

$$\|\dot{x}(t)\| \le \|F(x(t))\| \quad \text{for ae } t \in [0, \bar{t}]. \tag{5.82}$$

In order to prove that we can take $\bar{t} = T$ we consider the nonempty set

$$S := \{s \in [0, T] \mid x(t) \in C \text{ for all } t \in [0, s]\}, \tag{5.83}$$

which is obviously closed, due to the continuity of $x(\cdot)$ and the closedness of C.

Let the function η_2 be defined on $[0, \bar{t}]$ as

$$\eta_2(t) := \|x(t) - x_0\|.$$

Then, as we did with function η above, by (5.82) we have for ae $t \in [0, \bar{t}]$,

$$\dot{\eta}_2(t) \leq \|\dot{x}(t)\| \leq \|F(x(t))\| \leq \|F(x_0)\| + L\|x(t) - x_0\| = \|F(x_0)\| + L\eta_2(t),$$

so that (by Gronwall's Lemma),

$$\|x(t) - x_0\| = \eta_2(t) \leq \frac{\|F(x_0)\|\left(e^{Lt} - 1\right)}{L} < \frac{\|F(x_0)\|\left(e^{Lt} - 1\right)}{L} + \varepsilon;$$

that is, in particular, $x(t) \in C \cap \mathrm{B}(x_0, \frac{\|F(x_0)\|(e^{LT}-1))}{L} + \varepsilon)$. So, by arguing as in the paragraph above (to get (5.76)), we find $t_1 > 0$ such that $x(t + \bar{t}) = x(t; x(\bar{t})) \in C$ for all $t \in [0, t_1]$. Hence, the set S is also open and so $S = [0, T]$. Consequently, $x(t) \in C$ for all $t \in [0, T]$, as we wanted to prove. $\qquad\square$

Now, we give the characterizations of a-Lypaunov pairs for (5.72).

Theorem 5.27. *Let $V, W : H \to \mathbb{R} \cup \{+\infty\}$ be two proper lower semi-continuous functions such that $\mathrm{dom}V \subset C$, $W \geq 0$ and let $a \geq 0$, and let ∂V stand for either $\partial_P V$ or $\partial_F V$. Then the following are equivalent:*

(i) (V, W) is a strong a-Lyapunov pair for differential inclusion (5.72).

(ii) For every $x \in \mathrm{dom}V$,

$$\sup_{\xi \in \partial V(x)} \sup_{v \in F(x)} \langle \xi, v - \Pi_{\mathrm{N}_C(x)}(v) \rangle + aV(x) + W(x) \leq 0.$$

(iii) For every $x \in \mathrm{dom}V$,

$$\sup_{\xi \in \partial V(x)} \sup_{v \in F(x)} \inf_{x^* \in \mathrm{N}_C(x) \cap \mathrm{B}_{\|F(x)\|}} \langle \xi, v - x^* \rangle + aV(x) + W(x) \leq 0.$$

(iv) For every $x \in \mathrm{dom}V$,

$$\sup_{v \in F(x)} V'(x; v - \Pi_{\mathrm{N}_C(x)}(v)) + aV(x) + W(x) \leq 0.$$

(v) For every $x \in \mathrm{dom}V$,

$$\sup_{v \in F(x)} \inf_{x^* \in \mathrm{N}_C(x) \cap \mathrm{B}_{\|F(x)\|}} V'(x; v - x^*) + aV(x) + W(x) \leq 0.$$

Proof. The implications $(ii) \Rightarrow (iii)$ and $(iv) \Rightarrow (v)$ follow from the fact that, for all $x \in C$ and $v \in F(x)$,

$$\left\| \Pi_{\mathrm{N}_C(x)}(v) \right\| = \|v\| - \left\| v - \Pi_{\mathrm{N}_C(x)}(v) \right\| \leq \|v\| \leq \|F(x)\|.$$

The implications $(iv) \Rightarrow (ii)$ and $(v) \Rightarrow (iii)$ follow from the fact that

$$\sigma_{\partial_P V(x)}(\cdot) \leq \sigma_{\partial_F V(x)}(\cdot) \leq V'(x; \cdot).$$

$(i) \Rightarrow (iv)$. Let us fix $x_0 \in C$ and $v \in F(x)$. According to Theorem 5.26(i), there exists a solution $x(\cdot; x_0)$ of differential inclusion (5.72) with $x(0) = x_0$ such that $\frac{d^+ x(0)}{dt} = v - \Pi_{\mathrm{N}_C(x)}(v)$, and the proof follows the same way as the one of $(i) \Rightarrow (iv)$ in Corollary 5.19.

$(iii) \Rightarrow (i)$. Let us fix $T > 0$, $x_0 \in C$ and $m > e^{LT} \|F(x_0)\|$, and denote by $A_{C,m}$ an arbitrary maximal monotone extension of the monotone operator $\mathrm{N}_C(\cdot) \cap \mathrm{B}_m + \frac{m}{r}\mathrm{Id}$ (see Lemma 5.25), so that by Theorem 5.72(iii) every solution $x(\cdot; x_0)$ of (5.72) on $[0, T]$ is a solution of (5.76). Due to the Lipschitzianity of F we choose $k \in (0, m)$ and $\rho > 0$ such that $F(x) \in \mathrm{B}_k$ for all $x \in x \in \mathrm{B}(x_0, \rho)$. So, the current hypothesis reads, for all $x \in \mathrm{B}(x_0, \rho) \cap \mathrm{dom}V$

$$\sup_{\xi \in \partial V(x)} \sup_{v \in F(x)} \inf_{x^* \in \mathrm{N}_C(x) \cap \mathrm{B}_k} \langle \xi, v - x^* \rangle + aV(x) + W(x) \leq 0,$$

which can also be written as (see Lemma 5.25)

$$\sup_{\xi \in \partial V(x)} \sup_{v \in F(x) + \frac{m}{r}x} \inf_{x^* \in A_C(x) \cap \mathrm{B}_{k'}} \langle \xi, v - x^* \rangle + aV(x) + W(x) \leq 0,$$

where $k' := k + \sup_{x \in \mathrm{B}(x_0, \rho)} \frac{m}{r} \|x\|$. Consequently, (i) follows from Corollary 5.19. $\quad\square$

Theorem 5.28. *Let $V, W : H \to \mathbb{R} \cup \{+\infty\}$ be two proper lower semi-continuous functions such that $\mathrm{dom}V \subset C$, $W \geq 0$ and let $a \geq 0$. Let ∂V stand for either $\partial_P V$ or $\partial_F V$. Then the following statements are equivalent:*

(i) (V, W) is a weak a-Lyapunov pair for differential inclusion (5.72).

(ii) For every $x \in \mathrm{dom}V$,

$$\sup_{\xi \in \partial V(x)} \inf_{v \in F(x)} \inf_{x^* \in \mathrm{N}_C(x) \cap \mathrm{B}_{\|F(x)\|}}$$

(iii) For every $x \in \mathrm{dom}V$,

$$\inf_{v \in F(x)} \inf_{x^* \in \mathrm{N}_C(x) \cap \mathrm{B}_{\|F(x)\|}} V'(x; v - x^*) + aV(x) + W(x) \leq 0.$$

Proof. $(iii) \Rightarrow (ii)$ follows from inequality (5.7).

$(ii) \Rightarrow (i)$. Let us fix $x_0 \in \mathrm{dom}V \subset C$ and $T > 0$. We choose $m, \rho > 0$ such that $m > e^{LT} \|F(x_0)\| + L\rho$ and a maximal monotone extension $A_{C,m}$ of the monotone operator $\mathrm{N}_C(\cdot) \cap \mathrm{B}_m + \frac{m}{r}\mathrm{Id}$; hence, according to Theorem 5.26, differential inclusion (5.72) is equivalent to differential inclusion (5.76) on $[0, T]$.

Let us first show that for any $y \in \mathrm{B}(x_0, \frac{\|F(x_0)\|}{L}(e^{LT} - 1)) \cap \mathrm{dom}V$, there exists a solution $x(\cdot; y)$ such that

$$e^{at}V(x(t; y)) + \int_0^t W(x(\tau; y))d\tau \leq V(y) \quad \forall t \in [0, \tilde{T}]. \tag{5.84}$$

where $\tilde{T}$ is the positive number that satisfies the following equation in t

$$te^{ct}\left(\sup_{x \in \mathrm{B}(x_0, \frac{\|F(x_0)\|}{L}(e^{LT}-1))} (\|F(x)\| + \frac{m}{r}\|x\|)\right) = \frac{\rho}{6}.$$

Indeed, fix $y \in \mathrm{B}(x_0, \frac{\|F(x_0)\|}{L}(e^{Lt} - 1)) \cap \mathrm{dom}V$. Then for every $z \in \mathrm{B}(y, \rho) \cap \mathrm{dom}V$, we have $z \in \mathrm{B}(x_0, \frac{\|F(x_0)\|}{L}(e^{Lt} - 1) + \rho)$, so that by the current hypothesis (ii)

$$\sup_{\xi \in \partial V(z)} \inf_{v \in F(z) + \frac{m}{r}z} \inf_{z^* \in \mathrm{N}_C(z) \cap \mathrm{B}_{\|F(z)\|} + \frac{m}{r}z} \langle \xi, v - z^* \rangle + aV(z) + W(z) \leq 0.$$

But $\|F(z)\| \leq \|F(x_0)\| + L(\frac{\|F(x_0)\|}{L}(e^{Lt} - 1) + \rho) < m$, and so

$$\sup_{\xi \in \partial V(z)} \inf_{v \in F(z) + \frac{m}{r}z} \inf_{z^* \in A_{C,m}(z) \cap \mathrm{B}_{m' + \|F(z)\| + \frac{m}{r}z}} \langle \xi, v - z^* \rangle + aV(z) + W(z) \leq 0,$$

where

$$m' := \sup_{y \in \mathrm{B}(x_0, \frac{\|F(x_0)\|}{L}(e^{Lt}-1)+\rho)} \left(\|F(y)\| + \frac{m}{r}\|y\|\right).$$

Hence, (5.84) follows from Proposition 5.22 by taking into account that $\tilde{T} \leq T_{c,\rho}(y)$ for all $y \in \mathrm{B}(x_0, \frac{\|F(x_0)\|}{L}(e^{LT} - 1))$.

Now, by Theorem 5.26 we know that every solution $x(\cdot; x_0)$ of (5.76) satisfies

$$x(t; x_0) \in \mathrm{B}(x_0, \frac{\|F(x_0)\|}{L}(e^{LT} - 1)) \ \forall t \in [0, T],$$

and, since $\tilde{T}$ defined above does not depend on the points in $\mathrm{B}(x_0, \frac{\|F(x_0)\|}{L}(e^{LT} - 1)) \cap \mathrm{dom}V$, we prove as before the existence of a solution $x(\cdot; x_0)$ of (5.76) such that

$$e^{at}V(x(t; x_0)) + \int_0^t W(x(\tau; x_0))d\tau \leq V(x_0) \ \forall t \in [0, T].$$

$(i) \Rightarrow (iii)$. Let us fix $x_0 \in \mathrm{dom}V \subset C$ and $T > 0$. By the current hypothesis (i), there exists a solution $x(\cdot; x_0), t \in [0, T]$ of differential inclusion (5.72) such that

$$e^{at}V(x(t; x_0)) + \int_0^t W(x(\tau; x_0))d\tau \leq V(x_0). \tag{5.85}$$

But by Theorem 5.26(ii), $x(\cdot; x_0)$ is also a solution of differential inclusion

$$\dot{x}(t) \in F(x(t)) - \mathrm{N}_C(x(t)) \cap \mathrm{B}_{\|F(x(t))\|} \ \text{a.e.} \ t \geq 0, x(0) = x_0 \in C.$$

Hence, $x(\cdot; x_0)$ is Lipschitz on $[0, T]$, and we may take $v \in \mathrm{limsup}_{t\downarrow 0} \frac{x(t;x_0)-x_0}{t}$. So, by the Lipschitzian property of F, and the upper semicontinuity of the mapping $x \mapsto \mathrm{N}_C(x) \cap \mathrm{B}_{\|F(x)\|}$, we get $v \in F(x_0) - \mathrm{N}_C(x_0) \cap \mathrm{B}_{\|F(x_0)\|}$. Then, if $v = \lim_{t_n\downarrow 0} \frac{x(t_n;x_0)-x_0}{t_n}$, by using (5.85) we get

$$\begin{aligned}
V'(x_0; v) &\leq \liminf_{n\to\infty} \frac{V(x(t_n; x_0)) - V(x_0)}{t_n} \\
&\leq \liminf_{n\to\infty} \left(-\frac{e^{at_n} - 1}{t_n}V(x(t_n; x_0)) - \frac{1}{t_n}\int_0^{t_n} W(x(\tau; x_0)d\tau \right) \\
&\leq -aV(x_0) - W(x_0),
\end{aligned}$$

which shows that (iii) holds. $\qquad\square$

5.8   Examples

In this last section we consider a couple of examples, one on selector-linear differential inclusions, and the other one on the minimal time function.

First, we consider the following differential inclusion

$$
\begin{cases}
\dot{x}(t) \in F(x(t)) - \mathrm{N}_C(x(t)) \ \ \text{a.e.} \ \ t \geq 0, \\
x(0) = x_0 \in C;
\end{cases}
\tag{5.86}
$$

where $A_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, ..., k$ are linear mappings,

$$
F(x) = \mathrm{co}\{A_i x : i = 1, ..., k\} = \{Ax : A \in \mathrm{co}(A_1, ..., A_k)\},
$$

and C is an r-uniformly-prox-regular set of $\mathbb{R}^n$ such that . It is easy to see that F is an L-Lipschitz mapping with $L := \max\{\|A_i\|, i = 1, ..., k\}$.

We apply the results of the previous section to study $\theta \in C$ the stability of differential inclusion (5.86).

Proposition 5.29. *Let $\delta > 0$ and $\beta \in (0, \frac{r\delta}{L})$ be given.*

(i) *If for every $x \in C$, there exists $i \in \{1, ..., k\}$ such that*

$$
\langle x, (A_i + A_i^T)x \rangle \leq -\delta \|x\|^2,
$$

then for every $x_0 \in C \cap \mathrm{int}(\mathrm{B}_\beta)$, there exists a solution $x(\cdot; x_0)$ of (5.86) such that

$$
\|x(t; x_0)\| \leq e^{-\frac{1}{2}(\delta - \frac{\beta L}{r})t} \|x_0\| \ \ \forall t \geq 0.
$$

(ii) *If for every $i \in \{1, ..., k\}$*

$$
A_i + A_i^T \leq -\delta \mathrm{Id},
$$

then for every $x_0 \in C \cap \mathrm{int}(\mathrm{B}_\beta)$ and any solution $x(\cdot; x_0)$ of (5.86), one has

$$
\|x(t; x_0)\| \leq e^{-\frac{1}{2}(\delta - \frac{\beta L}{r})t} \|x_0\| \ \ \forall t \geq 0.
$$

Consequently, for any $\varepsilon > 0$, $x_0 \in \mathrm{int}(\mathrm{B}_\beta) \cap C$, and solution of (5.86), there exists $\bar{t} > 0$ such that

$$
\|x(t; x_0)\| \leq \varepsilon \ \forall t \geq \bar{t}.
$$

Proof. Let us consider the function

$$
V(x) := \frac{1}{2} \|x\|^2 + \mathrm{I}_C(x).
\tag{5.87}
$$

It is easy to see that V is lower semi-continuous and for every $x \in C$, one has

$$\partial_F V(x) = x + \mathrm{N}_C(x). \tag{5.88}$$

Now we chose $m > r\delta e^L$, so that

$$m > r\delta e^L \geq e^L \sup_{x \in \mathrm{B}_\beta} \|F(x)\|.$$

According to Theorem 5.26, for any $y \in \mathrm{int}(\mathrm{B}_\beta) \cap C$ differential inclusion (5.86) is equivalent to the following differential inclusion

$$\begin{cases} \dot{x}(t) \in F(x(t)) - A_{C,m}(x(t)) \text{ a.e } t \in [0,1], \\ x(0) = y \in \mathrm{int}(\mathrm{B}_\beta) \cap C; \end{cases}$$

where $A_{C,m}$ is a maximal monotone extension of the mapping $x \mapsto \mathrm{N}_C(x) \cap \mathrm{B}_m + \frac{m}{r}x$ (see Lemma 5.25).

(i) We take $x \in C$ and let $i \in \{1, \cdots, k\}$ be such that $\langle x, (A_i + A_i^T)x \rangle \leq -\delta \|x\|^2$. Fix $\xi \in \partial_F V(x)$ $(= x + \mathrm{N}_C(x))$. Since $A_i x - \Pi_{\mathrm{N}_C(x)}(A_i x) \in \mathrm{T}_C^B(x)$, from the definition of r-uniformly-prox-regularity, and the fact that $\theta \in C$, one gets

$$\langle -x, \Pi_{\mathrm{N}_C(x)}(A_i x) \rangle \leq \frac{\left\| \Pi_{\mathrm{N}_C(x)}(A_i x) \right\|}{2r} \|x\|^2 \leq \frac{\|A_i x\|}{2r} \|x\|^2, \tag{5.89}$$

and so, using (5.88),

$$\begin{aligned} \langle \xi, A_i x - \Pi_{\mathrm{N}_C(x)}(A_i x) \rangle &\leq \langle x, A_i x - \Pi_{\mathrm{N}_C(x)}(A_i x) \rangle \\ &\leq \frac{1}{2}\langle x, (A_i + A_i^T)x \rangle + \langle -x, \Pi_{\mathrm{N}_C(x)}(A_i x) \rangle \\ &\leq -\frac{\delta}{2}\|x\|^2 + \frac{\|A_i x\|}{2r}\|x\|^2 \\ &\leq \frac{1}{2}\left(\frac{L\|x\|}{r} - \delta \right) \|x\|^2; \end{aligned}$$

that is, for every $x \in \mathrm{int}(\mathrm{B}_\beta) \cap \mathrm{dom}V$ and $\xi \in \partial_P V(x)$

$$\langle \xi, A_i x - \Pi_{\mathrm{N}_C(x)}(A_i x) \rangle + (\delta - \frac{L\beta}{r})V(x) \leq 0; \tag{5.90}$$

moreover, since $\|A_i x\| \leq L\beta < m$, and so (by Lemma 5.25)

$$A_i x - \Pi_{\mathrm{N}_C(x)}(A_i x) = (A_i x + \frac{m}{r}x - A_{C,m}(x))^\circ \in A_i x + \frac{m}{r}x - A_{C,m}(x) \cap \mathrm{B}_k,$$

with $k := 2L\beta + \frac{m}{r}\beta$, we also have that

$$\inf_{v \in F(x) + \frac{m}{r}x} \inf_{x^* \in A_{C,m}(x) \cap B_k} \langle \xi, v - x^* \rangle + (\delta - \frac{L\beta}{r})V(x) \leq 0. \qquad (5.91)$$

Now, we choose $\overline{T} > 0$ such that

$$3(\|F(x_0)\| + \|A_C(x_0)\|)\overline{T}e^{c\overline{T}} < 3\beta(L + \frac{m}{r})\overline{T}e^{c\overline{T}} < \beta - \|x_0\|.$$

So, according to Proposition 5.22, by (5.91) there exists a Lipschitz solution $x(\cdot; x_0)$ of (5.76) on $[0, 1]$ such that

$$e^{\frac{1}{2}(\delta - \frac{L\beta}{r})t}\|x(t; x_0)\| \leq \|x_0\| \ \ \forall t \in [0, \overline{T}].$$

Also, since $x(\overline{T}; x_0) \in B_{\|x_0\|} \cap C \subset \text{int}(B_\beta) \cap C$, we can find (by extending the current solution) a solution $x(\cdot; x_0)$ of (5.76) such that

$$e^{\frac{1}{2}(\delta - \frac{L\beta}{r})t}\|x(t; x_0)\| \leq \|x_0\| \ \ \forall t \geq 0.$$

(ii). We take $x \in C$, $v \in F(x)$, and $\xi \in \partial_P V(x)$, where V is defined in (5.87); hence, $v = \sum_{i=1}^{k} \alpha_i A_i x$ for some α_i such that $\sum_{i=1}^{k} \alpha_i = 1$, and $\|v\| \leq L\|x\|$. As in the proof of statement (i) above we get

$$\langle \xi, v - \Pi_{N_C(x)}(v) \rangle \leq \langle x, v - \Pi_{N_C(x)}(v) \rangle$$

$$\leq \sum_{i=1}^{k} \frac{1}{2}\alpha_i \langle x, (A_i + A_i^T)x \rangle + \langle -x, \Pi_{N_C(x)}(v) \rangle$$

$$\leq -\frac{\delta}{2}\|x\|^2 + \frac{\|\Pi_{N_C(x)}(v)\|}{2r}\|x\|^2$$

$$\leq \frac{1}{2}\left(\frac{\|\Pi_{N_C(x)}(v)\|}{r} - \delta\right)\|x\|^2$$

$$\leq \frac{1}{2}\left(\frac{\|v\|}{r} - \delta\right)\|x\|^2 \leq \frac{1}{2}(\frac{L\|x\|}{r} - \delta)\|x\|^2,$$

and we conclude as above. $\qquad \square$

We recall differential inclusion (5.72),

$$\begin{cases} \dot{x}(t) \in F(x(t)) - N_C(x(t)) \ \text{ a.e. } \ t \geq 0, \\ x(0) = x_0 \in C. \end{cases}$$

where as before $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an L-Lipschitz Cusco mapping and C is an r-uniformly-prox-regular set of $\mathbb{R}^n$. Given a closed set $S \subset C$, we consider the minimum time function $T : C \to \mathbb{R} \cup \{+\infty\}$ defined as

$$T(z) := \inf\{t \geq 0 : \exists \text{ solution } x(\cdot) \text{ of } (5.72) \text{ st. } x(0) = z, \; x(t) \in S\}. \qquad (5.92)$$

The following lemma gathers some easy facts on the function T, which we call a minimal time function.

Lemma 5.30. (i) $T(z) = 0$ if $z \in S$, and $T(z) > 0$ if $z \in C \setminus S$.

(ii) If $T(z)$, $z \in C$, is finite, then there exists a solution $x(\cdot; z)$ of (5.72) such that $x(T(\alpha); x_0) \in S$.

(iii) For every solution $x(\cdot; x_0)$ of (5.72) we have

$$T(x(s; x_0)) + s \leq T(x(t; x_0)) + t, \quad \text{for all } 0 \leq s \leq t,$$

and the equality holds for optimal trajectories.

(iv) If $t > 0$ is such that $t < T(z) < +\infty$, $z \in C$, then there exists $\varepsilon > 0$ such that

$$[S + \varepsilon \mathbb{B}] \cap \{x(s; x_0), s \in [0, t], x(\cdot; x_0) \text{ solution of } (5.72)\} = \emptyset.$$

Proof. Statements (i) and (iii) are clear and follow easily from the definition of the function T.

To prove (ii) we assume that $T(z) < +\infty$, and let $x_n(\cdot; z)$ be a sequence of solutions of (5.72) such that $x_n(t_n; z) \in S$ for some sequence $t_n \downarrow T(z)$. Then, as in the proof of Lemma 5.15(ii), we may suppose that $x_n(\cdot; z)$ uniformly converges to a solution $x(\cdot; z)$ on $[0, T(z) + 1]$. Hence, from the closedness of the set S we obtain that $x(T(z); z) \in S$.

To prove (iv) we proceed by contradiction and assume that there are sequence of solutions $(x_n(\cdot; z))_n$ of (5.72) and sequence $(t_n)_n$ such that $t_n \leq t$ and $d(x_n(t_n; z), S) \to 0$. Without lost of generally, we can suppose that $(x_n(\cdot; z))_n$ uniformly converges to a solution $x(\cdot; z)$ on $[0, t]$ (see the proof of Lemma 5.15 (ii)). It follows that $x(s; z) \in S$ for some $s \in [0, t]$, which contradicts the fact that $t < T(x_0)$. $\qquad \square$

We now consider the differential inclusion

$$\begin{cases} (\dot{x}(t), \dot{y}(t)) \in (F(x(t)) - \mathrm{N}_C(x(t)), 1) \text{ a.e. } t \geq 0; \\ (x(0), y(0)) = (x_0, \alpha) \in C \times \mathbb{R}, \end{cases} \qquad (5.93)$$

so that from Lemma 5.30 it follows that the function $(x, y) \mapsto T(x) + y$ is a weak Lyapunov function for (5.93), while the function $(x, y) \mapsto -\widehat{T}(x) - y$, where

$$\widehat{T}(x) := \begin{cases} -T(x) & \text{if } x \in C \\ +\infty & \text{if } x \notin C, \end{cases} \tag{5.94}$$

is a strong Lyapunov function for differential inclusion (5.93).

We get the following result (also see [33]).

Proposition 5.31. *Suppose that the minimum time function T as defined in* (5.92) *is continuous on C. Then T is the unique continuous function such that*

$$\begin{aligned} T(x) &= 0 \ \ \textit{for all } x \in S, \\ T(x) &> 0 \ \ \textit{for all } x \in C \setminus S, \end{aligned} \tag{5.95}$$

and, for any $x \in C \setminus S$,

$$\sup_{\xi \in \partial_P \widehat{T}(x)} \sup_{v \in F(x)} \inf_{x^* \in \mathrm{N}_C(x) \cap \mathrm{B}_{\|F(x)\|}} \langle \xi, v - x^* \rangle - 1 \leq 0 \tag{5.96}$$

$$\sup_{\xi \in \partial_P T(x)} \inf_{v \in F(x)} \inf_{x^* \in \mathrm{N}_C(x) \cap \mathrm{B}_{\|F(x)\|}} \langle \xi, v - x^* \rangle + 1 \leq 0. \tag{5.97}$$

Proof. By Lemma 5.30 and the paragraph before the current proposition, the minimum time function T as defined in (5.92) satisfies $T(x) = 0$ for $x \in S$ and $T(x) > 0$ for $x \in C \setminus S$, and the functions $(x, y) \mapsto T(x) + y$ and $(x, y) \mapsto -\widehat{T}(x) - y$ (see (5.94)) are respectively weak and strong Lyapunov functions for (5.93). Then, $\widehat{T}$ and T also satisfy (5.96) and (5.97), thanks to Theorems 5.27 and 5.28.

Now, let V a continuous that satisfies (5.95), (5.96) and (5.97). We proceed by steps:

Step (1). We prove in this step that $V(x) \geq T(x)$ for all $x \in C \setminus S$. We fix $x_0 \in C \setminus S$ and denote

$$\bar{t} := \sup\{t \geq 0 : \exists x(\cdot; x_0) \text{ solution of } (5.72) \text{ st. } V(x(s; x_0)) + s \leq V(x_0) \ \forall s \in [0, t]\}.$$

Then, due to (5.97), by Theorem 5.28 there exist $t > 0$ and solution $x(\cdot; x_0)$ of (5.72) such that

$$V(x(s; x_0)) + s \leq V(z) \ \forall s \in [0, t],$$

so that $\bar{t} > 0$. Moreover, if sequences $t_n \nearrow \bar{t}$ and $(x_n(\cdot; x_0))_n$ are such $x_n(\cdot; x_0)$ is a

solution (5.72) and

$$V(x_n(s; x_0)) + s \leq V(x_0) \ \forall s \in [0, t_n],$$

then we may assume, without lost of generally, that $(x_n(\cdot; x_0))_n$ uniformly converges to a solution $\bar{x}(\cdot; x_0)$ on $[0, \bar{t}]$, so that, using the continuity of V,

$$V(\bar{x}(s; x_0)) + s \leq V(x_0) \ \forall s \in [0, \bar{t}]. \tag{5.98}$$

If $\bar{t} < T(x_0)$, then $\bar{x}(\bar{t}; x_0) \in C \setminus S$ (as $\bar{t} < T(x_0)$), and so, by applying again Theorem 5.28, there exist number $\delta > 0$ and solution $\widetilde{x}(\cdot; \bar{x}(\bar{t}; x_0))$ of (5.72) such that

$$V(\widetilde{x}(s; \bar{x}(\bar{t}; x_0))) + s \leq V(\bar{x}(\bar{t}; x_0)) \ \forall s \in [0, \delta].$$

It follows that the following solution of (5.72)

$$\hat{x}(t; x_0) := \begin{cases} \bar{x}(t; x_0) & \text{if } t \in [0, \bar{t}] \\ \widetilde{x}(t - \bar{t}; \bar{x}(\bar{t}; x_0)) & \text{if } t \in [\bar{t}, \infty), \end{cases}$$

satisfies $V(\hat{x}(s; x_0)) + s \leq V(x_0)$ for all $s \in [0, \bar{t} + \delta]$, which is a contradiction with the definition of $\bar{t}$. Hence, we have $\bar{t} = T(x_0)$ so that, by (5.98) and the fact that V equals 0 on S,

$$T(x_0) = V(\bar{x}(T(x_0); x_0)) + T(x_0) \leq V(x_0).$$

Step (2). We prove that $V(x) \leq T(x)$, for all $x \in C \setminus S$. We fix $x_0 \in C \setminus S$ and let $x(\cdot; x_0)$ be any solution of (5.72). Since $x(t; x_0) \in C \setminus S$ for any $t < T(x_0)$, by (5.96) Theorem 5.27 gives us

$$V(x(t; x_0)) + t \geq V(x_0) \ \forall t \in [0, T(x_0)].$$

In particular, if $x(\cdot; x_0)$ is an optimal trajectory we get $T(x_0) = V(\bar{x}(T(x_0); x_0)) + T(x_0) \geq V(x_0)$, as we wanted to prove. $\qquad \square$

Chapter 6

Boundary of maximal monotone operators values

We characterize the boundary of the values of maximal monotone operators defined in Hilbert spaces, by means only of the values at nearby points, which are closed enough to the reference point but distinct of it. This allows to write the values of such operators using finite convex (2-)combinations of the values at such nearby points. We also provide similar characterizations for the normal cone to prox-regular sets.

6.1 Introduction

Given a continuous convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$, according to [24, Theorem 3.1] the topological boundary of the Fenchel subdifferential of φ is completely characterized by means of the values of such subdifferential mapping at points, which are closed enough to the reference point but distinct of it. More specifically, for every $x \in \mathbb{R}^n$ we have that

$$\mathrm{bd}(\partial\varphi(x)) = \underset{y \xrightarrow{\;\;}_{\neq} x}{\mathrm{Limsup}} \; \partial\varphi(y). \tag{6.1}$$

This characterization has been shown useful for many stability purposes of parametrized semi-infinite linear programming problems, given in $\mathbb{R}^n$ as ([44])

$$P(c, a, b) : \quad \begin{aligned} &\text{minimize} && c'x \\ &\text{subject to} && a_t'x \leq b_t, \; t \in T, \end{aligned}$$

for compact space T and continuous functions a and b on T. The characterization above was the main ingredient in [23–25] to derive point-based explicit expressions

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for the so-called *calmness moduli* of the associated feasible and optimal solutions set-valued mappings; we refer to [50, 51, 53] for more details on this calmness property. For instance, if $\mathcal{F}_a : C(T, \mathbb{R}) \to \mathbb{R}^n$ denotes the feasible set-valued mapping,

$$\mathcal{F}_a(b) := \{x \in \mathbb{R}^n : a_t' x \leq b_t \; \forall t \in T\},$$

then the calmness modulus of $\mathcal{F}_a$ at a point $(\bar{b}, \bar{x})$ in its graph, given implicitly as

$$\mathrm{clm}\mathcal{F}_a(\bar{b}, \bar{x}) := \limsup_{\substack{x \to \bar{x}, \, b \to \bar{b} \\ x \in \mathcal{F}_a(b)}} \frac{d(x, \mathcal{F}_a(\bar{b}))}{d(b, \bar{b})},$$

is rewritten in the more explicit form (using the convention $\frac{1}{0} = +\infty$)

$$\mathrm{clm}\mathcal{F}_a(\bar{b}, \bar{x}) = \left(\liminf_{x \to \bar{x}, \, s(x) > 0} d_*(0, \partial s(x)) \right)^{-1},$$

where $s : \mathbb{R}^n \to \mathbb{R}$ is the convex continuous function given by

$$s(x) := \max_{t \in T} \{a_t' x - b_t\},$$

and whose subdifferential mapping can be easily estimated by means only of the data vectors a and b. From a qualitative point of view, the calmness of the mapping $\mathcal{F}_a$, say $\mathrm{clm}\mathcal{F}_a(\bar{b}, \bar{x}) > 0$, is equivalent to the fact that the function s has an (global) error bound at $\bar{x}$ (see [66, 67]).

At this stage, if, in addition, the set $\mathcal{F}_a(\bar{b})$ turns to be the singleton $\{\bar{x}\}$, in which case $s(x) > 0$ iff $x \neq \bar{x}$, then formula (6.1) goes into the play and entails a point-based expression of the calmness modulus of the mapping $\mathcal{F}_a$, that is given by

$$\mathrm{clm}\mathcal{F}_a(\bar{b}, \bar{x}) = (d_*(0, \mathrm{bd}(\partial s(\bar{x}))))^{-1}.$$

It is worth observing that in the framework of semi-infinite linear programming problems, this singleton's assumption is required for the solutions set-valued mapping and not for the feasible set-valued mapping (see [23–25] for more details).

For the aim of adapting this kind of analysis in a further research to more general semi-infinite linear programming problems with a non-necessarily compact index set T, so that the function s above lacks to be continuous, we extend in this paper formula (6.1) to the class of proper and lower semi-continuous convex functions. More generally, we establish similar characterizations for maximal monotone operators in the setting of Hilbert spaces. The first result given in

Theorem 6.5 asserts that given a maximal monotone operator $A : H \rightrightarrows H$, for all $x \in H$ we have that

$$\operatorname{bd}(A(x)) = \operatorname*{Limsup}_{y \to_{\neq} x} \operatorname{bd}(A(y)) = \operatorname*{Limsup}_{y \to_{\neq} x} A(y),$$

where the Limsup is taken with respect to the norm. As a consequence, we prove that the value of A at x can be expressed using only different nearby points in the sense that for every $x \in H$ such that $\operatorname{bd}(Ax) \neq \emptyset$ it holds (Theorem 6.12)

$$A(x) = \operatorname{N}_{\operatorname{cl}(\operatorname{dom}A)}(x) + \operatorname{co}_2 \left\{ \operatorname*{Limsup}_{y \to_{\neq} x} A(y) \right\},$$

where co_2 is the set of all the segments generated by the elements of the underlying set, and $\operatorname{N}_{\operatorname{cl}(\operatorname{dom}A)}(x)$ is the normal cone in the sense of convex analysis to the closure of the domain of the operator A. Characterizations of similar type are given for the faces of the values of A, see Theorem 6.9. Extensions to nonconvex objects, as prox-regular sets and functions, is also considered in Theorems 6.15 and 6.18.

This paper is organized as follows: After Section 6.2, dedicated to present the necessary notations and the preliminary tools, we give the main result in Section 6.3: Theorem 6.5 characterizes the boundary of the values of maximal monotone operators, while Theorem 6.12 recovers the values of such operators using these boundary points. Theorem 6.9 specifies such characterizations to the faces of the values of maximal monotone operators. In Section 6.4 we extend this analysis to non-convex objects, which are the normal cone to prox-regular sets (Theorem 6.15) and the subdifferential of prox-regular functions with uniform parameters (Theorem 6.18).

6.2   Notations and preliminary results

In this paper, H is a Hilbert space endowed with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|$. The weak topology on H is denoted by ω, while the strong and weak convergences in H are denoted by $\to$ and $\rightharpoonup$, resp. We denote by $\operatorname{B}(x, \rho)$ the closed ball with center $x \in H$ and radius $\rho > 0$; in particular, we write $\operatorname{B}_\rho := \operatorname{B}(\theta, \rho)$. The null vector in H is denoted θ. Given a set $S \subset H$, $\operatorname{co}(S)$ and $\operatorname{co}_2(S)$ are respectively the *convex hull* of S and the set

$$\operatorname{co}_2(S) := \{\alpha s_1 + (1 - \alpha)s_2 : \alpha \in [0, 1], s_1, s_2 \in S\}.$$

Observe that $co_2(S)$ coincides with $co(S)$ when $H = \mathbb{R}$, but the two sets may be different in general. By $int(S)$, $bd(S)$ and $cl(S)$ (or, indistinctly, $\overline{S}$), we denote the *interior,* the *boundary* and the *closure* of S, respectively. The *indicator, the support* and the *distance functions* to the set S are respectively given by

$$I_S(x) := 0 \text{ if } x \in S; +\infty \text{ otherwise, } \sigma_S(x) := \sup\{\langle x, s \rangle : s \in S\},$$
$$d_S(x) := \inf\{\|x - y\|: y \in S\}$$

(in the sequel we shall adopt the convention $\inf_\emptyset = +\infty$). We shall write $\overset{S}{\to}$ for the convergence when restricted to the set S, and $y \longrightarrow_{\neq} x$ when $y \to x$ with $y \neq x$. We denote Π_S the *(orthogonal) projection mapping* onto S defined as

$$\Pi_S(x) := \{y \in S : \|x - y\| = d_S(x)\}.$$

Next, we review some classical facts about convex functions and monotone operators; we refer to [21, 86] for more details. Given a function $\varphi : H \to \mathbb{R} \cup \{+\infty\}$, we say that φ is proper if its domain $dom\varphi := \{x \in H : \varphi(x) < +\infty\}$ is nonempty, lower semi-continuous if its epigraph $epi\varphi := \{(x, \lambda) \in H \times \mathbb{R} : \varphi(x) \leq \lambda\}$ is closed, and convex if its epigraph is convex. If φ is convex, the Fenchel subdifferential mapping of φ as $x \in dom\varphi$ is defined as

$$\partial\varphi(x) := \{x^* \in H : \langle x^*, y - x \rangle \leq \varphi(y) - \varphi(x) \; \forall y \in H\},$$

and $\partial\varphi(x) := \emptyset$ when $x \notin dom\varphi$. The normal cone to a closed convex set $S \subset H$ is $N_S(x) := \partial I_S(x)$ for $x \in H$.

Given a set-valued operator $A : H \rightrightarrows H$, the *domain* and the *graph* of A are given by

$$dom A := \{x \in H : A(x) \neq \emptyset\}, \; Gr(A) := \{(x, x^*) : x^* \in A(x)\}.$$

The operator A is said to be *monotone* if

$$\langle x_1 - x_2, x_1^* - x_2^* \rangle \geq 0 \quad \text{for all } (x_1, x_1^*), (x_2, x_2^*) \in Gr(A),$$

and *maximal monotone* if, in addition, A coincides with every monotone operator containing its graph. In such a case, it is known that $cl(dom A)$ is convex, and that Ax is convex and closed for every $x \in H$. Hence, the *minimal norm* element

of Ax; that is,

$$A^\circ(x) := \{x^* \in A(x) : \|x^*\| = \min_{z^* \in A(x)} \|z^*\|\},$$

is well-defined and unique.

Finally, given multifunction $F : H \rightrightarrows H$ we denote

$$\operatorname*{Limsup}_{y \to x} F(y) := \{x^* \in H : \exists\, y_n \to x,\ y_n^* \to x^*,\ \text{s.t. } y_n^* \in F(y_n)\ \forall n \geq 1\},$$

$$\operatorname*{Limsup}_{y \rightharpoonup x} F(y) := \{x^* \in H : \exists\, y_n \rightharpoonup x,\ y_n^* \to x^*,\ \text{s.t. } y_n^* \in F(y_n)\ \forall n \geq 1\},$$

$$\omega - \operatorname*{Limsup}_{y \to x} F(y) := \{x^* \in H : \exists\, y_n \to x,\ y_n^* \rightharpoonup x^*,\ \text{s.t. } y_n^* \in F(y_n)\ \forall n \geq 1\}.$$

6.3   Boundary of maximal monotone operators

In this section, we give the desired property which expresses a given maximal monotone operator $A : H \rightrightarrows H$, defined on a Hilbert space H, by means of its values at nearby points.

Definition 6.1. Given $x \in \mathrm{dom}A$ and $v \in H$, we define the set $A(x; v) \subset H$ as

$$A(x; v) := \left\{x^* \in A(x) : \langle x^*, v \rangle = \sigma_{A(x)}(v)\right\},$$

with the convention that $A(x, v) = \emptyset$ when $\sigma_{A(x)}(v) = +\infty$.

Since $A(x)$, $x \in \mathrm{dom}A$, is convex and closed, $A(x; \cdot)$ coincides with the subdifferential mapping of the proper, convex and lsc support function $\infty_{A(x)}$. As a consequence, the following remark resumes some easy properties of the set $A(x; v)$.

Remark 6.2. Given $x \in \mathrm{dom}A$ and $v \in H$, we have:

(i) $A(x; v)$ is convex and closed (possibly empty), and nonempty whenever the set $A(x)$ is bounded.

(ii) $A(x; \theta) = A(x)$, and if $v \neq \theta$ then $A(x; v)$ is a subset of $\mathrm{bd}(A(x))$. In the last case, we refer to $A(x; v)$ as the face of $A(x)$ with respect to the direction v.

(iii) $A(x; \alpha v) = A(x; v)$ for any $v \neq \theta$ and $\alpha > 0$; thus, the face $A(x; v)$ depends only on the direction v.

We shall need the following lemma.

Lemma 6.3. (see, e.g., [30]) *For any nonempty closed convex set $S \subset H$, the set of points $s \in \mathrm{bd}(S)$ such that $\mathrm{N}_S(s) \neq \{\theta\}$ is dense in $\mathrm{bd}(S)$.*

Proposition 6.4. *Let $x \in \mathrm{dom}A$ and $v \neq \theta$ be given. Then we have that*

$$\mathrm{bd}(A(x)) = \mathrm{cl}\left(\bigcup_{v \neq \theta} A(x; v)\right).$$

Proof. The inclusion "$\supset$" being obvious, due to the definition of the set $A(x; v)$, we only need to prove the inclusion "$\subset$". Take an arbitrary vector $\xi \in \mathrm{bd}(A(x))$. According to Lemma 6.3, there exists a sequence $(\xi_n)_n \subset \mathrm{bd}(A(x))$ such that $\xi_n \to \xi$ and $\mathrm{N}_{A(x)}(\xi_n) \neq \{\theta\}$. Hence, for each n there exists $v_n \neq \theta$ such that $v_n \in \mathrm{N}_{A(x)}(\xi_n) = \partial \mathrm{I}_{A(x)}(\xi_n)$, or, equivalently, $\xi_n \in \partial \mathrm{œ}_{A(x)}(v_n) = A(x; v_n)$; that is,

$$\xi \in \mathrm{cl}\left(\bigcup_{v \neq \theta} A(x; v)\right). \qquad \square$$

Theorem 6.5. *For every $x \in H$ we have*

$$\mathrm{bd}(A(x)) = \underset{y \to_{\neq} x}{\mathrm{Limsup}}\ \mathrm{bd}(A(y)) = \underset{y \to_{\neq} x}{\mathrm{Limsup}}\ A(y).$$

Proof. To prove the first statement of the theorem we proceed by verifying the following inclusions, for every fixed $x \in H$,

$$\mathrm{bd}(A(x)) \subset \underset{y \to_{\neq} x}{\mathrm{Limsup}}\ \mathrm{bd}(A(y)) \subset \underset{y \to_{\neq} x}{\mathrm{Limsup}}\ A(y) \subset \mathrm{bd}(A(x)). \qquad (6.2)$$

First, we observe that when $x \notin \mathrm{dom}A$, these inclusions follows since that, using the norm-weak upper semicontinuity of the (maximal monotone) operator A,

$$\emptyset = \mathrm{bd}(A(x)) \subset \underset{y \to_{\neq} x}{\mathrm{Limsup}}\ \mathrm{bd}(A(y)) \subset \underset{y \to_{\neq} x}{\mathrm{Limsup}}\ A(y) \subset A(x) = \emptyset.$$

So, we may assume that $x \in \mathrm{dom}A$. Also, if $\mathrm{bd}(A(x)) = \emptyset$, then we would have that $A(x) = H$, so that $\mathrm{dom}A = \{x\}$ and this leads to

$$\underset{y \to_{\neq} x}{\mathrm{Limsup}}\ \mathrm{bd}(A(y)) = \underset{y \to_{\neq} x}{\mathrm{Limsup}}\ A(y) = \emptyset;$$

that is, the conclusion of the first statement is also true in this case.

From the observation above we assume now that $\mathrm{bd}(A(x)) \neq \emptyset$. Take $x^* \in \mathrm{bd}(A(x))$ ($\subset A(x)$). According to Lemma 6.3, for each $n \geq 1$ there exists $x_n^* \in \mathrm{bd}(A(x))$ such that $\|x_n^* - x^*\| \leq \frac{1}{n}$ and $\mathrm{N}_{A(x)}(x_n^*) \neq \theta$; hence, $x_n^* = \Pi_{A(x)}(v_n)$ for

some $v_n \in H \setminus A(x)$. We fix $n \geq 1$ and consider the following differential inclusion

$$\dot{z}(t) \in v_n - A(z(t)) \ t \in [0, 1], \ z(0) = x,$$

which (see, e.g., [21]) possesses a unique solution $z_n(\cdot)$ that satisfies $z_n(t) \in \text{dom} A$ for all $t \in [0, 1]$, and such that the function

$$t \mapsto \frac{d^+ z_n(t)}{dt} = (v_n - A(z_n(t)))^\circ = v_n - \Pi_{A(z_n(t))}(v_n) \tag{6.3}$$

is right-continuous on $[0, 1)$. In particular, one has

$$\frac{d^+ z_n(0)}{dt} = (v_n - A(z_n(0)))^\circ = (v_n - A(x))^\circ = v_n - \Pi_{A(x)}(v_n) = v_n - x_n^*;$$

hence, since $v_n - x_n^* \neq \theta$, we get $z_n(t) \neq x$ for all small $t \in [0, 1)$. Then, from the right-continuity of $\frac{d^+ z_n(\cdot)}{dt}$ and the expressions in (6.3), there exists a sequence $t_k \downarrow 0$ such that $z_{n,k}^* := \Pi_{A(z_n(t_k))}(v_n) \to x_n^*$ as k goes to $+\infty$, and $z_n(t_k) \neq x$ for all $k \geq 1$. We observe that $z_{n,k}^* \in \text{bd}(A(z_n(t_k)))$ for all $k \geq 1$ is a cofinite set, because for otherwise, since $z_{n,k}^* \in A(z_n(t_k))$ we would have $z_{n,k}^* \in \text{int}(A(z_n(t_k)))$ for all k in a cofinite set K, and this would lead to $v_n \in A(z_n(t_k))$ for all $k \in K$. Consequently, as $z_n(t_k) \to x$ when k goes to $+\infty$, the maximal monotonicity of A would give us $v_n \in A(x)$, which is a contradiction. Now, we may choose a diagonal sequence $(z_{n,k_n}^*)_n$ such that $z_{n,k_n}^* \to x^*$ as $n \to +\infty$, and this shows that $x^* \in \underset{y \to_{\neq} x}{\text{Limsup}} \ \text{bd}(A(y))$, which yields the first inclusion in (6.2).

We take now $x^* \in \underset{y \to_{\neq} x}{\text{Limsup}} \ A(y)$, so that $x^* = \underset{n \to \infty}{\lim} x_n^*$ for some $x_n^* \in A(x_n)$ with $x_n \to x$ and $x_n \neq x$. Then by the norm-weak upper semi-continuity of the operator A, we deduce that $x^* \in A(x)$. Thus, it suffices to prove that $x^* \in H \setminus \text{int}(A(x))$. Proceeding by contradiction, we assume that $x^* + r\mathbb{B} \subset A(x)$ for some $r > 0$. Then, using the monotonicity of A, for every $n \geq 1$ one has that

$$\left\langle x_n^* - \left(x^* + r \frac{x_n - x}{\|x_n - x\|} \right), x_n - x \right\rangle \geq 0,$$

which gives

$$\|x_n^* - x^*\| \|x_n - x\| \geq \langle x_n^* - x^*, x_n - x \rangle \geq \langle r \frac{x_n - x}{\|x_n - x\|}, x_n - x \rangle = r\|x_n - x\|;$$

that is, $\|x_n^* - x^*\| \geq r$ for every $n \geq 1$, and this contradicts the convergence of (x_n^*) to x^*. Hence, $x^* \in \text{bd}(A(x))$ and we conclude the proof of (6.2). $\square$

It easily follows from Theorem 6.5 that

$$\text{bd}(A(x)) \subset \underset{y \to_{\neq} x}{\text{Limsup}} A(y) \subset \omega - \underset{y \to_{\neq} x}{\text{Limsup}} A(y),$$

but the last inclusion may be strict, as the following example shows.

Example 6.6. Assume that $(e_n)_{n \in \mathbb{N}}$ is an orthonormal basis for H, and consider the maximal monotone operator $A := \partial \|\cdot\|$. So,

$$A(\theta) = \text{B}(\theta, 1) \ \text{ and } \ A(x) = \frac{x}{\|x\|} \ \text{ for all } x \neq \theta.$$

We observe that the sequence $(\frac{e_n}{n})_{n \in \mathbb{N}}$ strongly converges to θ, and

$$A(\frac{e_n}{n}) = e_n \rightharpoonup \theta \in \text{int}(\text{B}(\theta, 1)) = \text{int}(A(\theta)).$$

We give an interesting corollary of Theorem 6.5.

Corollary 6.7. *For every $x \in H$ we have*

$$d(\theta, \text{bd}(A(x))) = \liminf_{y \to_{\neq} x} d(\theta, A(y)).$$

Consequently, if x is such that $\theta \notin \text{int}(A(x))$, then

$$\|A^\circ(x)\| = \liminf_{y \to_{\neq} x} \|A^\circ(y)\|.$$

Proof. It suffices to consider the case when $x \in \text{dom} A$, because otherwise both sides of the equality are equal to $+\infty$.

We may distinguish two cases: If $\theta \notin A(x)$, then $d(\theta, \text{bd}(A(x))) = d(\theta, A(x)) = \|A^\circ(x)\|$. Thus, according to Theorem 6.5 there are sequences $(y_n), (y_n^*) \subset H$ such that

$$y_n \to_{\neq} x, \ y_n^* \in A(y_n), \ \text{ and } y_n^* \to A^\circ(x) \text{ as } n \to +\infty.$$

Hence,

$$\|A^\circ(x)\| = \lim_{n \to \infty} \|y_n^*\| \geq \liminf n \to \infty d(\theta, A(y_n)) \geq \liminf_{y \to_{\neq} x} d(\theta, A(y)),$$

and so $d(\theta, \text{bd}(A(x))) = \|A^\circ(x)\| \geq \liminf_{y \to_{\neq} x} d(\theta, A(y))$. Hence, if $\liminf_{y \to_{\neq} x} d(\theta, A(y)) = +\infty$, then the first equality of the corollary obviously. Otherwise, we suppose that $\liminf_{y \to_{\neq} x} d(\theta, A(y)) < \alpha$ for some $\alpha \in \mathbb{R}$, and let sequences $(y_n), (y_n^*) \subset H$ be such

that

$$y_n \to_{\neq} x, \ y_n^* \in A(y_n), \ \text{and} \ \lim_{n \to \infty} \|y_n^*\| < \alpha.$$

Thus, taking into account Theorem 6.5, we may suppose that $y_n^* \to x^* \in \mathrm{bd}(A(x))$; that is,

$$d(\theta, \mathrm{bd}(A(x))) \leq \|x^*\| \leq \alpha.$$

We get the desired inequality " $\leq$ " when α goes to $\liminf_{y \to_{\neq} x} d(\theta, A(y))$, and this completes the proof of the first statement.

To prove the last statement, we observe that under the current assumption, we have that $\|A^\circ(x)\| = d(\theta, A(x)) = d(\theta, \mathrm{bd}(A(x)))$, and so it suffices to use the first statement of the theorem. $\qquad\square$

Corollary 6.8. *For every $x \in H$ such that $A(x)$ is a nonempty bounded set, we have*

$$\|A(x)\| \leq \limsup_{y \to_{\neq} x} \|A(y)\|,$$

and, when H is finite-dimensional,

$$\|A(x)\| = \limsup_{y \to_{\neq} x} \|A(y)\|.$$

Proof. Let $x \in H$ be as in the corollary. Then for any $\varepsilon > 0$ there exists $x^* \in \mathrm{bd}(A(x))$ such that $\|x^*\| \geq \|A(x)\| - \varepsilon$. According to Theorem 6.5, there exist sequences $y_n \to x$ and $y_n^* \in A(y_n)$ such that $y_n \neq x$ and $y_n^* \to x^*$ as $n \to +\infty$. Thus,

$$\limsup_{y \to_{\neq} x} \|A(y)\| \geq \limsup_{n \to +\infty} \|A(y_n)\| \geq \lim_{n \to \infty} \|y_n^*\| = \|x^*\| \geq \|A(x)\| - \varepsilon,$$

and the desired inequality follows when ε goes to 0.

We assume now that H is finite-dimensional, so that according to the first statement we only need to prove that

$$\|A(x)\| \geq \limsup_{y \to_{\neq} x} \|A(y)\|.$$

Indeed, if $\limsup_{y \to_{\neq} x} \|A(y)\| = +\infty$, then since A is locally bounded in $\mathrm{int}(\mathrm{cl}(\mathrm{dom}A))$ (when this set is nonempty), it follows that $x \in \mathrm{bd}(\mathrm{cl}(\mathrm{dom}A))$. Hence, $N_{\mathrm{cl}(\mathrm{dom}A)}(x) \neq \{\theta\}$ and the equality $A(x) = A(x) + N_{\mathrm{cl}(\mathrm{dom}A)}(x)$, which comes from the maximality of the operator A, entail the contradiction $\|A(x)\| = +\infty$.

Consequently, we may suppose that $\limsup\limits_{y\to_{\neq}x}\|A(y)\|< +\infty$. We let a sequence $(y_n, y_n^*)_n \subset \mathrm{Gr}(A)$ be such that $y_n \to x$, $y_n \neq x$ and $\limsup\limits_{y\to_{\neq}x}\|A(y)\|= \lim\limits_{n\to\infty}\|y_n^*\|$. We may also assume that the sequence $(y_n^*)_n$ converges to some $x^* \in A(x)$. Then

$$\|A(x)\|\geq \|x^*\|= \lim_{n\to\infty}\|y_n^*\|= \limsup_{y\to_{\neq}x}\|A(y)\|,$$

as we wanted to prove. $\qquad\qquad\square$

The following result concerns the faces of the values of maximal monotone operators.

Theorem 6.9. *For every $x \in \mathrm{dom}A$ and $v \neq \theta$ we have*

$$A(x; v) = \mathop{\mathrm{Limsup}}_{w\to v,\ t\downarrow0}A(x + tw) = \mathop{\mathrm{Limsup}}_{w\to v,\ t\downarrow0}A(x + tw) = \omega - \mathop{\mathrm{Limsup}}_{w\to v,\ t\downarrow0}A(x + tw).$$

Proof. We fix $x \in \mathrm{dom}A$ and $v \neq \theta$, and take $x^* \in A(x; v)$. From Definition 6.1, we have that $v \in (\partial\infty_{A(x)})^{-1}(x^*) = \mathrm{N}_{A(x)}(x^*)$, which ensures that $x^* = \Pi_{A(x)}(x^* + v)$. Let us consider the following differential inclusion

$$\dot{z}(t) \in x^* + v - A(z(t)) \ \ t \geq 0, \ \ z(0) = x.$$

As in the proof of Theorem 6.5, this differential inclusion has a unique solution $z(\cdot)$ such that

$$\lim_{t\downarrow0}\frac{d^+z(t)}{dt} = \lim_{t\downarrow0}(x^*+v-A(z(t)))^\circ = \frac{d^+z(0)}{dt} = (x^*+v-A(x))^\circ = (x^*+v)-x^* = v.$$
$$(6.4)$$

We denote
$$x_n^* := \Pi_{A(z(\frac{1}{n}))}(x^* + v), \ \ w_n := \frac{z(\frac{1}{n}) - x}{\frac{1}{n}};$$

hence, (6.4) ensures that $\frac{d^+z(\frac{1}{n})}{dt} = (x^* + v - A(z(\frac{1}{n})))^\circ = x^* + v - x_n^* \to \frac{d^+z(0)}{dt} = v$. Therefore, as $n \to +\infty$ we obtain that

$$x_n^* \to x^*, \ \ w_n \to \frac{d^+z(0)}{dt} = v,$$

and so

$$x^* = \lim_{n\to\infty}x_n^* \subset \mathop{\mathrm{Limsup}}_{n\to\infty}A(z(\frac{1}{n})) = \mathop{\mathrm{Limsup}}_{n\to\infty}A(x + \frac{1}{n}w_n) \subset \mathop{\mathrm{Limsup}}_{w\to v,\ t\downarrow0}A(x + tw),$$

showing that

$$A(x; v) \subset \underset{w \to v,\ t \downarrow 0}{\mathrm{Limsup}} A(x + tw) \subset \underset{w \rightharpoonup v,\ t \downarrow 0}{\mathrm{Limsup}} A(x + tw).$$

Thus, since $A(x; v) \subset \underset{w \to v,\ t \downarrow 0}{\mathrm{Limsup}} A(x + tw) \subset \omega - \underset{w \to v,\ t \downarrow 0}{\mathrm{Limsup}} A(x + tw)$, we only need to verify that

$$\underset{w \rightharpoonup v,\ t \downarrow 0}{\mathrm{Limsup}} A(x + tw) \subset A(x; v) \text{ and } \omega - \underset{w \to v,\ t \downarrow 0}{\mathrm{Limsup}} A(x + tw) \subset A(x; v). \qquad (6.5)$$

To see the first inclusion, we take $x^* \in \underset{w \rightharpoonup v,\ t \downarrow 0}{\mathrm{Limsup}} A(x + tw)$, so that $x^* = \lim_n x_n^*$ for some sequences $(x_n^*), (w_n) \in H, (t_n) \subset \mathbb{R}_+$, such that $x_n^* \in A(x + t_n w_n), w_n \rightharpoonup v$, and $t_n \downarrow 0$. It follows by the maximal monotonicity of A that $x^* \in A(x)$, and for all $\xi \in A(x)$

$$\langle x_n^* - \xi, w_n \rangle = \frac{1}{t_n} \langle x_n^* - \xi, x + t_n w_n - x \rangle \geq 0.$$

So, by taking the limit as $n \to +\infty$ we obtain that $\langle x^*, v \rangle \geq \sup_{\xi \in A(x)} \langle \xi, v \rangle \geq \langle x^*, v \rangle$, which shows that $x^* \in A(x; v)$, and the first inclusion in (6.5) follows. We conclude the proof of the theorem because the second inclusion in (6.5) can be obtained using the same arguments as in the first inclusion. $\qquad \square$

The following example shows the necessity of moving the vector v in the expression of Theorem 6.9.

Example 6.10. Consider the maximal monotone operator A defined on H as

$$A(x) := x + \mathrm{N}_{\mathrm{B}(\theta,1)}(x),$$

and let $x, v \in H \setminus \{\theta\}$ be such that

$$\|x\| = 1 \text{ and } \langle v, x \rangle = 0.$$

Then one can easily check that $A(x) = [1, +\infty[\, x$, and so

$$A(x; v) = \left\{ x^* \in A(x) : \langle x^*, v \rangle = \sup_{\xi \in A(x)} \langle \xi, v \rangle = \sup_{\alpha \in [1,+\infty[} \langle \alpha x, v \rangle = 0 \right\} = A(x).$$

But for any $t > 0$ we have that $A(x + tv) = \emptyset$, which shows that

$$\omega - \underset{t \downarrow 0}{\mathrm{Limsup}} A(x + tv) = \underset{t \downarrow 0}{\mathrm{Limsup}} A(x + tv) = \emptyset.$$

In Theorem 6.12 we give the expression of the values of maximal monotone operators by using the values at nearby points. We need first to check the following lemma.

Lemma 6.11. *Given $x \in \text{dom}A$, for every $x^* \in A(x)$ it holds*

$$\text{N}_{\text{cl}(\text{dom}A)}(x) = \{v \in H : x^* + tv \in A(x), \ \forall t \geq 0\} =: d_\infty(A(x)). \qquad (6.6)$$

Proof. Since the operator $A + \text{N}_{\text{cl}(\text{dom}A)}$ is monotone and $\text{Gr}(A) \subset \text{Gr}(A + \text{N}_{\text{cl}(\text{dom}A)})$, the maximality of A ensures that $A(x) + \text{N}_{\text{cl}(\text{dom}A)}(x) = A(x)$, which implies that $\text{N}_{\text{cl}(\text{dom}A)}(x) \subset d_\infty(A(x))$. Take now $v \in d_\infty(A(x))$, so that $x^* + tv \in A(x)$ for all $t \geq 0$. Then, by the monotonicity of A we get

$$\langle y^* - (x^* + tv), y - x \rangle \geq 0 \ \forall y^* \in A(y), \ \forall t \geq 0,$$

which in turn leads to

$$\langle y^* - x^*, y - x \rangle \geq t\langle v, y - x \rangle \ \forall y^* \in A(y), \ \forall t \geq 0.$$

Hence, $\langle v, y - x \rangle \leq 0$ for every $y \in \text{dom}A$, and we deduce that $v \in \text{N}_{\text{cl}(\text{dom}A)}(x)$. $\qquad \square$

Theorem 6.12. *For every $x \in \text{dom}A$ such that $\text{bd}(A(x)) \neq \emptyset$ we have that*

$$A(x) = \text{N}_{\text{cl}(\text{dom}A)}(x) + \text{co}_2 \left\{ \underset{y \to_{\neq} x}{\text{Limsup}} A(y) \right\}.$$

Proof. First, according to Theorem 6.5, ensuring that $\text{bd}(A(x)) = \underset{y \to_{\neq} x}{\text{Limsup}} A(y)$, and to the maximal monotonicity of the operator A, ensuring that $A = A + \text{N}_{\text{cl}(\text{dom}A)}$, we only need to prove the following inclusion when $\text{int}(A(x)) \neq \emptyset$,

$$\text{int}(A(x)) \subset \text{N}_{\text{cl}(\text{dom}A)}(x) + \text{co}_2 \{\text{bd}(A(x))\}. \qquad (6.7)$$

Given $x^* \in \text{int}(A(x))$, we fix $x_0^* \in \text{bd}(A(x))$ and introduce the set

$$S := \{x_0^* + t(x^* - x_0^*) : t \geq 1\}.$$

On the one hand, if $S \cap \text{bd}(A(x)) = \emptyset$, then $S \subset A(x)$ and, due to the convexity of $A(x)$, we obtain $x_0^* + \mathbb{R}_+(x^* - x_0^*) \subset A(x)$. Hence, thanks to Lemma 6.11 we

deduce that $x^* - x_0^* \in \mathrm{N}_{\mathrm{cl}(\mathrm{dom}A)}(x)$, and so we get

$$x^* \in x_0^* + \mathrm{N}_{\mathrm{cl}(\mathrm{dom}A)}(x) \subset \mathrm{N}_{\mathrm{cl}(\mathrm{dom}A)}(x) + \mathrm{co}_2\{\mathrm{bd}(A(x))\},$$

which yields (6.7). On the other hand, if $S \cap \mathrm{bd}(A(x)) \neq \emptyset$, then there exists some $t > 1$ such that $z^* = x_0^* + t(x^* - x_0^*) \in \mathrm{bd}(A(x))$. Thus, we get

$$x^* = \frac{1}{t}z^* + (1 - \frac{1}{t})x_0^* \in \mathrm{co}_2\{\mathrm{bd}(A(x))\} \subset \mathrm{N}_{\mathrm{cl}(\mathrm{dom}A)}(x) + \mathrm{co}_2\{\mathrm{bd}(A(x))\},$$

and this completes the proof of the theorem. $\qquad\square$

6.4 Prox-regular analysis

In this section, we extend the results of the previous section to two classes of operators of nonsmooth analysis, the normal cone to uniformly r-prox-regular sets, and the class of prox-regular extended-real-valued functions with uniform parameters. As before, we work in the setting of a given Hilbert space H.

We start by giving the definition of the proximal normal cone.

Definition 6.13.

([30]) Given a set $C \subset H$ and $x \in C$, the proximal normal cone to C at x, denoted by $\mathrm{N}_C^P(x)$, is the set of vectors $x^* \in H$ for which there exists $m > 0$ such that

$$\langle x^*, y - x \rangle \leq m \, \|y - x\|^2 \text{ for all } y \in C.$$

Definition 6.14. *([63])* For positive numbers r and α, a closed set C is said to be (r, α)-prox-regular at $\overline{x} \in C$ provided that one has $x = \Pi_C(x + v)$, for all $x \in C \cap \mathrm{B}(\overline{x}, \alpha)$ and all $v \in \mathrm{N}_C^P(x)$ such that $||v|| < r$. The set C is r-prox-regular (resp., prox-regular) at $\overline{x}$ when it is (r, α)-prox-regular at $\overline{x}$ for some real $\alpha > 0$ (resp., for some numbers $r, \alpha > 0$). The set C is said to be r-uniformly prox-regular when $\alpha = +\infty$.

The following theorem describes the boundary set of the normal cone of a uniformly r-prox-regular set, by means of its values at nearby points, which are different from the reference point. We also characterize such normal cone by means of their boundaries points. Recall that the Bouligand tangent cone of a

prox-regular closed set C at $x \in C$ is given by

$$T_C(x) := (N_C^P(x))^*.$$

Theorem 6.15. *Let $C \subset H$ be a uniformly r-prox-regular set. Then for every $x \in C$ we have that*

$$\mathrm{bd}(N_C^P(x)) = \operatorname*{Limsup}_{y \to_{\neq} x} \mathrm{bd}(N_C^P(y)) = \operatorname*{Limsup}_{y \to_{\neq} x} N_C^P(y). \tag{6.8}$$

If $\mathrm{int}(T_C(x)) \neq \emptyset$*, then*

$$N_C^P(x) = \mathrm{co}_2\left\{\mathrm{bd}(N_C^P(x))\right\} = \mathrm{co}_2\left\{\operatorname*{Limsup}_{y \to_{\neq} x} N_C^P(y)\right\}. \tag{6.9}$$

Proof. First, we observe that the inclusions

$$\mathrm{bd}(N_C^P(x)) \subset \operatorname*{Limsup}_{y \to_{\neq} x} \mathrm{bd}(N_C^P(y)) \subset \operatorname*{Limsup}_{y \to_{\neq} x} N_C^P(y), \tag{6.10}$$

follow as in the the proof of Theorem 6.5, since the following differential inclusion,

$$\dot{z}(t) \in f(z(t)) - N_C^P(z(t)) \ \ t \in [0,1], \ z(0) = x \in C,$$

for a given Lipschitz function $f : H \to H$, also possesses a unique solution $z(\cdot)$ such that the function $\frac{d^+ z(\cdot)}{dt}$ is right-continuous on $[0,1[$ and $\frac{d^+ z(t)}{dt} = \left(f(z(t)) - N_C^P(z(t))\right)^\circ$ for all $t \in [0,1[$ (see [5, Theorem 4.6] for more details).

We are going to prove the converse inclusions of (6.10). We take $\xi \in \operatorname*{Limsup}_{y \to_{\neq} x} N_C^P(y)$, and let the sequences (y_n) and (ξ_n) be such that

$$\xi_n \in N_C^P(y_n), \ y_n \to x, \ \xi_n \to \xi \text{ as } n \to +\infty;$$

hence, we may suppose that for some $M > 0$ we have that $\xi_n \in N_C^P(y_n) \cap B_M$ for all $n \in \mathbb{N}$. Next, using the r-uniform prox-regularity of the set C, we obtain that $\xi \in N_C^P(x)$ ([63]). We claim that $\xi \in \mathrm{bd}(N_C^P(x))$. Proceeding by contradiction, we assume that for some positive number ρ such that $\rho < M$ it holds $\xi + B_\rho \subset N_C^P(x)$; that is,

$$\xi + \rho \frac{y_n - x}{\|y_n - x\|} \in N_C^P(x) \ \forall n \in \mathbb{N}.$$

Now, using the monotonicity of the mapping $x \to N_C^P(x) \cap B_{2M} + \frac{2M}{r}x$ (see [63]),

we get

$$\langle \xi_n + \frac{2M}{r} y_n - (\xi + \rho \frac{y_n - x}{\|y_n - x\|} + \frac{2M}{r} x), y_n - x \rangle \geq 0 \ \text{ for all } n \geq 1,$$

which implies that

$$\|\xi_n - \xi\| \|y_n - x\| + \frac{2M}{r} \|y_n - x\|^2 \geq \langle \xi_n - \xi, y_n - x \rangle + \frac{2M}{r} \|y_n - x\|^2 \geq \rho \|y_n - x\|,$$

and, dividing by $\|y_n - x\|$,

$$\|\xi_n - \xi\| + \frac{2M}{r} \|y_n - x\| \geq \rho,$$

which is a contradiction. Hence, $\xi \in \mathrm{bd}(\mathrm{N}_C^P(x))$ and (6.10) holds as equalities.

In this last part of the proof, we assume that $\mathrm{int}(\mathrm{T}_C(x)) \neq \emptyset$; that is, there exist $v \in H$ and $\eta > 0$ such that $v + \mathrm{B}_\eta \subset \mathrm{int}(\mathrm{T}_C(x))$. According to the first statement of the theorem we only need to prove that

$$\mathrm{int}(\mathrm{N}_C^P(x)) \subset \mathrm{co}_2 \left\{ \mathrm{bd}(\mathrm{N}_C^P(x)) \right\}. \tag{6.11}$$

We take $\xi \in \mathrm{int}(\mathrm{N}_C^P(x)) \setminus \{\theta\}$, so that $-\xi \notin \mathrm{N}_C^P(x)$ by [76, Exercise 9.42] (the proof of [76, Exercise 9.42] can be easily extended to the current infinite-dimensional setting), and hence we can choose $z^* \in \mathrm{bd}(\mathrm{N}_C^P(x)) \setminus \{\theta\}$. Let us show that for some $t_0 > 0$ we have that $\xi + t_0(\xi - t_0 z^*) \notin \mathrm{N}_C^P(x)$. Otherwise, $\xi + t(\xi - tz^*) \in \mathrm{N}_C^P(x)$ for all $t \geq 0$, and we get

$$\frac{1+t}{t^2} \xi - z^* \in \mathrm{N}_C^P(x) \ \forall t > 0,$$

which as $t \to +\infty$ gives us $-z^* \in \mathrm{N}_C^P(x)$, which contradicts the nonemptyness of the set $\mathrm{int}(\mathrm{T}_C(x))$ (again by [76, Exercise 9.42]). Then, there exists some $\beta \in (0,1)$ such that $w^* := \xi + \beta t_0(\xi - t_0 z^*) \in \mathrm{bd}(\mathrm{N}_C^P(x))$, and hence $\xi = \frac{1}{1+\beta t_0} w^* + \frac{\beta t_0}{1+\beta t_0}(t_0 z^*) \in \mathrm{co}_2 \left\{ \mathrm{bd}(\mathrm{N}_C^P(x)) \right\}.$ $\qquad \square$

In this last part of the paper, we extend the results of Section 6.3 to the proximal subdifferential mapping of lower semi-continuous functions.

Definition 6.16. *[19, Definition 3.1.]* Given a lower semi-continuous function $f : H \to \mathbb{R} \cup \{+\infty\}$ and $x \in \mathrm{dom} f$, a vector $x^* \in H$ is called proximal subgradient

of f at x, written $x^* \in \partial_P f(x)$, if there are $\rho, \delta > 0$ such that

$$f(y) \geq f(x) + \langle x^*, y - x \rangle - \delta \|y - x\|^2, \ \forall y \in \mathrm{B}(x, \rho).$$

A vector $x^* \in H$ is called limiting subgradient of f at x, written $\xi \in \partial_L f(x)$, if there are sequence $(x_k), (x_k^*) \subset H$ such that

$$x^* = \omega - \lim_{k \to \infty} x_k^*, \ x_k \longrightarrow x, \ f(x_k) \longrightarrow f(x), \ x_k^* \in \partial_P f(x_k).$$

Definition 6.17. *[19, Definition 3.1.]* A function $f : H \to \mathbb{R} \cup \{+\infty\}$ is said to be prox-regular at $\bar{x} \in \mathrm{dom} f$ with uniform parameters if there exist $\varepsilon, r > 0$ such that for any $\bar{v} \in \partial_L f(\bar{x})$, one has, for all $(x, v) \in \mathrm{Gr}(\partial_L f)$ satisfying $\|x - \bar{x}\| < \varepsilon$, $|f(x) - f(\bar{x})| < \varepsilon$ and $\|v - \bar{v}\| < \varepsilon$,

$$f(x') \geq f(x) + \langle v, x' - x \rangle - \frac{r}{2}\|x' - x\|^2 \ \forall x' \in \mathrm{B}(\bar{x}, \varepsilon).$$

It is worth observing that for prox-regular functions with uniform parameters f at $x \in \mathrm{dom} f$, we have that $\partial_P f(\bar{x}) = \partial_L f(\bar{x})$, and, in particular, if f is convex, then $\partial_P f(\bar{x}) = \partial f(\bar{x})$. In the following result, we give the counterpart of Theorem 6.5 to the proximal subdifferential mapping of prox-regular functions.

Theorem 6.18. *Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function and let $x \in \mathrm{dom} f$. If f is prox-regular with uniform parameters on a neighborhood of x with the same parameter $r > 0$, then*

$$\mathrm{bd}(\partial_P f(x)) = \mathop{\mathrm{Limsup}}_{y \to_{\neq} x} \partial_P f(y),$$

and, provided that $\mathrm{bd}(\partial_P f(x)) \neq \emptyset$,

$$\partial_P f(x) = \mathrm{N}_{\mathrm{dom} f}(x) + \mathrm{co}_2 \left\{ \mathop{\mathrm{Limsup}}_{y \to_{\neq} x} \partial_P f(y) \right\}.$$

Proof. According to [19, Proposition 3.6], the current prox-regularity assumption entails the existence of an open convex neighborhood U of x and a lsc convex function g such that

$$f(y) = g(y) - \frac{r}{2}\|y\|^2 \ \forall y \in U; \tag{6.12}$$

hence, $\partial_P f(y) = \partial g(y) - ry$ for all $y \in U$. Thus, since ∂g is a maximal monotone

operator [75], by applying Theorem 6.5 we get

$$
\begin{aligned}
\mathrm{bd}(\partial_P f(x)) &= \mathrm{bd}(\partial g(x) - rx) \\
&= \mathrm{bd}(\partial g(x)) - rx \\
&= \underset{y \to_{\neq} x}{\mathrm{Limsup}} \partial g(y) - rx \\
&= \underset{y \to_{\neq} x}{\mathrm{Limsup}} (\partial g(y) - ry) \\
&= \underset{y \to_{\neq} x}{\mathrm{Limsup}} (\partial_P f(y)),
\end{aligned}
$$

which yields the first conclusion.

To prove the second statement we observe that $\mathrm{dom} f \cap U = \mathrm{dom} g \cap U$, which yields $\mathrm{N}_{\mathrm{dom} f}(x) = \mathrm{N}_{\mathrm{dom} g}(x)$. Thus, sine $\mathrm{bd}(\partial g(x)) = \mathrm{bd}(\partial_P f(x)) + rx \neq \emptyset$ due to the current assumption, by applying Theorem 6.12 and taking into account (6.12) we get

$$
\begin{aligned}
\partial_P f(x) &= \partial g(x) - rx \\
&= \mathrm{N}_{\mathrm{cl}(\mathrm{dom} \partial g)}(x) + \mathrm{co}_2 \left\{ \underset{y \to_{\neq} x}{\mathrm{Limsup}} (\partial g(y) - ry) \right\} \\
&= \mathrm{N}_{\mathrm{dom} f}(x) + \mathrm{co}_2 \left\{ \underset{y \to_{\neq} x}{\mathrm{Limsup}} (\partial_P f(y)) \right\},
\end{aligned}
$$

where we used the fact that $\mathrm{cl}(\mathrm{dom} \partial g) = \mathrm{cl}(\mathrm{dom} g)$ (see, e.g. [86]). $\qquad \square$

Chapter 7

Future work

We are interested in the Lyapunov stability of the following differential inclusion

$$\dot{x}(t) \in -A(t)(x(t)) + f(x(t)), \;\; t \geq 0, x(0) = x_0 \in \mathrm{dom}A(0), \qquad (7.1)$$

where $f : H \to H$ is a Lipschitz mapping, and for each $t \geq 0$, $A(t) : H \rightrightarrows H$ is maximal monotone operator and $A(\cdot)$ is absolutely continuous.

Existence and unicity solutions of (7.1) have been already studied by S. Saïdi and M. Yarou [84].

Recently, Colombo and Palladino [33] provided strong and weak invariant characterizations for the following differential inclusion which is called sweeping process

$$\dot{x}(t) \in -\mathrm{N}_{C(t)}(x(t)) + f(t, x(t)) \text{ a.e. } t \geq 0, x(0) = x_0 \in C(0),$$

where $C(t)$ is uniformly prox-regular. We see that if all $C(t)$ are closed convex sets, then it becomes a special case of (7.1). If $C(t) = C$ for all $t \geq 0$ and f does not depend on time t, then the results of [33] and the results of Section 6, Chapter 5 coincide.

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Titre thèse français: Contribution à la stabilité de Lyapunov non-régulière des inclusions différentielles avec opérateurs monotones maximaux

Dans cette thèse de doctorat, nous apportons quelques contributions à la stabilité de Lyapunov non-régulière des inclusions différentielles de premier ordre avec opérateurs monotones maximaux, dans un cadre Hilbertien de dimension infini. Nous fournissons des caractérisations explicites, primales et/ou duales, des paires de Lyapunov faibles et fortes, dont les fonctions sont semi-continues inférieurement à valeurs réelles étendues, et associées à des inclusions différentielles dont la partie de droite est gouvernée par des perturbations Lipschitziennes des operateurs dits Cusco F, ou des opérateurs monotones maximaux A, ou les deux à la fois

$$\dot{x}(t) \in F(x(t)) - A(x(t)) \quad t \geq 0,\, x(0) \in \overline{\mathrm{dom}A}.$$

De manière équivalente, nous étudions l'invariance faible et forte des ensembles fermés pour ces inclusions différentielles. Comme dans L'approche classique de Lyapunov à la stabilité des équations différentielles, les résultats présentés dans cette thèse n'utilisent que les données du système différentiel; c'est-à-dire, l'opérateur A et la multifonction F, et donc pas besoin de connaître les solutions, ni les semi-groupes générés par les opérateurs monotones en question. Parce que les paires de Lyapunov sont formées pars des fonctions qui sont simplement semi-continues nférieurement, et les ensembles invariants ne sont que ensembles fermés, nous faisons usage dans cette thèse à des outils de l'analyse non-lisse, afin de fournir des critères du premier ordre, utilisant des sous-différentiels généraux et des cônes normaux.
Nous fournissons une analyse similaire pour les inclusions différentielles gouvernées par le cône normal proximal à des ensembles prox-réguliers. Notre analyse ci-dessus, nous a permis de présenter ces systèmes prox-réguliers d'apparence plus générale, comme des inclusions différentielles avec opérateurs monotones maximaux. Nous utilisons aussi nos résultats pour étudier la géométrie des opérateurs monotones maximaux, et plus précisément, la caractérisation de la frontière des valeurs de ces opérateurs seulement au moyen des valeurs situées à proximité, distinctes du point de référence. Ce résultat a des applications dans la stabilité des problèmes de la programmation semi-infinie. Nous utilisons également nos résultats sur les paires de Lyapunov et les ensembles invariants pour établir une étude systématique des observateurs de type Luenberger pour des inclusions différentielles avec des cônes normaux à des ensembles prox-réguliers.
La thèse est organisée comme suit: Au chapitre 1, nous expliquons les principaux objectifs de la thèse, la méthodologie que nous suivons et nous donnons un aperçu des principaux résultats. Nous faisons aussi dans ce chapitre un aperçu général de la théorie de Lyapunov, et nous présentons les principales réalisations et les différents résultats que nous avons trouvé dans littérature et qui ont, en quelques sortes, guidé les travaux de cette thèse. Au chapitre 2, nous présentons les principaux outils et résultats préliminaires dont nous avons besoin dans notre

analyse. Au chapitre 3, nous donnons les caractérisations souhaitées des paires de Lyapunov et des ensembles invariants pour des inclusions différentielles avec des perturbations Lipschitzienne des opérateurs monotones maximaux, Quant au Chapitre 4, nous étudions les inclusions différentielles avec des perturbations Lipschitzienne des cônes normaux proximaux. Ce chapitre comprend l'application à la conception des observateurs de type Luenberger. Au chapitre 5, nous étudions les inclusions différentielles avec des perturbations Lipschitziennes de type Cusco des opérateurs monotones maximaux. Au Chapitre 6, nous donnons un résultat sur la géométrie des opérateurs monotones maximaux, et nous décrivons la limite de leurs valeurs. Enfin, nous donnons au chapitre 7 un résumé des résultats obtenus.

Mots clés : Inclusions différentielles, operateurs monotones maximaux, fonctions de Lyapunov, ensembles invariants, ensembles prox-réguliers, opérateurs de type Cusco.

Titre thèse anglais: Contribution to Nonsmooth Lyapunov Stability of Differential Inclusions with Maximal Monotone Operators

In this PhD thesis, we make some contributions to nonsmooth Lyapunov stability of first-order differential inclusions with maximal monotone operators, in the setting of infinite-dimensional Hilbert spaces. We provide primal and dual explicit characterizations for parameterized weak and strong Lyapunov pairs of lower semicontinuous extended-real-valued functions, referred to as $a-$Lyapunov pairs, associated to differential inclusions with right-hand-sides governed by Lipschitz or Cusco perturbations F of maximal monotone operators A,

$$\dot{x}(t) \in F(x(t)) - A(x(t)) \;\; t \geq 0, \, x(0) \in \overline{\text{dom}A}.$$

Equivalently, we study the weak and strong invariance of sets with respect to such differential inclusions. As in the classical Lyapunov approach to the stability of differential equations, the presented results make use of only the data of the differential system; that is, the operator A and the multifunction F, and so no need to know about the solutions, nor the semi-groups generated by the monotone operators. Because our Lyapunov pairs and invariant sets candidates are just lower semicontinuous and closed, respectively, we make use of nonsmooth analysis to provide first-order-like criteria using general subdifferentials and normal cones. We provide similar analysis to non-convex differential inclusions governed by proximal normal cones to prox-regular sets. Our analysis above allowed to prove that such apparently more general systems can be easily coined into our convex setting. We also use our results to study the geometry of maximal monotone operators, and specifically, the characterization of the boundary of the values of such operators by means only of the values at nearby points, which are distinct of the reference point. This result has its application in the stability of semi-infinite programming problems. We also use our results on Lyapunov pairs and invariant sets to provide a systematic study of Luenberger-like observers design for differential inclusions with normal cones to prox-regular sets.

The thesis is organized as follows: In chapter 1, we explain the main objectives of the thesis, the methodology that we follow, and we give a preview of the main results. We also make in this chapter a general overview of Lyapunov's theory, and present the main previous achievements on the subject. In Chapter 2, we present the main tools and preliminary results that we need in our analysis. In Chapter 3, we give the desired characterizations of Lyapunov pairs and invariant sets for differential inclusions with Lipschitz perturbations of maximal monotone operators, while in Chapter 4, we investigate differential inclusions with Lipschitz perturbations of proximal normal cones. This chapter includes the application to Luenberger-like observers design. In Chapter 5, we study differential inclusions with Lipschitz Cusco perturbations of maximal monotone operators. In Chapter 6, we give a result on the geometry of maximal monotone operators, and describe the boundary of their values. Finally, we give in Chapter 7 a resume of the results we obtained.