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Sub-gradient diffusion equations

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To my parents.

Résumé

Ce mémoire de thèse est consacrée à l'étude des problèmes d'évolution où la dynamique est régi par l'opérateur de diffusion de sous-gradient. Nous nous intéressons à deux types de problèmes d'évolution.

Le premier problème est régi par un opérateur local de type Leray-Lions avec un domaine borné. Dans ce problème, l'opérateur est maximal monotone et ne satisfait pas la condition standard de contrôle de la croissance polynomiale. Des exemples typiques apparaît dans l'étude de fluide non-Neutonian et aussi dans la description de la dynamique du flux de sous-gradient. Pour étudier le problème nous traitons l'équation dans le contexte de l'EDP non linéaire avec le flux singulier. Nous utilisons la théorie de gradient tangentiel pour caractériser l'équation d'état qui donne la relation entre le flux et le gradient de la solution. Dans le problème stationnaire, nous avons l'existence de la solution, nous avons également l'équivalence entre le problème minimisation initial, le problème dual et l'EDP. Dans l'équation de l'évolution, nous proposons l'existence, l'unicité de la solution.

Le deuxième problème est régi par un opérateur discret. Nous étudions l'équation d'évolution discrète qui décrivent le processus d'effondrement du tas de sable. Ceci est un exemple typique de phénomènes auto-organisés critiques exposées par une slope critique. Nous considérons l'équation d'évolution discrète où la dynamique est régie par sous-gradient de la fonction d'indicateur de la boule unité. Nous commençons par établir le modèle, nous prouvons existence et l'unicité de la solution. Ensuite, en utilisant arguments de dualité nous étudions le calcul numérique de la solution et nous présentons quelques simulations numériques.

Mots-clés : la diffusion de sous-gradient, l'opérateur de type Leray-Lions, flux singulier, gradient tangentielle, la dualité, équation elliptique, équation parabolique, la contraction, sandpile effondrement, l'équation de l'évolution discrète.

Abstract

This thesis is devoted to the study of evolution problems where the dynamic is governed by sub-gradient diffusion operator. We are interest in two kind of evolution problems.

The first problem is governed by local operator of Leray-Lions type with a bounded domain. In this problem, the operator is maximal monotone and does not satisfied the standard polynomial growth control condition. Typical examples appears in the study of non-Neutonian fluid and also in the description of sub-gradient fows dynamics. To study the problem we handle the equation in the context of nonlinear PDE with singular flux. We use the theory of tangential gradient to characterize the state equation that gives the connection between the flux and the gradient of the solution. In the stationary problem, we have the existence of solution, we also get the equivalence between the initial minimization problem, the dual problem and the PDE. In the evolution problem, we provide the existence, uniqueness of solution.

The second problem is governed by a discrete operator. We study the discrete evolution equation which describe the process of collapsing sandpile. This is a typical example of Self-organized critical phenomena exhibited by a critical slop. We consider the discrete evolution equation where the dynamic is governed by sub-gradient of indicator function of the unit ball. We begin by establishing the model, we prove the existence and uniqueness of solution. Then by using dual arguments we study the numerical computation of the solution and we present some numerical simulations.

Keywords : sub-gradient diffusion, Leray-Lions operator, singular flux, tangential gradient, duality, elliptic equation, parabolic equation, contraction, collapsing sandpile, discrete evolution equation.

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Contents

Introduction

This thesis is devoted to the study of evolution problems where the dynamic is governed by sub-gradient diffusion operator. We are interest in two kind of evolution problems. The first problem is governed by local operator of Leray-Lions type with a bounded domain and the second one is governed by a discrete operator.

Let $\Omega \subseteq R^n$ be an open bounded set with Lipschitz boundary. Let us consider the following evolution problem :

$$\left\{\begin{array}{ll} \left.\begin{array}{l} \partial_t u(t) - \nabla\cdot\Phi = \mu(t) \\ \\ \Phi(x) \in \partial_\xi J(x, \nabla u(x)) \end{array}\right\} & in \quad \Omega,\ \text{for } t \in (0,T) \\ \\ u = 0 & \text{on } \Sigma := (0,T)\times\Gamma, \\ \\ u(0) = u_0 & \text{in } \Omega \end{array}\right. \tag{1}$$

where $\partial_t u$ denotes the partial derivative of u with respect to t, and $\partial_\xi J(x,\xi)$ denotes the subdifferential of J with respect to ξ. The functions $u_0 = u_0(x)$ and $\mu = \mu(t,x)$ are given. This problem is well studied in the case where $\partial_\xi J(x,\xi)$ is replaced by a Leray-Lions operator $a(x,\xi)$. That is a is a Carathéodory function, i.e. a is a vector valued mapping from $\Omega \times I\!R^n$ into $I\!R^n$, (Carathéodory mean continuous w.r.t. $x \in \Omega$ and measurable w.r.t. $\xi \in I\!R^n$), there exists $1 < p < \infty$, such that :

(L1) for any $\xi,\ \eta \in I\!R^n$, $\xi \neq \eta$

$$\big(a(x,\ \xi) - a(x,\ \eta)\big).(\xi - \eta) \geq 0 \ \text{ a.e. } \ x \in \Omega$$

(L2) there exists $C > 0$ such that, for any $\xi \in I\!R^n$

$$a(x,\ \xi).\xi \geq C|\xi|^p \ \text{ for a.e. } x \in \Omega$$

(L3) there exists $\sigma > 0$ and $k \in L^{p'}(\Omega)$ where $p' = \dfrac{p}{p-1}$, for any $\xi \in I\!R^n$

$$|a(x,\xi)| \leq \sigma(k(x) + |\xi|^{p-1}) \ \text{ a.e. } \ x \in \Omega$$

Leray-Lions type operators motivated many studies and new developments in the theory of nonlinear elliptic and parabolic PDE.

In the case where $1 < p < +\infty$, the existence and uniqueness of solution of this type equation is studied widely. If $\mu \in L^{p'}(Q)$, $u_0 \in L^p(\Omega)$ we can apply the Leray-Lions theorem, there is a unique solution in $L^p(0, T, W_0^{1,p}(\Omega))$ (cf. Theorem 1.2 page 162 [70]). If μ is a measure, the existence theorems for weak solutions have been given [31]. More about the summability with respect to space and time of the gradients of solutions are obtained [30].
To provide the uniqueness, the new concepts of solution have been defined. If $\mu \in L^1(Q)$, $u_0 \in L^1(\Omega)$, the authors studied the existence and uniqueness of entropy solution [6]. In the framework of renormalized solution, the notion was introduced by Lions and Di Perna [71], the existence and the uniqueness has been proved, see [27], [28]. And in [53], the author prove that the notion of renormalized solution and entropy solution for parabolic equation of Leray-Lions type are equivalent.

The case where $p = 1$, the equation (1.2) appear as models for heat, mass transfer in turbulent fluids or in the theory of phase transitions (see [9]). Some variant appears also in the context of image denoising and reconstruction (see [9]). In this situations the equation (1.2) appears as a border case with respect to the standard assumptions (L1-L3). Its study has developed many new theoretical and numerical tools (see [9]) currently essential for nonlinear PDEs analysis in the spaces BV, the set of function of bounded variation. Indeed, due to the linear growth condition, the natural energy space to study (1.2) in this case is the space of functions of bounded variation and the flux is a bounded function.

Typical examples for the opposite borderline case $p \to \infty$ ($p' = 1$), appears in the study of sub-gradient flow dynamics. In [16], the authors interpret the limit problem as a simple physical model for growing sandpile.

The aim of the first part of this thesis is to study the case where $J : \Omega \times I\!R^n \to [0, \infty]$; $J(x, \xi)$ is continuous with respect to x, l.s.c. with respect to ξ, and satisfies $J(x, 0) = 0$, for any $x \in \Omega$ and, moreover

(J1) There exists $M(x)$ in $L^\infty(\Omega)$ such that, the domain of $J(x, .) \subseteq \mathcal{B}(0, M(x))$ for all x in Ω.
This condition is equivalent to : for any $p \geq 1$, $J(x, \xi) \geq ((|\xi| - M(x))^+)^p$, for any $(x, \xi) \in \Omega \times I\!R^n$.

(J2) For any $x \in \Omega$, $J(x, .)$ is convex.

(J3) $0 \in Int(\mathcal{D}(J(x, .)))$, for any $x \in \Omega$.

Where, the domain of $J(x, .)$ is the set

$$D(x) := \mathcal{D}(J(x, .)) := \{\xi \in I\!R^n : J(x, \xi) < \infty\}.$$

To understand our situations, we consider the stationary problem

$$\begin{cases} \left.\begin{array}{l} -\nabla \cdot \phi = \mu \\ \phi \in \partial_\xi J(x, \nabla u(x)) \end{array}\right\} & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We start by minimizing the following integral functional, for $1 < p < \infty$:

$$\min I[u] = \min_{u \in W_0^{1,p}(\Omega)} \left\{ \int_\Omega J(x, \nabla u) dx - \int_\Omega f u dx \right\}, \qquad (2)$$

where $J : \Omega \times I\!R^n \to [0, +\infty)$; $J(x, \xi)$ is continuous with respect to x, l.s.c. with respect to ξ, $f \in L^2(\Omega)$.
There exist many standard studies of this type equation. If we suppose that $J(x, .)$ is convex, satisfies the coercive inequality $J(x, \nabla u) \geq \alpha |\nabla u|^p - \beta$, then there exists a solution of minimization problem. By using the standard duality argument we get

the dual problem as the following :

$$\max I^*[\phi] = \max_{\phi \in L^{p'}(\Omega)^n} \left\{ -\int_\Omega J^*(x,\phi)dx \ : -\nabla\cdot\phi = f \right\}.$$

In term of PDE, the Euler- Lagrange equation associated to this problem is elliptic equation, with Dirichlet boundary condition :

$$\begin{cases} \left.\begin{array}{l} -\nabla\cdot\phi = f \\ \phi(x) \in \partial_\xi J(x,\nabla u) \end{array}\right\} & \text{in} \quad \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

Moreover, if we assume that $J(x,.)$ satisfies the growth condition

$$\sup\{|w|; w \in \partial_\xi J(x,\xi)\} \leq C(1+|\xi|^{p-1}), \tag{4}$$

then we have "the equivalence between the three problems". (see section 2 of chapter 1 for details).
In the case where the assumptions (L3) fails to be true, the operator $A(x,.)$ may grows rapidly with respect to ξ. In this situation, the flux is not a Lebesgue function in general. It is a vector valued Radon measure. So we need to handle the equation (1) in the context of nonlinear PDE with singular flux. The questions are, can we apply the standard approach to solve this problem ? Even in the stationary case, an elliptic equation, how can we define the solution and have the existence of solution ? How can we characterize the singular flux ? How can we get the equivalence between the initial minimization problem, the dual problem and the PDE ? The questions are more difficult in the evolution problem. How can we get the uniqueness of solution ? Those questions will be answered in chapter 1, chapter 2 and chapter 3.

The aim of the second part of this thesis is to study the evolution problem where the dynamic is governed by a discrete operator. We study the discrete evolution equation which describe the process of collapsing sandpile. This is a typical example of Self-organized critical phenomena exhibited by a critical slop. We consider the discrete evolution equation where the dynamic is governed by the indicator function of the unit ball in $\mathbb{R}^n$ and we also give some numerical results. This is the aim of chapter 4.

0.1 Chapter 1

In the section 1 of this chapter, we recall some useful tools that we use in the thesis. In section 2, by starting from the minimization problem, we summarize the standard Euler-Lagrange approach to solve the stationary problem. From this, we understand the role of each assumption. Another approach to solve problem is using the dual arguments and the PDE come from the extremality relations. At the end of this chapter, we show why we can't apply those approach to our situation.

0.2 Chapter 2

In this chapter, we consider the stationary problem associated with (1), with Dirichlet boundary condition. Let $\mu \in \mathcal{M}_b(\Omega)$ be a given Radon and $g \in \mathcal{C}(\partial\Omega)$ be given, we consider the following equation

$$(P_1) \qquad \begin{cases} \left.\begin{array}{l} -\nabla \cdot \Phi = \mu \\ \Phi \in \partial_\xi J(x, \nabla u) \end{array}\right\} & \text{in } \Omega \\ \\ u = g & \text{on } \partial\Omega. \end{cases}$$

where $J : \Omega \times I\!R^n \to [0, \infty)$; $J(x, \xi)$ is continuous with respect to x, and l.s.c. with respect to ξ, and satisfies $J(x, 0) = 0$, for any $x \in \Omega$. Moreover, we assume that J satisfies the following assumptions

- (J1) There exists $M(x)$ in $L^\infty(\Omega)$ such that $D(x) \subseteq \mathcal{B}(0, M(x))$ for all x in Ω.
- (J2) For any $x \in \Omega$, $J(x, .)$ is convex.
- (J3) $0 \in Int(D(x))$

The aim of this chapter is to prove existence of a solution to problem (P_1). Moreover, we give the connection between the PDE, the minimization problem and the dual problem.

To solve the problem, we approximate the function J by Yosida approximation J_λ. We consider the regularization problem in $W^{1,p}(\Omega)$ and we get the approximate solutions u_λ. We prove that u_λ in $W^{1,\infty}(\Omega)$. Using the compactness arguments we get a solution u in $W^{1,\infty}(\Omega)$ and a flux Φ. Actually, we show that the flux is a vector valued measure. The regular part Φ_r (with respect to Lebesgue measure) leaves in $\partial_\xi J(x,\nabla u)$ and, the singular part Φ_s is concentrated on the boundary of $D(x)$ and is connected to the tangential gradient of u, $\nabla_{|\Phi_s|}u$ where $|\Phi_s|$ is the total variation of Φ_s, through the support function of $D(x)$. A reminder on all these tools is given in chapter 2. To set our first main result, we denote by

$$K=\Big\{z\in W^{1,\infty}(\Omega)\ ;\ \nabla z(x)\in D(x),\ \text{a.e. } x\in\Omega,\ z=g \text{ in } \partial\Omega\Big\}$$

and

$$\mathcal{H}_g=\big\{u\in W^{1,p}(\Omega) \text{ such that } u=g \text{ on } \partial\Omega\big\}.$$

For any $x\in\Omega$, let us denote by $S_{D(x)}$ the support function of $D(x)$, given by

$$S_{D(x)}(\phi)=\sup\Big\{\phi\cdot q\ ;\ q\in D(x)\Big\},\qquad \text{for any } (x,\phi)\in\Omega\times I\!R^n.$$

We consider the function g in the following set :

$$\mathcal{G}=\{g\in C(\partial\Omega),\exists g_0\in W^{1,\infty}(\Omega):\nabla g_0\in D(x);g_0=g \text{ in } \partial\Omega\}.$$

Theorem 0.1. *Assume that J satisfies the assumptions (J1)-(J2). For any $\mu\in\mathcal{M}_b(\Omega)$, $g\in\mathcal{G}$, the problem*

$$(P_2)\qquad \min\left\{\int_\Omega J(x,\nabla z(x))\,dx-\int_\Omega z\,d\mu\ ;\ z\in\mathcal{H}_g\right\}$$

has a solution u. If, moreover J satisfies (J3), then u is a solution of (P_2) if and only if $u\in K$ and, there exists $\Phi\in\mathcal{M}_b(\Omega)^n$ such that

$$\Phi_r(x)\in\partial_\xi J(x,\nabla u(x)),\quad \mathcal{L}^n\, a.e.\ x\in\Omega \tag{5}$$

$$\frac{\Phi_s}{|\Phi_s|}(x)\cdot\nabla_{|\Phi_s|}u(x)=S_{D(x)}\left(\frac{\Phi_s}{|\Phi_s|}(x)\right),\quad |\Phi_s|\text{- } a.e.\ in\ \Omega \tag{6}$$

and

$$\int_\Omega\Phi_r\cdot\nabla\xi\,dx+\int_\Omega\nabla_{|\Phi_s|}\xi\,d\Phi_s=\int_\Omega\xi\,d\mu,\qquad for\ any\ \xi\in C_0^1(\Omega). \tag{7}$$

If $\nabla_{|\Phi_s|}u(x) \in \overline{D(x)}$, $|\Phi_s|$- a.e. $x \in \Omega$, then (6) is equivalent to

$$\frac{\Phi_s}{|\Phi_s|}(x) \in \partial I\!I_{\overline{D(x)}}(\nabla_{|\Phi_s|}u(x)) \quad |\Phi_s|\text{- a.e.} x \in \Omega, \tag{8}$$

where $\partial I\!I_{\overline{D(x)}}$ denotes the subdifferential of the indicator function of $\overline{D(x)}$. Roughly speaking (6) with the fact that $\nabla u(x) \in D(x)$, $\mathcal{L}^n$-a.e. $x \in \Omega$, is a generalized formulation of the standard formulation (8). Formally, we can say that the problem (P_1) is governed by the following formulation

$$(P_1') \qquad \begin{cases} \left.\begin{array}{l} -\nabla \cdot \Phi = \mu \\ \Phi_r \in \partial_\xi J(x, \nabla u), \quad \dfrac{\Phi_s}{|\Phi_s|} \in \partial_\xi I\!I_{\overline{D(x)}}(\nabla_{|\Phi_s|}u) \end{array}\right\} & \text{in } \Omega \\ \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Throughout this chapter, the couple $(u, \Phi) \in W^{1,\infty}(\Omega) \times \mathcal{M}_b(\Omega)^n$ given by Theorem 0.1 will be called the weak solution of (P_1), and (P_1') will be called the weak formulation of (P_1). As to the problem (P_2), thanks to Theorem 0.1, it is the minimization problem associated with (P_1).
In particular, by using the definition $\partial I\!I_{\overline{D(x)}}$, we can deduce the existence of a solution for the variational formulation associated with the problem (P_1)

$$\int_\Omega \Phi \cdot \nabla(u - \xi) \leq \int_\Omega (u - \xi)\, d\mu, \qquad \text{for any } \xi \in K, \tag{9}$$

as well as its equivalence with a weak formulation and the minimization problem. The equivalence between the three formulations is summarized in the following Corollary

Corollary 0.1. *Under the assumptions (J1-J3), let $\mu \in \mathcal{M}_b(\Omega)$ and $(u, \Phi) \in K \times M_b(\Omega)^n$ be given. The following propositions are equivalent :*

1. *(u, Φ) is a weak solution of (P_1).*
2. *(u, Φ_r) is a variational solution of (P_1).*
3. *u is a solution of the minimization problem (P_2).*

Another important fact is the connection between (P_1), (P_2) and the dual problem for (P_1). To this aim, we denote by $J^* : \Omega \times I\!R^n \to I\!R$ the Legendre-Fenchel conjugate function of $J(.,\xi)$ which is defined by

$$J^*(x, y^*) = \sup\left\{\langle y^*, \xi\rangle - J(x,\xi) : \xi \in I\!R^n\right\}; \quad \text{for any } x \in \Omega.$$

We fix $\tilde{g} \in \mathcal{C}^1(\Omega)$ be such that $\tilde{g} = g$ in $\partial\Omega$ and we denote by

$$T(g,\psi) = \int_\Omega \nabla\tilde{g}d\psi - \int_\Omega \tilde{g}d\mu$$

Remark 0.1. *We will see that in our results, the value of $T(g,\psi)$ not depends on the choice of $\tilde{g}$ inside Ω and depends only on the trace of $\tilde{g}$ on $\partial\Omega$ which is equal to g. Indeed, $T(g,\psi) = \int_{\partial\Omega} g\,\psi\cdot ndS$. In this direction we can see the works of G.Q. Chen and H. Frid where the authors give directly a sense of the trace of a measure $\psi \in \mathcal{M}_b(\Omega)^n$ such that $div(\psi) \in \mathcal{M}_b(\Omega)$ (see c.f. [42] [43], [44]).*

We get the following theorem

Theorem 0.2. *Let $\mu \in \mathcal{M}_b(\Omega)$. Under the assumptions (J1-J3), the problem (P_3)*

$$\min\left\{\int_\Omega J^*(x,\psi_r(x))\,dx + \int_\Omega S_{D(x)}\left(\frac{\psi_s(x)}{|\psi_s(x)|}\right)\,d|\psi_s(x)| - T(g,\psi)\;;\;\psi\in\mathcal{S}(\mu)\right\}$$

has a solution $\Phi \in \mathcal{S}(\mu)$. Moreover, Φ is a solution of problem (P_3) if and only if there exists $u \in K$ such that (u,Φ) is a weak solution of the problem (P_1).

(See Section 1.1 for more details about definition of $\mathcal{S}(\mu)$).

0.3 Chapter 3

In this chapter we consider the evolution equation

$$\left\{\begin{array}{ll} \left.\begin{array}{l} \partial_t u(t) - \nabla\cdot(\Phi(t)) = \mu(t) \\ \\ \Phi \in \partial_\xi J(x,\nabla u) \end{array}\right\} & in \quad \Omega,\ \text{for } t\in(0,T) \\ \\ u = 0 & \text{on } \Sigma := (0,T)\times\partial\Omega, \\ \\ u(0)=u_0 & \text{in } \Omega. \end{array}\right. \tag{10}$$

where, $0 < T < \infty$, $Q = (0,T)\times\Omega$ and recall that J satisfies the following assumptions (the same as chapter 2)

(J1) There exists $M(x)$ in $L^\infty(\Omega)$ such that $D(x) \subseteq \mathcal{B}(0, M(x))$ for all x in Ω.
(J2) For any $x \in \Omega$, $J(x,.)$ is convex.
(J3) $0 \in Int(D(x))$.

Here μ in $BV(0,T;w^* - \mathcal{M}_b(\Omega))$ which is a subspace of $L^1(0,T;w^* - \mathcal{M}_b(\Omega))$ (for more about this space, see 1.1.3). We denote by

$$K = \Big\{ z \in W_0^{1,\infty}(\Omega)\ ;\ \nabla z(x) \in D(x),\ \text{a.e. } x\in\Omega \Big\}.$$

$$K_T = \Big\{ z \in \mathcal{C}([0,T);L^2(\Omega)) \cap L^\infty(0,T;W_0^{1,p}(\Omega))\ ;\ z(t)\in K \text{ for any } t\in[0,T) \Big\}.$$

So, for any $u \in K_T$ and $\mu \in L^1(0,T;w^* - \mathcal{M}_b(\Omega))$ the quantity $\displaystyle\iint_Q u\,d\mu$ is well defined.

Thanks to chapter 2, the weak formulation of this equation is given by :

$$\begin{cases} \left.\begin{array}{l} \partial_t u(t) - \nabla\cdot(\Phi_r(t)+\Phi_s(t)) = \mu(t) \\ \\ \Phi_r(t) \in \partial_\xi J(x,\nabla u(t)), \\ \dfrac{\Phi_s(t)}{|\Phi_s(t)|}\cdot\nabla_{|\Phi_s|}u(t) = S_{D(x)}\left(\dfrac{\Phi_s(t)}{|\Phi_s(t)|}\right) \end{array}\right\} & \text{in} \quad (0,T)\times\Omega, \\ \\ u = 0 & \text{on} \quad \Gamma, \text{ for } \quad (0,T)\times\partial\Omega, \\ \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{11}$$

For the existence, we consider the regularization problem and using the compactness arguments, we get a weak solution of (10). Then we prove that this solution also gives a variational solution. For the uniqueness, we use the doubling and dedoubling variables techniques to get the quadratic contraction of variational solutions. By passing to the limit in the approximate solutions, we prove the L^1-contractions of our solutions.
To begin with, we give our first main result is the coming of the existence and uniqueness of variational solution.

Theorem 0.3. *For any $u_0 \in K$ and $\mu \in BV(0,T;w^*-\mathcal{M}_b(\Omega))$, (10) has a variational solution (u,Φ) ; i.e. $u \in K_T$, $u(0)=u_0$, $\Phi \in L^1(Q)^n$, and for any $\xi \in K$*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u(t)-\xi|^2 + \int_\Omega \Phi(t)\cdot\nabla(u(t)-\xi) \le \int_\Omega (u(t)-\xi)\,d\mu(t) \quad \text{in } \mathcal{D}'(0,T). \tag{12}$$

Moreover, if (u_i,Φ_i) is a variational solution of (P_{μ_i}), for $i\in\{1,\ 2\}$, then

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_1(t)-u_2(t)|^2 \le \int_\Omega (u_1(t)-u_2(t))\,d(\mu_1(t)-\mu_2(t)) \quad \text{in } \mathcal{D}'(0,T).$$

In particular, we have the uniqueness of u, such that (u,Φ) is a variational solution of (10).

For the weak formulation (11), we prove the following result

Theorem 0.4. *Let $u_0 \in K$ and $\mu \in BV(0,T;w^*-\mathcal{M}_b(\Omega))$. Then, (u,Φ_r) is the variational solution of (10) if and only if $u \in K_T$, $u(0)=u_0$, $\partial_t u \in L^\infty(0,T;w^*-$*

$\mathcal{M}_b(\Omega)$), $\Phi_s \in L^\infty(0,T; w^* - \mathcal{M}_b(\Omega)^n)$ $\Phi_s(t) \perp \mathcal{L}^n$, and for $\mathcal{L}^1 -$ a.e. $t \in (0,T)$ we have

$$\Phi_r(t) \in \partial_\xi J(.,\nabla u), \quad \mathcal{L}^n\text{-}a.e.\ \Omega,$$

$$\frac{\Phi_s(t)}{|\Phi_s(t)|} \cdot \nabla_{|\Phi_s|} u(t) = S_{D(x)}\left(\frac{\Phi_s(t)}{|\Phi_s(t)|}\right) \quad |\Phi_s(t)|\text{ - }a.e.\ in\ \Omega.$$

Moreover, for any $\xi \in C_0^1(\Omega)$, we have

$$\frac{d}{dt}\int_\Omega u(t)\,\xi + \int_\Omega \Phi_r(t)\cdot\nabla\xi + \int_\Omega \nabla_{|\Phi_s(t)|}\xi\, d\Phi_s(t) = \int_\Omega \xi\, d\mu(t), \qquad in\ \mathcal{D}'(0,T).$$

0.4 Chapter 4

In this chapter, we contribute to the study of the dynamic of a sandpile. We study the discrete model for the collapse of a pile. We begin by establishing the model, we prove existence and uniqueness of the solution. Then by using dual arguments we study the numerical computation of the solution and we present some numerical simulations.

We consider the surface of the pile to be divided into cubes of integer point $i \in \mathbb{Z}^n$. The model here consists in finding for each $t > 0$ the application application $u : \mathbb{Z}^n \to \mathbb{R}$, where $u(i)$ describes the density of cubes at the position i , satisfying the following discrete equation

$$(DM) \qquad \begin{cases} \partial_t u(t,i) + \displaystyle\sum_{j:j\sim i} \sigma(t,i,j) = 0,\ t>0,\ i \in \mathbb{Z}^n, \\[2em] |u(t,i) - u(t,j)| \le c(t) \text{ for } i \sim j, \\[2em] \sigma(t,i,j) = -\sigma(t,j,i) \text{ and } \mathit{support}(\sigma(t,.,.)) \subseteq X_{c(t)}(u(t)). \end{cases}$$

where $i \sim j$ means $|i-j| \le 1$, the function $c : [0,T) \to \mathbb{R}^+$ satisfying $\lim\limits_{t\to T} c(t) = 1$ and

$$X_r(v) := \{(i,j) \in \mathbb{Z}^n \times \mathbb{Z}^n : |v(i) - v(j)| = r \text{ and } i \sim j\}.$$

We get the unique solution of (DM) in the following theorem

Theorem 0.5. *Assume that* $c \in W^{1,\infty}(0,\, T)$, $u_0 \in K(c(0))$ *and* $0 < T < \infty$. *Then the problem (DM) has a unique solution* $u \in W^{1,1}\Big(0, T;\, \ell^2(\mathbb{Z}^n)\Big)$ *and* u *satisfies*

$$\begin{cases} u_t(t) + \partial I\!I_{K(c(t))}(u(t)) \ni 0 \text{ for } t \in (0, T) \\ \\ u(0) = u_0. \end{cases}$$

Moreover, if u_λ *is a* λ*- approximate solution, then*

$$u_\lambda \longrightarrow u \quad \text{in} \quad \mathcal{C}([0, T]; \ell^2(\mathbb{Z}^n)) \qquad \text{as } \lambda \longrightarrow 0.$$

For the numerical computation, we attempt to discretize (DM) by the Euler implicit scheme, we have the generic problem is given by

$$(DSP) \qquad \begin{cases} v(i) + \displaystyle\sum_{j:j\sim i} \sigma(i,j) = g(i) & \text{for any } i \in \mathbb{Z}^n, \\ \\ v \in K(r), \quad \sigma(i,j) = -\sigma(j,i) & \text{for any } (i,j) \in \mathbb{Z}^n \times \mathbb{Z}^n, \\ \\ \text{and } \ support(\sigma) \subseteq X_r(v), \end{cases}$$

where $r \geq 1$ is a given constant and $g : \mathbb{Z}^2 \to I\!R$ is a given application.
For a given $r > 0$, we introduce the convex set

$$K(r) = \{z \in \ell^2(\mathbb{Z}^n) : |z(i) - z(j)| \leq r \quad \text{for} \quad i \sim j\}.$$

We prove the following main result

Theorem 0.6. *Let* $g \in \ell^2(\mathbb{Z}^n)$ *and* $v \in K(r)$. *Then* $v = I\!P_{K(r)}(g)$ *if and only if, there exists* $\sigma \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)$, *such that the couple* (v, σ) *satisfies (DSP).*

Remember that $v = I\!P_{K(r)}(g)$ if and only if $v \in K(r)$ and

$$J(v) = \frac{1}{2}\|v - g\|^2_{\ell^2(\mathbb{Z}^n)} = \min_{z \in K(r)} J(z). \tag{13}$$

Let $G:\ \ell^1(\mathbb{Z}^n\times\mathbb{Z}^n)\longrightarrow\mathbb{R}$ is defined by

$$G(\eta)=-\frac{1}{2}\sum_i\Big(\sum_{j:j\sim i}\eta(i,j)-\eta(j,i)\Big)^2-\sum_i\Big(\sum_{j:j\sim i}\eta(i,j)-\eta(j,i)\Big)g(i)-r\sum_{i,j}|\eta(i,j)|. \tag{14}$$

We denote by $\mathcal{S}_{as}=\Big\{\hat{\mu}\in\ell^1_{as}(\mathbb{Z}^n\times\mathbb{Z}^n)\ ;\hat{\mu}(i,j)=0 \text{ for } |i-j|>1\Big\}$, where

$$\ell^1_{as}(\mathbb{Z}^n\times\mathbb{Z}^n)=\Big\{\hat{\mu}\in\ell^1(\mathbb{Z}^n\times\mathbb{Z}^n)\ ;\hat{\mu}(i,j)=-\hat{\mu}(j,i),\ \text{for any } (i,j)\in\mathbb{Z}^n\times\mathbb{Z}^n\Big\}$$

We prove that

Theorem 0.7. *Let $g\in\ell^2(\mathbb{Z}^n)$ and $v:=\mathbb{P}_{K(r)}(g)$. Then, there exists $w\in\mathcal{S}_{as}$ and $v\in K(r)$ such that*

$$G(w)=\max_{\eta\in\mathcal{S}_{as}}G(\eta)=\min_{z\in K(r)}J(z)=J(v).$$

Moreover, for any $i\in\mathbb{Z}^n$, $v(i)=g(i)+\sum_{j:j\sim i}\Big(w(i,j)-w(j,i)\Big)$.

From this, we give the numerical method to minimize the functional G. Then we get some numerical simulations for our model.

1 Preliminaries

1.1 Notation, spaces and basic tools

Let $\Omega \subset I\!R^n$ be a bounded open domain. We denote by $\mathcal{L}^n$ the $n-$dimensional Lebesgue measure of $I\!R^n$. For $1 \leq p < +\infty$, $L^p(\Omega)$, $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ denote respectively, with respect to $\mathcal{L}^n$, the standard Lebesgue space, Sobolev space and the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$. Otherwise, we denote by $L^p_\mu(\Omega)$, the standard L^p space with respect to the measure μ.

1.1.1 Measure space

We denote by $\mathcal{M}(\Omega)$ the space of Radon measures in Ω. We recall that $\mathcal{M}(\Omega)$ can be identified with the dual space of the set of continuous functions with compact support in Ω ; i.e. $\mathcal{M}(\Omega) = \Big(\mathcal{C}_c(\Omega)\Big)^*$, in the sense that, every $\mu \in \mathcal{M}(\Omega)$ is equal to $\xi \in \mathcal{C}_c(\Omega) \to \displaystyle\int_\Omega \xi \, d\mu$.

For $\mu \in \mathcal{M}(\Omega)$, we denote by μ^+, μ^- and $|\mu|$ the positive part, negative part and the total variation measure associated with μ, respectively. Then we denote, $\mathcal{M}_b(\Omega)$ the space of Radon measures with bounded total variation $|\mu|(\Omega)$. Recall that $\mathcal{M}_b(\Omega)$ equipped with the norm $|\mu|(\Omega)$ is a Banach space.

We denote by $\mathcal{M}(\Omega)^n$ the space of $I\!R^n$-valued Radon measures of Ω ; i.e. $\mathcal{X} \in \mathcal{M}(\Omega)^n$ if and only if $\mathcal{X} = (\mathcal{X}_1, ..., \mathcal{X}_n)$ with $\mathcal{X}_i \in \mathcal{M}(\Omega)$. We recall that the total variation measure associated with $\mathcal{X} \in \mathcal{M}(\Omega)^n$, denoted again by $|\mathcal{X}|$, is defined by

$$|\mathcal{X}|(B) = \sup \left\{ \sum_{i=1}^{\infty} |\mathcal{X}(B_i)| \ ; \ B = \cup_{i=1}^{\infty} B_i, \ B_i \text{ a Borel set} \right\}$$

and belongs to $\mathcal{M}^+(\Omega)$, the set of nonnegative Radon measure. The subspace $\mathcal{M}_b(\Omega)^n$ equipped with the norm $\|\mathcal{X}\| = |\mathcal{X}|(\Omega)$ is a Banach space. It is clear that $\mathcal{M}(\Omega)^n$ endowed with the norm $\| \ \|$ is isometric to the dual of $\mathcal{C}_c(\Omega)^n$. The duality is given by

$$\left\langle \mathcal{X}, \xi \right\rangle = \sum_{i=1}^{n} \int_\Omega \xi_i \, d\mathcal{X}_i,$$

for any $\mathcal{X} = (\mathcal{X}_1, .., \mathcal{X}_n) \in \mathcal{M}(\Omega)^n$ and $\xi = (\xi_1, ..., \xi_n) \in \mathcal{C}_c(\Omega)^n$.

Theorem 1.1 (Lebegue-Radon-Nicodym decomposition). *[Theorem 6.10 [82]] Let μ be a positive σ-finite measure on a σ-agebra $\mathfrak{M}$ in a set X, and let λ be a complex measure on $\mathfrak{M}$*

- *There is unique a pair of complex measures λ_a and λ_s on $\mathfrak{M}$ such that :*

$$\lambda = \lambda_a + \lambda_s, \qquad \lambda_a \ll \mu, \qquad \lambda_s \perp \mu$$

  If λ is positive and finite, then so are λ_a, λ_s
- *There is unique $h \in L^1(\mu)$ such that*

$$\lambda_a(A) = \int_A h d\mu$$

  for everay $A \in \mathfrak{M}$

For any $\mathcal{X} \in \mathcal{M}_b(\Omega)^n$ and $\nu \in \mathcal{M}_b(\Omega)^+$, $\mathcal{X}$ is absolutely continuous with respect to ν ; denoted by $\mathcal{X} \ll \nu$, provided $\nu(A) = 0$ implies $|\mathcal{X}|(A) = 0$, for any $A \subset \Omega$. Thanks to Radon-Nicodym decomposition Theorem, we know that for any $\mathcal{X} \in \mathcal{M}_b(\Omega)^n$ and $\nu \in \mathcal{M}_b(\Omega)$ such that $\mathcal{X} \ll \nu$, there exists unique bounded $I\!R^n-$valued Radon measure denoted by $\dfrac{d\mathcal{X}}{d\nu}$, such that

$$\mathcal{X}(A) = \int_A \frac{d\mathcal{X}}{d\nu} \, d\nu \quad \text{for any } A \subseteq \Omega \ ;$$

$\dfrac{d\mathcal{X}}{d\nu} \in \mathcal{M}_b(\Omega)^n$ is the density of $\mathcal{X}$ with respect to ν, that can be computed by differentiating. In particular, it is not difficult to see that, for any $\mathcal{X} \in \mathcal{M}(\Omega)^n$, we have $\mathcal{X} \ll |\mathcal{X}|$ and $\dfrac{d\mathcal{X}}{d|\mathcal{X}|} \in L^1_{|\mathcal{X}|}(\Omega)^n$ and $\left|\dfrac{d\mathcal{X}}{d|\mathcal{X}|}\right| = 1$, $|\mathcal{X}|$-a.e. in Ω (see for instance

[82]). In connection with the polar factorization, in general, $\dfrac{d\mathcal{X}}{dvert\mathcal{X}|}$ is denoted by $\dfrac{\mathcal{X}}{|\mathcal{X}|}$. So, for any $\mathcal{X} \in \mathcal{M}_b(\Omega)^n$, we have

$$\mathcal{X}(A) = \int_A \frac{\mathcal{X}}{|\mathcal{X}|}\, d|\mathcal{X}|, \qquad \text{for any } A \subseteq \Omega,$$

and, every $\mathcal{X} \in \mathcal{M}(\Omega)^n$ can be identified with the linear application

$$\xi \in \mathcal{C}_c(\Omega)^n \to \int_\Omega \frac{\mathcal{X}}{|\mathcal{X}|} \cdot \xi \;\, d|\mathcal{X}|.$$

For any $\mathcal{X} \in \mathcal{M}_b(\Omega)^n$ and $\mu \in \mathcal{M}_b(\Omega)$, we say that $-\nabla \cdot \mathcal{X} = \mu$ in $\mathcal{D}'(\Omega)$ provided

$$-\int_\Omega \frac{\mathcal{X}}{|\mathcal{X}|} \cdot \nabla\xi \, d|\mathcal{X}| = \int_\Omega \xi \;\, d\mu \quad \text{ for any } \xi \in \mathcal{D}(\Omega) \; ;$$

in particular this remains true for any $\xi \in \mathcal{C}_0^1(\Omega)$, where $\mathcal{C}_0^1(\Omega)$ is the set of $\mathcal{C}^1$ function in Ω, such that ξ and $\nabla\xi$ are null on the boundary of Ω. In particular, $-\nabla \cdot \mathcal{X} = \mu$ in $\mathcal{D}'(\Omega)$ is equivalent to $-\nabla \cdot \left(\dfrac{\mathcal{X}}{|\mathcal{X}|} |\mathcal{X}| \right) = \mu$ in $\mathcal{D}'(\Omega)$. To simplify the presentation we'll use the notation

$$\int_\Omega \eta \, d\mathcal{X} := \int_\Omega \frac{\mathcal{X}}{|\mathcal{X}|} \cdot \eta \;\, d|\mathcal{X}|, \text{ for any } \eta \in \mathcal{C}_c(\Omega)^n.$$

For a given $\mu \in \mathcal{M}_b(\Omega)$, we say that $\mathcal{X} \in \mathcal{M}_b(\Omega)^n$ satisfies the PDE

$$-\nabla \cdot \mathcal{X} = \mu, \quad \text{ in } \Omega \tag{1.1}$$

if and only if

$$\int_\Omega \nabla\xi \, d\mathcal{X} = \int_\Omega \xi \, d\mu \quad \text{ for any } \xi \in \mathcal{C}_0^1(\Omega).$$

We denote by $\mathcal{S}(\mu)$ the set of vector valued Radon measure $\mathcal{X} \in \mathcal{M}_b(\Omega)^n$ satisfying the PDE (1.1). For any $\mathcal{X} \in \mathcal{M}_b(\Omega)^n$, we denote by $\mathcal{X}_r\mathcal{L}^n + \mathcal{X}_s$ the Radon-Nicodym decomposition of the vector valued measure $\mathcal{X}$ with respect to $\mathcal{L}^n$. So, $\mathcal{X} \in \mathcal{S}(\mu)$ is equivalent to say that

$$\int_{\Omega} \nabla \xi \cdot \mathcal{X}_r \, dx + \int_{\Omega} \nabla \xi \; d\mathcal{X}_s = \int_{\Omega} \xi \; d\mu \quad \text{for any } \xi \in \mathcal{C}_0^1(\Omega).$$

1.1.2 Sobolev embedding

Theorem 1.2 (Rellich-Kondrachov Theorem). *[Theorem 9.16 [41]] Suppose that Ω is a bounded subset of $I\!R^n$, class C^1. We have the following compact injection :*

$$\begin{aligned} & W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [1, p^\star], \text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, && \text{if } p < n, \\ & W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [p, \infty], && \text{if } p = n, \\ & W^{1,p}(\Omega) \hookrightarrow C(\bar{\Omega}) && \text{if } p > n. \end{aligned}$$

Theorem 1.3 (Poincaré inequality). *[5.8.1 [57]] Assume Ω is bounded, connected, open subset of $I\!R^n$ with a C^1 boundary $\partial\Omega$. Assume $1 \leq p \leq \infty$. Then there exists $C(n,p,\Omega)$, such that*

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq C \, \|\nabla u\|_{L^p(\Omega)}$$

for each function $u \in W^{1,p}(\Omega)$

1.1.3 Space involving time

For any $1 \leq q < \infty$, $L^q(0,T;V)$ is the space of measurable functions

$$\psi : \quad [0,T] \to V$$

such that

$$\|\psi\|_{L^q(0,T;V)} = \left(\int_0^T \|\psi(t)\|_V^q \, dt \right)^{\frac{1}{q}} < \infty$$

and $L^\infty(0,T;V)$ is

$$\|\psi\|_{L^q(0,T;V)} = \sup_{(0,T)} \|\psi(t)\|_V .$$

For any $1 \leq q \leq \infty$, $L^q(0,T;V)$ is Banach space. If V' is a dual space of V, separable then the dual space of $L^q(0,T;V)$ can be identified with $L^{q'}(0,T;V')$.
Since $\mathcal{M}_b(\Omega) = (\mathcal{C}_c(\Omega))^*$ and $\mathcal{C}_c(\Omega)$ is separable, then, for a given $T > 0$, any weak*-

measurable function $\psi : (0,T) \to \mathcal{M}_b(\Omega)$ is such that $t \in (0,T) \to |\psi(t)|(\Omega)$ is measurable (see [49]). So, for any $1 \le q \le \infty$, we define

$$L^q(0,T;w^*{-}\mathcal{M}_b(\Omega)) = \Big\{\psi:\ (0,T) \to \mathcal{M}_b(\Omega)\ \text{ weak}^*\text{-measurable };\ \int_0^T |\psi(t)|^q(\Omega)dt < \infty\Big\}.$$

Recall that the space $L^q(0,T;w^* - \mathcal{M}_b(\Omega))$ equipped with the norm

$$\|\psi\|_{L^q(0,T;w^*-\ \mathcal{M}_b(\Omega))} = \begin{cases} \left(\displaystyle\int_0^T (|\psi(t)|(\Omega))^q\, dt\right)^{\frac{1}{q}} & \text{if } q < \infty \\ \\ \text{ess-sup}_{t\in(0,T)} |\psi(t)|(\Omega) & \text{if } q = \infty \end{cases}$$

is a Banach space. If $q > 1$, then (cf. [48]) $L^q(0,T;w^* - \mathcal{M}_b(\Omega))$ can be identified with $\Big(L^{q'}(0,T;\mathcal{C}_0(\Omega))\Big)^*$ the dual space of $L^{q'}(0,T;\mathcal{C}_0(\Omega))$, where $q' = \dfrac{q}{q-1}$. The identification is given by the application

$$\mathcal{I}\ :\ L^q(0,T;w^* - \mathcal{M}_b(\Omega)) \to \Big(L^{q'}(0,T;\mathcal{C}_0(\Omega))\Big)^* \text{ with } \mathcal{I}(\mu)(\xi) = \int_0^T \int_\Omega \xi(t)\, d\mu(t).$$

The set $BV(0,T;w^* - \mathcal{M}_b(\Omega))$ is the subspace of $L^1(0,T;w^* - \mathcal{M}_b(\Omega))$ defined by $\mu \in BV(0,T;w^* - \mathcal{M}_b(\Omega))$ if and only if $\mu \in L^1(0,T;w^* - \mathcal{M}_b(\Omega))$ and

$$V(\mu,T) := \limsup_{h\to 0} \frac{1}{h} \int_0^{T-h} |\mu(\tau+h) - \mu(\tau)|(\Omega)\, d\tau\ <\ \infty.$$

If $\mu \in BV(0,T;w^* - \mathcal{M}_b(\Omega))$, then it is essentially bounded and has an essential limit from the right, denoted by $\mu(t+)$, for every $t \in [0,T)$. We also use the notation

$$V(\mu,t+) = \limsup_{h\to 0} \frac{1}{h} \int_0^t |\mu(\tau+h) - \mu(\tau)|(\Omega)\, d\tau \quad \text{for } 0 \le t < T.$$

We recall the following classical embedding result. Let H be a Hilbert space such that :$V \hookrightarrow H \hookrightarrow V'$.

Theorem 1.4. *Let $u \in L^q(0,T;V)$ be such that $\dfrac{\partial u}{\partial t}$ (is defined in the distributional sense) belongs to $L^q(0,T;V')$. Then u belongs to $C([0,T];H)$.*

For more detailed proofs and general results, see Theorem 1 [Chapter XVIII [48]]. In our situation, we consider the embedding

$$W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow W^{-1,p'}(\Omega).$$

1.1.4 Parabolic equation of Leray-Lions type

We consider the following evolution equation

$$\begin{cases} \partial_t u(t) - \nabla \cdot (a(x, \nabla u(t)) = \mu(t) & \text{in} \quad \Omega, \text{ for } t \in (0,T), \\ u = 0 & \text{on } \Sigma := (0,T) \times \Gamma, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

where a is a Leray-Lions operator.
Under the conditions $(L1, L2, L3)$, the problem (1.2) falls into the scope of the abstract evolution problem

$$\begin{cases} \dfrac{du}{dt} + A(u) = f, \\ u(0) = u_0. \end{cases} \tag{1.3}$$

where A is a nonlinear operator from $L^p(0,T; W_0^{1,p}(\Omega))$ to $L^{p'}(0,T; W^{-1,p'}(\Omega))$.

Theorem 1.5 (cf. Theorem 1.2 page 162 [70])**.** *Let A nonlinear operator satisfies*

- $\|A\| \leq c\,\|u\|^{p-1}$.
- *A is monotone.*
- $(A(u), u) \geq \alpha\,\|u\|^p$ *where* $\alpha > 0$, *for all* $v \in W_0^{1,p}(\Omega)$.

then there exists a unique solution $u \in L^p(0,T; W_0^{1,p}(\Omega))$ *of (1.3) for initial data* u_0 *in* $W_0^{1,p}(\Omega)$.

We recall the following theorem which is useful for our regularization problem

Theorem 1.6 (see cf. Proposition 5.7 [21])**.** *Let* $f \in W^{1,1}([0,T]; L^2(\Omega))$, $u_0 \in W_0^{1,p}(\Omega)$ *be such that* $\nabla \cdot \eta_0 \in L^2(\Omega)$ *for* $\eta_0 \in L^q(\Omega)$ *and* $\eta_0 \in a(x, \nabla u_0)$ *a.e. in* Ω. *Then there is a unique strong solution of* (1.2) *such that*

- $u \in L^\infty(0,T; W_0^{1,p}(\Omega)) \cap W^{1,\infty}([0,T]; L^2(\Omega))$,
- $\dfrac{d^+}{dt} u(t) - \nabla \cdot \eta(t) = f(t)$ *for all* $t \in [0,T]$,

where $\eta \in L^\infty([0,T];L^2(\Omega))$, $\eta(x,t) \in a(x,\nabla u(x,t))$ *a.e.* $(x,t) \in Q$.

1.1.5 Chain rule differentiation

Theorem 1.7 (cf. Proposition 9.5 [41])**.** *Let* $G : \mathbb{R} \to \mathbb{R}$ *be a Lipschitz function such that* $G(0) = 0$*. Then for every* $u \in W^{1,p}(\Omega)$*, we have* $G(u) \in W^{1,p}(\Omega)$ *and*

$$\nabla G(u) = G'(u)\nabla u \ a.e.\ \Omega.$$

Hence, we are able to consider the composition of functions in $W^{1,p}(\Omega)$ with some useful function. One is the truncation function $T_k(r) = max(-k, min(k,r))$ and another one is, for any r in $\mathbb{R}$

$$H_\epsilon(r) = \begin{cases} 1 \text{ if } r \geq \epsilon, \\ \dfrac{r}{\epsilon} \text{ if } -\epsilon \leq r \leq \epsilon, \\ -1 \text{ if } -\epsilon \geq r. \end{cases}$$

1.1.6 Variation calculus tools

Let $J : \Omega \times \mathbb{R}^n \to (-\infty, +\infty]$ be a function.
The domain of $J(x,.)$ is the set :

$$dom(J(x)) := \{\xi \in \mathbb{R}^n : J(x,\xi) < \infty\}.$$

The sub-level set of $J(x,.)$ at level $\gamma \in \mathbb{R}$ is defined by :

$$lev_\gamma J(x) := \{\xi \in \mathbb{R}^n : J(x,\xi) \leq \gamma\}.$$

$J(x,.)$ is lower-semicontinuous a.e $x \in \Omega$ if sub-level sets $lev_\gamma J(x)$ are closed for any $\gamma \in \mathbb{R}$.

Definition 1.1. *The function* $J(x,.)$ *is called proper convex if it is not identically equal to* $+\infty$ *and satisfies :*

$$J(x, t\xi + (1-t)y) \leq tJ(x,\xi) + (1-t)J(x,y),$$

for all ξ and y in $I\!R^n$ when $0 < t < 1$.

Definition 1.2. *A subgradient of J(x,.) at $\xi \in I\!R^n$ is a vector $\phi \in I\!R^n$ such that*

$$J(x,y) \geq J(x,\xi) + \langle \phi, y-\xi \rangle \qquad \text{for all } y \in I\!R^n.$$

For each ξ, we denote by $\partial_\xi J(x,\xi)$ the set of all subgradients of $J(x,.)$ at ξ. The subdifferential of $J(x,.)$ is a multi-valued mapping $\partial_\xi J(x,.)$ which assigns the set $\partial_\xi J(x,\xi)$ to each ξ.

Proposition 1.1. *(cf. Theorem A [80]) If $J(x,.)$ is lower-semicontinuous proper convex function then for any x, $\partial_\xi J(x,.)$ is maximal monotone in $I\!R^n$.*

Definition 1.3. *The Legendre-Fenchel conjugate associated with J is the function $J^* : \Omega \times I\!R^n \to I\!R$ defined by :*

$$J^*(x,y^*) = \sup_{y^* \in I\!R^n} \{\langle y^*, \xi \rangle - J(x,\xi) : \xi \in domJ(x)\}; \qquad \text{for any } x \in \Omega.$$

Proposition 1.2. *(cf. [58]) For all $\phi, \xi \in I\!R^n$, we have $\langle \phi, \xi \rangle \leq J^*(x,\phi) + J(x,\xi)$. Moreover, we get the equality if and only if $\phi \in \partial_\xi J(x,\xi)$.*

1.2 Minimization problem and PDE

1.2.1 Convex minimization problem

We start by the following definition

Definition 1.4. *The function $f : X \to I\!R \cup \{+\infty\}$ is coercive if*

$$\lim_{\|x\| \to +\infty} f(x) = +\infty.$$

Recall that, in the reflexive Banach space, every coercive function is weakly inf-compact. So, we have the following theorem

Theorem 1.8. *(cf. Theorem 3.3.4 [17]) Let X be a reflexive Banach space and $f : X \to I\!R \cup \{+\infty\}$ be weakly $-$ l.s.c. and coercive. Then there exist $x^* \in X$ such* *that*

$$f(x^*) \leq f(x) \qquad \textit{for all } x \in X.$$

Specially, when a function is convex the $l.s.c$ is equivalent to the $weakly - l.s.c.$ Then we also use the fact that every closed convex and bounded subset of a reflexive Banach space is weakly compact, we get

Theorem 1.9. *(cf. Theorem 2.11 [22]) Let X be a reflexive Banach space and $f : X \to I\!R \cup \{+\infty\}$ is a l.s.c., proper and convex function. If M is closed convex subset of X and f satisfies the coercive condition*

$$\lim_{x\in M, \|x\|\to+\infty} f(x) = +\infty,$$

then there exist $x^* \in M$ such that

$$f(x^*) \leq f(x) \qquad \textit{for all } x \in M.$$

1.2.2 Minimization of integral functionals

Let $J : \Omega \times I\!R^n \to (-\infty, +\infty]$ be a convex, proper function. For $1 < p < \infty$ and $\mu \in W^{-1,p'}(\Omega)$, we consider the following integral functional :

$$\begin{aligned} \mathcal{I} : \quad W^{1,p}(\Omega) \to \quad & [0, +\infty) \\ u \mapsto \quad \mathcal{I}(u) \quad & = \begin{cases} \displaystyle\int_\Omega J(x, \nabla u)dx - \int_\Omega u d\mu & \text{if } J(x, \nabla u) \in L^1(\Omega) \\ +\infty & \text{otherwise} \end{cases} \end{aligned}$$

where $\mu \in W^{-1,p'}(\Omega)$.
We summarize the results related to the minimization problem

$$\min \{\mathcal{I}(z) \,;\, z \in \mathcal{A}\}, \tag{1.4}$$

in the case where either $\mathcal{A} = W_0^{1,p}(\Omega)$ or $\mathcal{A} = \{u \in W^{1,p}(\Omega), \displaystyle\int_\Omega u = 0\}$.

The direct method give us the way to obtain the minimizer, by proving the l.s.c. and the coercivity of $\mathcal{I}$ with respect to a suitable topology. Among the tools, the l.s.c. of the integral functionals play an important role and has been studied wildly. For detail results, see [75], [76], [47], [72], [73].

The following definition and theorem give the necessary and sufficient conditions for the l.s.c. of general integral functional. We recall here the definition of quasi-convex functional, which is given the first time by Morrey (see [76]) then by Meyers (see [75]).

Definition 1.5. *Let $g : I\!R^{N\times n} \to I\!R$ be a locally bounded integrand. g is called quasi-convex if and only if for $u(x) = Ax + b$, where $a \in I\!R^N$ and $A \in M^{N\times n}$, and $w \in u + W_0^{1,\infty}(e, I\!R^N)$, e is unitary cube in $I\!R^n$, we have*

$$g(A) \leq \int_e g(\nabla w(x))dx.$$

Theorem 1.10. *(cf. see [73]) Let $1 \leq p < \infty$ and f be a Caratheodory integrand satisfying*

$$0 \leq f(x, u, \xi) \leq C(1 + |u|^q + |\xi|^p)$$

for $(u, \xi) \in I\!R^n \times I\!R^{N\times n}$, $q \leq \dfrac{np}{n-p}$ if $p<n$ and $q \geq 1$ if $p \geq n$. Then, the functional $F : W^{1,p}(\Omega, I\!R^N) \to [0, \infty)$ given by

$$F = \int_\Omega f(x, u(x), \nabla u(x))dx.$$

is well defined. Moreover, F is weakly $-$ l.s.c. in $W^{1,p}(\Omega, I\!R^N)$ if and only if F is quasi-convex with respect to the last variable ξ.

The quasi-convex notion restricted to the scalar case (N=1) is equivalent to convex one. For N>1, it is very hard to check that a given function is quasi-convex or not. Then Ball ([18]) introduced the polyconvexity (convex for all minors of a matrix) to get sufficient condition for quasi-convexity.
In our situation, it is easier to get the l.s.c of $\mathcal{I}$, because we are in the scalar case and the function J does not depend on u. Before giving the necessary condition for l.s.c., we recall the Mazur's Lemma. This is an important result that allows a passage from weak to strong convergence.

Lemma 1.1 (Mazur). *(cf. Theorem 2, section V [83]) Let $(X, \|.\|)$ is a normed linear space and let $x_n \rightharpoonup x$ in X. Then there exists a sequence $\{y_k\}$ of convex combination of $x_j's$ i.e.*

$$y_k = \sum_{i=1}^{j} a_i^k x_i \qquad where \qquad \sum_{i=1}^{j} a_i^k = 1, \qquad a_i^k \geq 0$$

such that $y_k \to x$ in X.

Theorem 1.11. *(c.f. see [47]) Let $1 < p < \infty$. If $J(x,.)$ be a Caratheodory function, l.s.c, convex and bounded below. Then, $\mathcal{I}$ is weakly $-$ l.s.c. in $W^{1,p}(\Omega)$*

Remark 1.1. *We see that the weak convergence in $W^{1,p}(\Omega)$ can be replaced by the weak* convergence in $W^{1,\infty}(\Omega)$. Therefore, the convexity of $J(x,.)$ implies the weakly* $-$ l.s.c. of $\mathcal{I}$ in $W^{1,\infty}(\Omega)$.*

Another ingredient to get minimum is the coercivity, it depends on the space and its topology

Definition 1.6. *An integral funtional $\mathcal{I}(u)$ satisfies the coercive condition in a subset $\mathcal{A} \subseteq W^{1,p}(\Omega)$ if and only if*

$$\lim_{u \in \mathcal{A}, \|u\|_{W^{1,p}} \to \infty} \mathcal{I}(u) = +\infty.$$

Remark 1.2. *If we assume that*

$$J(x,\xi) \geq C(1 + |\xi|^p) \text{ for } \xi \in I\!R^n. \tag{1.5}$$

By using Poincaré's inequality in $\mathcal{A}$ we get $\mathcal{I}[u] \geq C \|\nabla u\|^p_{L^p(\Omega)} - C \|u\|_{W^{1,p}(\Omega)}$. Then $\mathcal{I}(u)$ satisfies the coercive condition.

Now, we have the following theorem

Theorem 1.12. *Let $1 < p < \infty$ and J be a Caratheodory function,l.s.c., convex, bounded from below satisfying the coercive condition, then $\mathcal{I}$ attains its minimum in $\mathcal{A}$.*

Proof : Set $m = \inf_{\mathcal{A}} \mathcal{I}[u]$. If $m = \infty$ then we do nothing and we can suppose $m < \infty$. So, there exist $\{u_k\}$ such that

$$\mathcal{F}[u_k] \to m.$$

Using the coercive condition we have $\|\nabla u_k\|_{L^p} \leq M$. So lucky, we can apply Poincaré's inequality in $\mathcal{A}$, then $\|u_k\|_{W^{1,p}} \leq C$. Therefore,

$$u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega).$$

Finally, we have, $u \in \mathcal{A}$ and $\mathcal{I}[u] = m$, which ends the proof. □

1.2.3 Euler-Lagrange equation

In this section, we want to connect the problem (1.4) with the partial differential equation. This come very naturally from the following general result

Theorem 1.13. *Let $(X, \|.\|)$ be a Banach space and $g : X \to \mathbb{R}$ be a C^1 function, bounded from below that satisfies the Palais-Smale condition, i.e. if $\{u_k\}_{k=1}^{\infty}$ is a sequence such that : $\{g(u_k)\}_{k=1}^{\infty}$ is bounded, and $g'(u_k) \to 0$, then it has a convergent subsequence. Then, g attains the inf at some point $\bar{x} \in X$. Moreover, $\bar{x}$ is critical point of g, i.e. $Dg(\bar{x}) = 0$.*

In the case where function J is "good" enough so that the integral functional $I[u]$ satisfies all the conditions of this theorem, the critical point of

$$I[u] = \int_{\Omega} J(x, \nabla u)dx - \int_{\Omega} fudx,$$

where $u \in C_c^{\infty}(\Omega)$, will give us the Euler-Lagrange equation. Indeed, formally for any $w \in C_c^\infty(\Omega)$ we consider the real-value function, $i(\epsilon) := I[u+\epsilon w]$. We have

$$\begin{aligned} \frac{d}{d\epsilon} I[u+\epsilon w]_{\epsilon=0} &= \frac{d}{d\epsilon}[\int_\Omega J(x, \nabla u + \epsilon \nabla w)dx - \int_\Omega f(u+\epsilon w)]_{\epsilon=0} \\ &= \int_\Omega \nabla_\xi J(x,\nabla u)\cdot \nabla w - \int_\Omega fw \\ &= -\int_\Omega (\nabla \cdot \nabla_\xi J(x,\nabla u) - f)w. \end{aligned}$$

From $i'(0)=0$ we get the associated Euler- Lagrange equation of our problem is the following elliptic equation

$$-\nabla \cdot \nabla_\xi J(x,\nabla u) = f \quad \text{in } \Omega. \tag{1.6}$$

If $\mathcal{I}$ is not differentiable, the question is how does a solution of minimization problem become a solution of Euler-Lagrange system ? Let us explain in the case of Dirichlet boundary condition. In the standard Leray-Lions situation, it is usually assumed that J has a derivative which satisfies the growth condition :

$$|\nabla_\xi J(x,\xi)| \le \sigma(k(x) + |\xi|^{p-1}) \qquad \text{where } k(x) \in L^{p'}(\Omega). \tag{1.7}$$

Under that assumption we have :

$$|\nabla_\xi J(x,\nabla u)|^{p'} = |\nabla_\xi J(x,\nabla u)|^{\frac{p}{p-1}} \le \sigma(k(x) + |\nabla u|^{p-1})^{\frac{p}{p-1}} \le \sigma(|k(x)|^{p'} + |\nabla u|^p).$$

So that $\nabla_\xi J(x,\nabla u) \in L^{p'}(\Omega)^n$, for any u in $W^{1,p}(\Omega)$. Then we have the following definition :

Definition 1.7. *We say that $u \in W^{1,p}(\Omega)$ is a weak solution of Euler-Lagrange equation if and only if*

$$\int_\Omega \nabla_\xi J(x,\nabla u)\cdot \nabla v dx = \int_\Omega f v dx,$$

for all $v \in W^{1,p}(\Omega)$.

Theorem 1.14. *Suppose that J is convex, satisfies the growth condition* (1.7)*, then the minimizer of* $\min\left\{\mathcal{I}(z)\,;\, z\in W^{1,p}(\Omega)\right\}$ *is a weak solution of Euler-Lagrange* *equation* (1.6).

Proof : In the following proof, we see that the growth condition plays an important role to get the differential under the integration. First, J is convex and satisfies condition (1.7), so

$$|J(x,\nabla u)| \leq |\nabla_\xi J(x,\nabla u)|\,|\nabla u| \leq \sigma(k(x)+|\nabla u|^{p-1})|\nabla u| \leq \sigma|k(x)||\nabla u| + |\nabla u|^p.$$

This implies that the function $I[u+\epsilon w]$ is well defined and is finite for all u, w in $W^{1,p}(\Omega)$.
If we denote

$$g_\epsilon(x) = \frac{J(x,\nabla u+\epsilon\nabla w) - J(x,\nabla u)}{\epsilon},$$

then

$$\begin{aligned} g_\epsilon(x) &= \frac{1}{\epsilon}\int_0^\epsilon \frac{d}{dt}J(x,\nabla u + t\nabla w)dt \\ &= \frac{1}{\epsilon}\int_0^\epsilon \nabla_\xi J(x,\nabla u+t\nabla w)\cdot\nabla w\, dt. \end{aligned}$$

Now, by using the growth condition we have

$$\begin{aligned} |g_\epsilon(x)| &\leq \frac{1}{\epsilon}\int_0^\epsilon \sigma(|\nabla u+t\nabla w|^{p-1}+k(x))|\nabla w|\,dt \\ &\leq \frac{1}{\epsilon}\int_0^\epsilon \sigma|\nabla u+t\nabla w|^{p-1}|\nabla w|\,dt + \sigma|k(x)|\,|\nabla w| \\ &\leq C_1|\nabla u|^{p-1}|\nabla w| \ + C_2\epsilon^{p-1}|\nabla w|^p + \sigma|k(x)||\nabla w|. \end{aligned}$$

Since u, w in $W^{1,p}(\Omega)$, $k(x)$ in $L^{p'}(\Omega)$, this implies that $|g_\epsilon(x)| \in L^1(\Omega)$. Using Dominated convergence theorem, we have

$$\begin{aligned} \lim_{\epsilon\to 0}\frac{i(\epsilon)-i(0)}{\epsilon} &= \lim_{\epsilon\to 0}\int_\Omega \frac{J(x,\nabla u+\epsilon\nabla w)-J(x,\nabla u)}{\epsilon}dx - \int_\Omega fw \\ &= \int_\Omega \lim_{\epsilon\to 0}\frac{J(x,\nabla u+\epsilon\nabla w)-J(x,\nabla u)}{\epsilon}dx - \int_\Omega fw \\ &= \int_\Omega \nabla_\xi J(x,\nabla u)\cdot\nabla w\,dtdx - \int_\Omega fw \\ &= -\int_\Omega(\nabla\cdot\nabla_\xi J(x,\nabla u)-f)w + \int_{\partial\Omega}\frac{\partial}{\partial n}\nabla_\xi J(x,\nabla u)w \end{aligned}$$

Using the fact that u is a minimizer, we have this limit is equal to 0, in another words, u is a weak solution of Euler-Lagrange equation (1.6). □

Remark 1.3. *See that under the assumption* (1.7)*, setting* $a(x,\xi) = \nabla_\xi J(x,\xi)$*, the problem (*1.6*) falls into the scope of Leray-Lions type elliptic equation. So, the equation has a unique solution.*

For $f \notin L^2(\Omega)$, the problem is more difficult. In [31], by approximating f and using the compactness arguments, the authors get the existence of weak solution $u \in W_0^{1,1}(\Omega)$ for also $f \in L^1(\Omega)$ and $f \in \mathcal{M}_b(\Omega)$.
In the case where $f \in L^1(\Omega)$, the authors proved the existence and the uniqueness of entropy solutions of (1.6) [26] . Review the author in [7] for more general results. In the case where $f \in \mathcal{M}_b(\Omega)$ and if $p > n$ we have the embedding $\mathcal{M}_b(\Omega)$ in $W^{-1,p'}$ and solutions turn out to become continuous. Then the theory of entropy solutions is easily adapted. The case where $p \leq n$, the authors introduced the notation $p-capacity$ of a set [32]. They considered space $\mathcal{M}_b^p(\Omega)$ of all signed Radon measure $\mu \in \mathcal{M}_b(\Omega)$ such that $\mu(E) = 0$ for all set E of $p-capacity$ equal to 0. Then they decomposed this space as $L^1(\Omega) + W^{-1,p'}(\Omega)$. Finally, they obtained the existence and the uniqueness of entropy solutions.

1.2.4 Maximal monotone type operator

Remark 1.4. *We consider the following equation :*

$$-\nabla \cdot a = f \ in \ \Omega \tag{1.8}$$

when a *is a multi-valued graph in* $(\Omega, I\!R^n) \times I\!R^n$*. The conditions* $(L2)$ *and* $(L3)$ *can be replaced by*

(L_2^*) *there exists* $C > 0$ *such that, for any* $((x,\xi);w) \in a$

$$w.\xi \geq C|\xi|^p \ for \ a.e. \ x \in \Omega$$

(L_3^*) *there exists* $\sigma > 0$ *and* $k \in L^{p'}(\Omega)$ *where* $p' = \dfrac{p}{p-1}$, *for any* $\xi \in I\!R^n$

$$\sup\{|w|; w \in a(x,\xi) \leq \sigma(k(x) + |\xi|^{p-1}) \ \ a.e.\ \ x \in \Omega.$$

We have the following theorem

Theorem 1.15. *(c.f. theorem 2.17 [21]) Suppose that a is maximal monotone graph, satisfies condition* (L_2^*) *and* (L_3^*). *There is unique weak solution* $u \in W_0^{1,p}(\Omega)$ *of* (1.8) *in the following sense*

$$\int_\Omega w \cdot \nabla v dx = \int_\Omega f v dx,$$

for all $v \in W_0^{1,p}(\Omega)$, *where* $w(x) \in a(x, \nabla u)$ *a.e* x *in* Ω.

Remark 1.5. *Let* $J : \Omega \times I\!R^n \to (-\infty, +\infty]$, *we want to connect the hypothesis on* J *and the hypothesis of Leray-Lions operator*

- $J(x,.)$ *is convex,* $J(x,.) = 0$.
  We have $\partial_\xi J(x,.)$ *is a maximal monotone operator, for any* $\xi, \eta \in I\!R^n$, $\xi \neq \eta$

$$\big(\partial_\xi J(x,\ \xi) - \partial_\xi J(x,\ \eta)\big).(\xi - \eta) \geq 0 \ \ a.e.\ \ x \in \Omega.$$

- *For any* $\xi \in I\!R^n$, *there exist* $C > 0$ *such that* $J(x,\xi) \geq C(1 + |\xi|^p)$.
  By using the convexity of J *and* $J(.,0) = 0$ *we have* $w \cdot \xi \geq J(x,\xi)$, *where* $w \in \partial_\xi J(x,\ \xi)$. *This implies that*

$$w.\xi \geq C(1 + |\xi|^p) \ \ a.e.\ \ x \in \Omega.$$

- *The* $\partial_\xi J$ *satisfies the growth condition*

$$\sup\{|w|; w \in \partial_\xi J(x,\xi) \leq \sigma(k(x) + |\xi|^{p-1}) \ \ a.e.\ \ x \in \Omega. \tag{1.9}$$

Finally, we get :

Theorem 1.16. *Suppose that* J *is convex, satisfies the coercivity and the growth condition* (1.9). *Then* u *is the minimizer of* $\min\big\{\mathcal{I}(z)\ ;\ z \in W_0^{1,p}(\Omega)\big\}$ *if and only if* *u is a weak solution of Euler-Lagrange equation*

$$-\nabla \cdot \partial_\xi J(x, \nabla u) = f \text{ in } \Omega.$$

(c.f. see [21] for more details)

1.2.5 Dual approach

Let V, Y are Banach space, in this section we denote by

$$I(u,p) = G(p) + F(u)$$

where $p \in Y$ and $u \in V$, we consider the following optimize problem (P)

$$\inf_{u \in V} \{G(\Lambda u) + F(u)\}.$$

Using the standard duality argument (cf. [58]) , we get easily the dual problem (P^*)

$$\sup_{p^* \in Y^*} \{-G^*(-p^*) - F^*(\Lambda^* p^*)\}.$$

Theorem 1.17. *(cf. Theorem 4.2 and Remark 4.2, Chapter III [58]) Let us assume that V is reflexive Banach space, if F, G is convex and*

(I_1) $\lim_{\|u\| \to \infty} I(u, \Lambda u) = +\infty$,

(I_2) *there exist $u_0 \in V$, $F(u_0) < +\infty$ and $G(\Lambda u_0) < +\infty$, the function $G(\Lambda u_0)$ being continuous at Λu_0.*

Then P and P^* have at least one solution, $\inf P = \sup P^*$. More over, $\bar{u} \in V$, the solution of P, and $\bar{p}^* \in V^*$, the solution of P^*, satisfy the extremality relations

$$\Lambda^* \bar{p}^* \in \partial F(\bar{u}),$$

$$-\bar{p}^* \in \partial G(\Lambda \bar{u}).$$

Remark 1.6.

- *Condition $[I_1]$ is the coercive condition, with the convexity of F and G are* *necessaries conditions to get the existence of solution of primary problem.*

- $[I_2]$ *is called stability criterion, is very important hypothesis, sometime can be called the qualification hypothesis. This condition implies the existence of solution of the dual problem.*

See [58] for more detailed proofs, general results and examples.

Return to our situation, we have the following proposition

Proposition 1.3. *Suppose J is convex, coercive, $0 \in int(domJ)$ and satisfies the growth condition (1.9), we denote by P_1*

$$\inf_{u \in W_0^{1,p}(\Omega)} \left\{ \int_\Omega J(x, \nabla u) - \int_\Omega fu \right\},$$

then, the dual problem is

$$\sup_{\substack{\phi \in L^{p'}(\Omega)^n \\ -\nabla \cdot \phi = f}} \left\{ -\int_\Omega J^*(x, \phi) \right\}$$

Then P_1 and P_1^* have at least one solution, respectively u and ϕ. Moreover we $\inf P_1 = \sup P_1^*$ *an the extremality relation reads*

$$\phi \in \partial_\xi J(x, \nabla u). \tag{1.10}$$

Proof : We have

$$G(p) = \int_\Omega J(x,p) \text{ and } F(u) = -\int_\Omega fu.$$

The operator Λ now is the gradient ∇, V is $L^p(\Omega)$ and Y is $L^p(\Omega)^n$. Now we apply the dual argument to calculate

$$\begin{aligned} G^*(p^*) &= \sup_{p \in L^p(\Omega)^n} \{ < p^*, p > - \int_\Omega J(x,p) \} \\ &= \sup_{p \in L^p(\Omega)^n} \{ \int_\Omega p^* \cdot p - \int_\Omega J(x,p) \} \\ &= \int_\Omega J^*(x, p^*). \end{aligned}$$

$$F^*(\Lambda^* u^*) = \begin{cases} 0 & \text{if } -\Lambda^* u^* = f \\ +\infty & \text{otherwise} . \end{cases}$$

Finally, the dual problem is

$$\sup_{\phi \in L^{p'}(\Omega)^n} \left\{ -\int_\Omega J^*(x,\phi); -\nabla\cdot\phi = f \right\}$$

Now we check the condition $[I_2]$, the more difficult one is the continuous of G at ∇u_0. We prove that G is continuous at 0. Using the fact that $0 \in int(domJ)$ then J is continuous at 0. Let $\{p_n\}$ is a consequence convergent to 0. Using the growth condition (1.9), we get

$$|J(x,p_n)| \leq \sigma|k(x)||p_n| + |p_n|^p.$$

Since p_n in $L^p(\Omega)^n$, $k(x)$ in $L^{p'}(\Omega)$, this implies that $|J(x,p_n)| \in L^1(\Omega)$. Then

$$\lim_{\epsilon\to 0}\int_\Omega J(x,p_n) = \int_\Omega J(x,0).$$

This implies the continuous of G at 0. Finally, we easily get the extremality relations.

□

In the following examples, we have the equivalence between the PDE, the minimization and the dual problem

Example 1 (*Dirichlet Problem*) For $f \in L^2(\Omega)$, the PDE is

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The minimization problem is

$$\min_{u\in H_0^1(\Omega)} \int_\Omega \frac{1}{2}|\nabla u|^2\,dx - \int_\Omega fudx.$$

The dual problem is

$$\max_{\phi\in L^2(\Omega)^n}\left\{\int_\Omega -\frac{1}{2}|\phi|^2\,dx;\ \text{s.t.}\ -\text{div}\,(\phi) = f\right\}.$$

Example 2 (*p-Laplace*) For $f \in L^{p'}(\Omega)$, the PDE is

$$\begin{cases} -\nabla \cdot (|\nabla u|^{p-1} \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

The minimization problem

$$\min_{u \in W_0^{1,p}(\Omega)} \int_\Omega \frac{1}{p} |\nabla u|^p \, dx - \int_\Omega f u dx.$$

The dual problem is

$$\max_{\phi \in L^{p'}(\Omega)^n} \left\{ \int_\Omega -\frac{1}{p'} |\phi|^{p'} \, dx; \text{ s.t. } -\operatorname{div}(\phi) = f \right\}.$$

2 Sub-gradient Diffusion Leray-Lions Operator

Let $\mu \in \mathcal{M}_b(\Omega)$ be a given Radon and $g \in \mathcal{G}$ be given. We consider the following equation, with Dirichlet boundary condition

$$(P_1) \qquad \begin{cases} \left.\begin{array}{l} -\nabla \cdot \Phi = \mu \\ \Phi(x) \in \partial_\xi J(x, \nabla u(x)) \end{array}\right\} & \text{in } \Omega \\ \\ u = g & \text{on } \partial\Omega. \end{cases}$$

More precisely, we are interested in the case where, for any $x \in \Omega$, $\Phi(x,.)$ is a maximal monotone graph in $I\!R^n$ given by

$$\Phi(x,\xi) = \partial_\xi J(x,\xi), \tag{2.1}$$

where $J : \Omega \times I\!R^n \to [0,\infty)$; $J(x,\xi)$ is continuous with respect to x, and l.s.c. with respect to ξ, and satisfies $J(x,0) = 0$, for any $x \in \Omega$. Moreover, we assume that J satisfies the following assumptions

(J1) There exists $M(x)$ in $L^\infty(\Omega)$ such that $D(x) \subseteq \mathcal{B}(0, M(x))$ for all x in Ω.

(J2) For any $x \in \Omega$, $J(x,.)$ is convex.

(J3) $0 \in Int(D(x))$.

In the following section, we begin with some preliminaries, recalling the main tools we use to handle a PDE with singular flux, like tangential measure and tangential gradient. Then, we prove two technical results that will be useful for the proof of our main result. In Section 3, we present our main results. Under the assumptions (J1)-(J3), we begin with the characterization of the solution of the optimization problem (P_2) as a solution of the PDE (P_1) with Dirichlet boundary condition. Actually, we show that the flux is a vector valued measure. The regular part (with respect to Lebesgue measure) leaves in $\Phi(x, \nabla u)$ and the singular part is concentrated on the boundary of $D(x)$ and is connected to the tangential gradient of u through a support function of $D(x)$. Then, we present an equivalent characterization using the notion of variational solution and duality. Then, we give the proof of our main results. We consider a regularization of the problem (P_2) by taking the Yosida approximation of J, and we use compactness arguments for the proofs. Finally, we give some corollaries.

2.1 Tangential gradient

Nowadays, the Monge-Kantorovich equation which corresponds to the limit as $p \to \infty$, in the $p-$Laplacien operator is extensively used in the study of optimal mass transportation problem (cf. [13], [56]) as well as in the optimal mass transfer problem (cf. [37]). It is also used in the description of the dynamics of granular matter like the sandpile (cf. [77], [56] and [51]) and also in the deformation of polymer plastic during compression molding (cf. [15]). In this situation $A(x,\xi) = \partial I\!I_{\overline{B}(0,1)}(\xi)$,. Its study allowed the development of new useful tools like tangential gradient with respect to a Radon measure. The pioneering work in this direction which opened a possible way to manage the difficulties related to PDE with singular flux is [36], where Bouchitté, Buttazzo and Seppecher introduced a new notion of tangent space to a measure on $I\!R^n$. They use these tools in order to model the elastic energy of low-dimensional structures. One can see also the paper [39] where these tools was used for the first time in the study of the limit as $p \to \infty$ in the $p-$Laplacien equation.

As we notice in the introduction, the PDE (1) involves Lipschitz continuous functions as an energy space and vector valued measure flux. So, the standard Sobolev space as well as the standard gradient defined with respect to Lebesgue measure is not enough to handle the state equation (2.1). To overcome this difficulties, we will use the notion of tangentiel gradient introduced by Bouchitté, Buttazzo and Seppecher in [36]. For a given $\Phi \in \mathcal{M}_b(\Omega)^n$, let us consider $\gamma \in \mathcal{M}_b(\Omega)^+$ and

$\sigma \in L^1(\Omega, d\gamma)^N$ be such that $\Phi = \sigma\gamma$. Notice that this is always possible, since one can take $\gamma = |\Phi|$ and $\sigma = \dfrac{\Phi}{|\Phi|}$. Among the objective of the tangential gradient theory is to give a sense to the variation of a Lipschitz continuous function in the Lebesgue space with respect to γ, so that, if $\nabla \cdot \Phi =: \nu \in \mathcal{M}_b(\Omega)$, the integration by parts formula has a sense; i.e.

$$\int u \, d\nu = \int {}''\nabla u'' \cdot \sigma \, d\gamma,$$

for a suitable $''\nabla u''$. Thanks to [36], this is possible if the measure Φ is a tangential measure. That is $\sigma(x) \in \mathcal{T}_\gamma(x)$, $\gamma-$a.e, where $\mathcal{T}_\gamma(x) \subseteq I\!\!R^n$ is the tangential space with respect to γ. In the case where γ coincides with the $k-$dimensional Hausdorff measure on a smooth $k-$dimensional manifold $S \subset I\!\!R^n$, $\mathcal{T}_\gamma(x)$ coincides $\gamma-$a.e. with the usual tangent bundle T_S given by differential geometry. In general, it coincides with

$$\mathcal{T}_\gamma(x) = \gamma - ess \cup \left\{ \sigma(x) \, ; \, \sigma \in L^1_\gamma(\Omega)^n, \ \nabla \cdot (\sigma \, \gamma) \in \mathcal{M}_b(\Omega) \right\}.$$

Here, the $\gamma-$essential union is defined as a $\gamma-$measurable closed multifunction given by

- if $\sigma \in L^1_\gamma(\Omega)^n$ and $\nabla \cdot (\sigma\gamma) \in \mathcal{M}_b(\Omega)$, then $\sigma(x) \in \mathcal{T}_\gamma(x)$, for $\gamma-$a.e. $x \in \Omega$.
- between all the multi-functions with the previous property, the γ essential union is minimal with respect to the inclusion $\gamma-$a.e.

Now, denoting by $P_\gamma(x)$ the orthogonal projection on $\mathcal{T}_\gamma(x)$, for $\gamma-$a.e $x \in \Omega$, we have

Proposition 2.1 (cf. [38]). *The linear operator $u \in C^1(\Omega) \to P_\gamma(x)\nabla u(x) \in L^\infty_\gamma(\Omega)^n$ can be extended uniquely to a continuous linear operator :*

$$\nabla_\gamma \ : \ Lip(\Omega) \to \nabla_\gamma u \in L^\infty_\gamma(\Omega)^n,$$

where $Lip(\Omega)$ is equipped with the uniform convergence on a bounded subsets of $Lip(\Omega)$ and $L^\infty_\gamma(\Omega)^n$ with the weak star topology. Then, $\nabla_\gamma u$ is called the tangential gradient of u with respect to γ.

For the integration by parts formula, we have

Proposition 2.2 (cf. [38]). *For any $\gamma \in \mathcal{M}_b(\Omega)^+$ and $\sigma \in L^1(\Omega, d\gamma)^n$ such that*

$\sigma(x) \in \mathcal{T}_\gamma(x)$, $\gamma-a.e$ and $\nabla \cdot (\sigma\, \gamma) =: \mu \in \mathcal{M}_b(\Omega)$, we have

$$\int u\, d\mu = \int_\Omega \sigma \cdot \nabla_\gamma u\, d\gamma, \quad \text{for any } u \in Lip(\Omega).$$

The question now is to identify the set of vector valued Radon measure for which the integration by parts formula is true. Thanks to the previous proposition, let us define

$$\mathcal{M}_{\mathcal{T}}(\Omega) = \left\{ \lambda = \sigma\gamma\, ;\, \gamma \in \mathcal{M}^+(\Omega), \sigma(x) \in \mathcal{T}_\gamma(x),\ \gamma - a.e \right\}.$$

The so called tangential space of Ω.

Proposition 2.3. *[cf. [38]] Let $\lambda \in \mathcal{M}_b(\Omega)^n$ be given. Then, $\lambda \in \mathcal{M}_{\mathcal{T}}(\Omega)$ if and only if there exists $\Phi \in L^1(\Omega)^n$ such that $\nabla \cdot \lambda = \nabla \cdot \Phi$ in $\mathcal{D}'(\Omega)$.*

For any $\mathcal{X} \in \mathcal{M}_b(\Omega)^n$, we denote by $\mathcal{X}_r\mathcal{L}^n + \mathcal{X}_s$ the Radon-Nicodym decomposition of the vector valued measure $\mathcal{X}$ with respect to $\mathcal{L}^n$. So, $\mathcal{X} \in \mathcal{S}(\mu)$ is equivalent to say that

$$\int_\Omega \nabla\xi \cdot \mathcal{X}_r\, dx + \int_\Omega \nabla\xi\ \ d\mathcal{X}_s = \int_\Omega \xi\ \ d\mu \quad \text{for any } \xi \in \mathcal{C}^1_0(\Omega).$$

Using the previous propositions, we have the following integration by parts formula for the vector valued measure of $\mathcal{S}(\mu)$, which involves the singular part.

Lemma 2.1. *Let $\mu \in \mathcal{M}_b(\Omega)$ and $\mathcal{X} \in \mathcal{M}_b(\Omega)^n$ be given. Then, $\mathcal{X} \in \mathcal{S}(\mu)$ if and only if*

$$\int_\Omega \nabla\xi(x) \cdot \mathcal{X}_r(x)\, dx + \int_\Omega \nabla_{|\mathcal{X}_s|}\xi\, d\mathcal{X}_s = \int_\Omega \xi\, d\mu, \quad \text{for any } \xi \in C^1_0(\Omega).$$

2.2 Technical lemmas

Thanks to the assumption (J2), the set $D(x)$ is convex for any $x \in \Omega$. For any $x \in \Omega$, let us denote by $S_{D(x)}$ the support function of $D(x)$, given by

$$S_{D(x)}(p) = \sup \Big\{ p \cdot q\, ;\, q \in D(x) \Big\}, \quad \text{for any } (x,p) \in \Omega \times I\!R^n.$$

Recall that, for any $x \in \Omega$, the function $\xi \in I\!\!R^n \to S_{D(x)}(\xi)$ is a nonnegative, convex and positively homogeneous function. So, thanks to [14] (see also [12]), for any $\Phi \in \mathcal{M}_b(\Omega)^n$, the Radon measure $S_{D(.)}(\Phi) \in \mathcal{M}_b(\Omega)$ is well defined by the following formula

$$S_{D(.)}(\Phi)(B) = \int_B S_{D(x)}(\Phi_s(x))\, dx = \int_B S_{D(x)}\left(\frac{\Phi_s(x)}{|\Phi_s(x)|}\right) d|\Phi_s(x)|,$$

for any Borel set $B \subseteq \Omega$.

Moreover, if $\Phi \ll \lambda$, for a given $\lambda \in \mathcal{M}_b(\Omega)^+$, then

$$S_{D(.)}(\Phi)(B) = \int_B S_{D(x)}\left(\frac{d\Phi}{d\gamma}(x)\right) d\gamma(x) \quad \text{for any Borel set } B \subseteq \Omega.$$

In particular, for any $\Phi \in \mathcal{M}_b(\Omega)^n$, we have

$$S_{D(.)}(\Phi)(B) = \int_B S_{D(x)}\left(\frac{\Phi(x)}{|\Phi(x)|}\right) d|\Phi(x)| \quad \text{for any Borel set } B \subseteq \Omega.$$

Proposition 2.4. *Let $\gamma \in \mathcal{M}_b(\Omega)^+$, $g \in \mathcal{C}(\partial\Omega)$ and $\sigma \in L^1(\Omega, d\gamma)^n$ be such that $\sigma(x) \in \mathcal{T}_\gamma(x)$, $\gamma-$a.e $x \in \Omega$. If $u \in W^{1,\infty}(\Omega)$, $u = g$, $\mathcal{L}^{n-1}$-a.e. on $\partial\Omega$, and $\nabla u(x) \in D(x)$, $\mathcal{L}^n-$a.e. $x \in \Omega$, then*

1. *There exists a sequence $(u_\epsilon)_{\epsilon>0}$ in $\mathcal{D}(\Omega)$, such that $\nabla u_\epsilon(x) \in D(x)$, for any $x \in \Omega$ and $u_\varepsilon \to u$ in $W^{1,\infty}$-weak.*
2. *We have*

$$\sigma(x) \cdot \nabla_\gamma u(x) \leq S_{D(x)}(\sigma(x)), \quad \gamma - a.e\ x \in \Omega. \tag{2.2}$$

Proof : First, let us prove the result for $g = 0$. Following the same idea of the proof of Lemma 3.2 [68], for a given $\epsilon > 0$, we consider the application $I_\epsilon : I\!\!R \to I\!\!R$, defined by

$$I_\epsilon(r) = \begin{cases} 0 & \text{if } |r| \leq \epsilon \\ r - sign(r)\,\epsilon & \text{if } |r| > \epsilon. \end{cases}$$

Then, we choose

$$\tilde{u}_\epsilon = I_\epsilon(u), \qquad \text{a.e. in } \Omega.$$

One sees that $\tilde{u}_\epsilon$ is compactly supported in Ω. Moreover, there exists $0 < \alpha < 1$ and

$\epsilon_0 > 0$, such that

$$z_\epsilon = \tilde{u}_\epsilon * \rho_{\alpha\epsilon} \quad \in \mathcal{D}(\omega), \qquad \text{for any } 0 < \epsilon < \epsilon_0,$$

where $\omega \subset\subset \Omega$. Now, for any $x \in I\!R^n$, let us consider the dual function of $S_{D(x)}$ given by

$$S^*_{D(x)}(q) = \max\Big\{ q \cdot p \,;\, S_{D(x)}(p) \leq 1 \Big\}.$$

Recall that $q \in D(x)$ if and only if $S^*_{D(x)}(q) \leq 1$. Now, arguing like in the proof of Lemma 3.1 [25]) (see also the proof of Lemma 1 of [65]), we consider

$$\omega(\delta) := \sup\left\{ \left| S^*_{D(x)}(A) - S^*_{D(y)}(A) \right| \; ; \; |x - y| \leq \delta \text{ and } |A| \leq \|\nabla u\|_\infty \right\},$$

the uniform modulus of continuity $x \to S^*_{D(x)}(A)$. Then, we set

$$u_\epsilon := \frac{1}{1 + \omega(\alpha\epsilon)} z_\epsilon \in \mathcal{D}(\Omega).$$

Then, it is not difficult to see that $u_\varepsilon \to u$ in $W^{1,\infty}$-weak. And, moreover

$$S^*_{D(x)}(\nabla u_\varepsilon(x)) \leq 1.$$

Indeed, using Jensen inequality, we have

$$\begin{aligned} S^*_{D(x)}(\nabla u_\varepsilon(x)) &\leq \frac{1}{1 + \omega(\alpha\epsilon)} \int \rho_{\alpha\epsilon}(x - y) S^*_{D(x)}(\nabla u(y))\, dy \\ &\leq \frac{1}{1 + \omega(\alpha\epsilon)} \int \rho_{\alpha\epsilon}(x - y) S^*_{D(y)}(\nabla u(y))\, dy \\ &\quad + \frac{1}{1 + \omega(\alpha\epsilon)} \int \rho_{\alpha\epsilon}(x - y) \left(S^*_{D(x)}(\nabla u(y)) - S^*_{D(y)}(\nabla u(y)) \right) \, dy \\ &\leq 1. \end{aligned}$$

Now, we see that, for any open subset $B \subset \Omega$, we have

$$\begin{aligned}\int_B \sigma\cdot\nabla_\gamma u\,d\gamma &= \lim_{\varepsilon\to 0}\int_B \sigma\cdot\nabla_\gamma u_\varepsilon\,d\gamma\\ &= \lim_{\varepsilon\to 0}\int_B \sigma\cdot\nabla u_\varepsilon\,d\gamma\\ &\le \lim_{\varepsilon\to 0}\int_B S_{D(x)}(\sigma(x))\,S^*_{D(x)}(\nabla u_\varepsilon(x))\,d\gamma(x)\\ &\le \int_B S_{D(x)}(\sigma(x))d\gamma(x).\end{aligned}$$

Now, for general g, we take $\tilde g \in \mathcal{C}^1(\Omega)$ be such that $\tilde g = g$ in $\partial\Omega$. Then $\tilde u = u - \tilde g \in W_0^{1,\infty}(\Omega)$.
Denote by $\tilde D(x) := \{q - \nabla g(x); q \in D(x)\}$. Then, $\tilde D(x)$ is convex and $\nabla\tilde u \in \tilde D(x)$. It's easy to see that

$$S_{\tilde D(x)}(\sigma(x)) + \sigma(x)\cdot\nabla\tilde g(x) = S_{D(x)}(\sigma(x).$$

Apply the result when $g = 0$, there exists a sequence $(\tilde u_\epsilon)_{\epsilon>0}$ in $\mathcal{D}(\Omega)$, such that $\nabla\tilde u_\epsilon(x) \in \tilde D(x)$, for any $x \in \Omega$ and $\tilde u_\varepsilon \to \tilde u$ in $W^{1,\infty}$-weak.
Now, take $u_\epsilon = \tilde u_\epsilon + \tilde g$ we get $\nabla u_\epsilon(x) \in D(x)$, for any $x \in \Omega$ and $u_\varepsilon \to u$ in $W^{1,\infty}$-weak. Moreover,

$$\begin{aligned}\int_B \sigma\cdot\nabla_\gamma u\,d\gamma &= \lim_{\varepsilon\to 0}\int_B \sigma\cdot\nabla u_\varepsilon\,d\gamma\\ &= \lim_{\varepsilon\to 0}\int_B \sigma\cdot(\nabla\tilde u_\varepsilon + \nabla g)\,d\gamma\\ &= \lim_{\varepsilon\to 0}\int_B \sigma\cdot\nabla\tilde u_\varepsilon\,d\gamma + \int_B \sigma\cdot\nabla\tilde g\,d\gamma\\ &\le \int_B \big(S_{\tilde D(x)}(\sigma(x)) + \sigma(x)\cdot\nabla\tilde g(x)\big)d\gamma(x).\\ &\le \int_B S_{D(x)}(\sigma(x))d\gamma(x).\end{aligned}$$

This ends up the proof. □

Proposition 2.5. *Let $\gamma \in \mathcal{M}_b(\Omega)^+$ and $\sigma \in L^1(\Omega, d\gamma)^n$ be such that $\sigma(x) \in \mathcal{T}_\gamma(x)$, $\gamma-$a.e $x \in \Omega$. If $u \in W^{1,\infty}(\Omega)$ and $\nabla u(x) \in D(x)$, $\mathcal{L}^n-$a.e. $x \in \Omega$, then the following assertions are equivalent :*

1. $\sigma(x) \cdot \nabla_\gamma u(x) = S_{D(x)}\left(\sigma(x)\right), \quad \gamma$-a.e. $x \in \Omega$.
2. $\displaystyle\int S_{D(x)}\left(\sigma(x)\right)\, d\gamma(x) \leq \int \nabla_\gamma u \cdot \sigma \, d\gamma$.

Moreover, if $\nabla_\gamma u(x) \in \overline{D(x)}$, $\mathcal{L}^n$-a.e. $x \in \Omega$, then 1) and 2) are equivalent to

$$\sigma(x) \in \partial I\!\!I_{\overline{D(x)}}(\nabla_\gamma u(x)) \quad \gamma\text{- a.e. } x \in \Omega.$$

Proof : The proof is a simple consequence of Proposition 2.4 and the definition of $\partial I\!\!I_{\overline{D(x)}}$. □

2.3 Main results

For any $x \in \Omega$, let us denote by $S_{D(x)}$ the support function of $D(x)$, given by

$$S_{D(x)}(\phi) = \sup\Big\{\phi \cdot q\ ;\ q \in D(x)\Big\}, \quad \text{for any } (x, \phi) \in \Omega \times I\!\!R^n.$$

To set our first main result, we denote by

$$K = \Big\{z \in W^{1,\infty}(\Omega)\ ;\ \nabla z(x) \in D(x),\ \text{a.e. } x \in \Omega;\ z = g \text{ in } \partial\Omega\Big\}$$

and

$$\mathcal{H}_g = \big\{u \in W^{1,p}(\Omega) \text{ such that } u = g \text{ on } \partial\Omega\big\}.$$

See here that, in general K could be an empty set. Then it is important to assume that the function g in the following set :

$$\mathcal{G} = \{g \in C(\partial\Omega), \exists g_0 \in W^{1,\infty}(\Omega) : \nabla g_0 \in D(x); g_0 = g \text{ in } \partial\Omega\}.$$

This implies $K \neq \emptyset$.

Remark 2.1. *In general, $\mathcal{G} \neq \emptyset$. Indeed, $0 \in \mathcal{G}$.*

Remark 2.2. *If $D(x)$ is closed set then*

$$\mathcal{G} := \{g \in C(\partial\Omega), g(x) - g(y) \leq S_J(y,x)\}.$$

For any $y, x \in \Omega$, we define $S_J(y,x) = \min_{\varphi \in L_{y,x}} \int_0^1 S^*_{D(\varphi(t))}(\dot{\varphi}(t))dt$ where

$$L_{y,x} = \{\varphi \in \mathcal{C}^1[0,1], \varphi(0) = y, \varphi(1) = x\}.$$

Thanks to [60], we know that S_J is quasi-metric. The nonempty of $\mathcal{G}$ is deduced from the following proposition

Proposition 2.6. *Let $g \in \mathcal{G}$, then function g_0 defined by*

$$g_0(x) = \inf_{y \in \partial\Omega} \{S(y,x) + g(y)\}$$

satisfies

- $g_0 \in W^{1,\infty}(\Omega)$, $g_0 = g$ *in* $\partial\Omega$.
- $\nabla g_0 \in D(x)$.

Proof : *The proof of this Proposition follows by showing that g_0 is a subsolution of the Hamilton-Jacobi equation :*

$$\begin{cases} S_{D(x)}(\nabla u) = 1, \\ u = g \text{ on } \partial\Omega, \end{cases}$$

For the details we refer the reader to the Proposition 4.7 [60]. □

Remark 2.3. *If $D(x)$ is a open set, we do not know which kind of geometrical condition characterize the element of $\mathcal{G}$.*

Theorem 2.1. *Assume that J satisfies the assumptions (J1)-(J2). For any $\mu \in \mathcal{M}_b(\Omega)$ and $g \in \mathcal{G}$, the problem*

$$(P_2) \qquad \min\left\{\int_\Omega J(x, \nabla z(x))\, dx - \int_\Omega z\, d\mu\, ;\; z \in \mathcal{H}_g\right\}$$

has a solution u. If, moreover J satisfies (J3), then u is a solution of (P_2) if and *only if $u \in K$ and, there exists $\Phi \in \mathcal{M}_b(\Omega)^n$ such that*

$$\Phi_r(x) \in \partial_\xi J(x, \nabla u(x)), \quad \mathcal{L}^n \text{ a.e. } x \in \Omega \tag{2.3}$$

$$\frac{\Phi_s}{|\Phi_s|}(x) \cdot \nabla_{|\Phi_s|} u(x) = S_{D(x)}\left(\frac{\Phi_s}{|\Phi_s|}(x)\right), \quad |\Phi_s|\text{- a.e. in } \Omega \tag{2.4}$$

and

$$\int_\Omega \Phi_r \cdot \nabla\xi \, dx + \int_\Omega \nabla_{|\Phi_s|}\xi \, d\Phi_s = \int_\Omega \xi \, d\mu, \qquad \text{for any } \xi \in C_0^1(\Omega). \tag{2.5}$$

If $\nabla_{|\Phi_s|} u(x) \in \overline{D(x)}$, $|\Phi_s|$- a.e. $x \in \Omega$, then (2.4) is equivalent to

$$\frac{\Phi_s}{|\Phi_s|}(x) \in \partial I\!I_{\overline{D(x)}}(\nabla_{|\Phi_s|} u(x)) \quad |\Phi_s|\text{- a.e.} x \in \Omega. \tag{2.6}$$

Roughly speaking (2.4) with the fact that $\nabla u(x) \in D(x)$, $\mathcal{L}^n$-a.e. $x \in \Omega$, is a generalized formulation of the standard formulation (2.6).
We fix $\tilde{g} \in \mathcal{C}^1(\Omega)$ be such that $\tilde{g} = g$ in $\partial\Omega$ and we denote by

$$T(g, \psi) = \int_\Omega \nabla\tilde{g} d\psi - \int_\Omega \tilde{g} d\mu$$

Theorem 2.2. *Let $\mu \in \mathcal{M}_b(\Omega)$, $g \in \mathcal{G}$. Under the assumptions (J1-J3), the problem (P_3)*

$$\min\left\{\int_\Omega J^*(x, \psi_r(x))\, dx + \int_\Omega S_{D(x)}\left(\frac{\psi_s(x)}{|\psi_s(x)|}\right)\, d|\psi_s(x)| - T(g, \psi)\, ;\ \psi \in \mathcal{S}(\mu)\right\}$$

has a solution $\Phi \in \mathcal{S}(\mu)$. Moreover, Φ is a solution of problem (P_3) if and only if there exists $u \in K$ such that (u, Φ) is a weak solution of the problem (P_1).

Remark 2.4. *Roughly speaking, the solution of (P_3) depends only on the trace of $\tilde{g}$ on $\partial\Omega$ which is equal to g. Indeed, $T(g, \psi) = \int_{\partial\Omega} g\, \psi \cdot n dS$. See Remark 0.1 in Chapter 0 Introduction.*

2.4 Regularization problem and estimates

We fix $p > n$, we consider the Yosida approximation of J :

$$J_\lambda(x,\delta) = \min_{y\in R^n}\left\{ J(x,y) + \frac{1}{p\lambda^{p-1}}\left\| y-\delta \right\|^p \right\}.$$

Lemma 2.2. *We have $J_\lambda(x,.)$ is convex, C^1-function and its gradient $\nabla_\xi J_\lambda$ is Lipchitzs continuous. Moreover, for each $\delta \in \mathbb{R}^n$ we have*

$$\lim_{\lambda\to 0} J_\lambda(x,\delta) = J(x,\delta) \qquad \textit{for all } x \in \Omega.$$

Proof : We consider

$$g(\delta,y) := J(x,y) + \frac{1}{p\lambda^{p-1}}\left\| y-\delta \right\|^p,$$

for each δ, $g(\delta,.)$ is a strictly convex function, so it has at most one minimum. Since $g(\delta,.)$ is 1-coercive and l.s.c, it has one minimum. There exists y_λ^δ such that

$$J_\lambda(x,\delta) = J(x,y_\lambda^\delta) + \frac{1}{p\lambda^{p-1}}\left\| y_\lambda^\delta-\delta \right\|^p. \tag{2.7}$$

Therefore, $J_\lambda(x,.)$ is differentiable and its gradient is

$$\nabla_\xi J_\lambda(x,\delta) = \frac{1}{\lambda^{p-1}}\left\| y_\lambda^\delta-\delta \right\|^{p-2}(\delta - y_\lambda^\delta),$$

(c.f. [63] Theorem 3.4.1 and the same argument as Example 3.4.4).

Let $\delta \in D(x)$, remark that $J(x,\xi) \geq 0$ and using (2.7) we get

$$\frac{1}{p\lambda^{p-1}}\left\| y_\lambda^\delta-\delta \right\|^p \leq J_\lambda(x,\delta) \leq J(x,\delta),$$

which implies that $\left\| y_\lambda^\delta-\delta \right\|^p \leq J(x,\delta)p\lambda^{p-1}$. Then $\lim_{\lambda\to 0} y_\lambda^\delta = \delta$. So, using the l.s.c of $J(x,.)$ we get

$$J(x,\delta) \leq \liminf_{\lambda\to 0} J(x,y_\lambda^\delta).$$

Moreover, from (2.7) we also have

$$J(x, y_\lambda^\delta) \leq J_\lambda(x,\delta) \leq J(x,\delta).$$

Then we can conclude that $\lim\limits_{\lambda\to 0} J_\lambda(x,\delta) = J(x,\delta)$.

Let $\delta \notin D(x)$, we prove that $\lim\limits_{\lambda\to 0} J_\lambda(x,\delta) = +\infty$. We argue by contradiction, if we assume that there is a subsequence $\{\lambda_k\}$ such that $J_{\lambda_k}(x,\delta) \leq C$. Again, from (2.7) we get $\lim\limits_{\lambda_k\to 0} y_{\lambda_k}^\delta = \delta$. Then

$$+\infty = J(x,\delta) \leq \liminf_{\lambda_k\to 0} J(x, y_{\lambda_k}^\delta) \leq \liminf_{\lambda_k\to 0} J_{\lambda_k}(x,\delta) \leq C.$$

This is a contradiction. □

Now, we consider the functional

$$\begin{aligned} \mathcal{J}_\lambda :\ L^2(\Omega) \to\ & [0,+\infty) \\ u \mapsto\ \mathcal{J}_\lambda(u) &= \begin{cases} \displaystyle\int_\Omega J_\lambda(x,\nabla u)dx & \text{if } J_\lambda(.,\nabla u) \in L^1(\Omega) \\ +\infty & \text{otherwise} \end{cases} \end{aligned}$$

First, we begin with the regular minimization problem :

$$\min\left\{\int_\Omega J_\lambda(x,\nabla z) - \int_\Omega z d\mu\ ;\ z \in \mathcal{H}_g\right\}. \tag{2.8}$$

Lemma 2.3. *For any $\lambda > 0$, there exists $u_\lambda \in \mathcal{H}_g$ solution of the problem (2.8). Moreover $w_\lambda := \nabla_\xi J_\lambda(x,\nabla u) \in L^1(\Omega)^n$ satisfies the PDE*

$$-\nabla\cdot w_\lambda = \mu \quad \text{in } \Omega. \tag{2.9}$$

Proof : Let us consider the functional

$$z \in W^{1,p}(\Omega) \to \mathcal{I}(z) = \int_\Omega J_\lambda(x,\nabla z) - \int_\Omega z d\mu.$$

Since J_λ is convex, C^1, bounded below and is coercive, the functional $z \in W^{1,p}(\Omega) \to$

$\int_\Omega J_\lambda(x, \nabla z(x))\, dx$ is lower semi-continuous. Thus $\mathcal{I}$ is l.s.c. Moreover, since $\mathcal{H}_g$ is closed, the minimization problem (2.8) has a solution $u_\lambda \in \mathcal{H}_g$. The function J_λ satisfies the growth condition then we have the second assertion in the thesis of Lemma. See Theorem 1.14 . □

2.5 Compactness and passage to the limit

Lemma 2.4. *The sequences $(u_\lambda)_{\lambda>0}$ and $(w_\lambda)_{\lambda>0}$ are bounded in $W^{1,p}(\Omega)$ and $L^1(\Omega)^n$, respectively. Moreover,*

1. *there exists $C = C(\Omega, p, \mu, g_0)$ bounded as $p \to \infty$, such that*

$$\frac{1}{2^p}\int_\Omega (|\nabla u_\lambda| - M(x))^{+p} \leq C(\Omega, p, \mu, g_0). \tag{2.10}$$

2. *for any $\xi \in \mathcal{C}(\Omega)$, such that $\xi(x) \in D(x)$, for any $x \in \Omega$, we have*

$$\int_\Omega \xi\, d\Phi_\lambda \leq \int_\Omega J(x,\xi)\, dx + \int_\Omega (u_\lambda - g_0)\, d\mu. \tag{2.11}$$

Proof : First, let us see that

$$\frac{1}{2^{p-1}}\,|\xi|^p \leq p\lambda^{p-1} J_\lambda(x,\xi) + |M(x)|^p\,. \tag{2.12}$$

We see that

$$J_\lambda(x,\xi) \geq \frac{1}{p\lambda^{p-1}}((|\xi| - M(x))^+)^p. \tag{2.13}$$

Indeed, for a given $x \in \Omega$, any $\xi \in \mathbb{R}^n$, there exist $y \in \mathbb{R}^n$, $|y| \leq M(x)$, such that

$$J_\lambda(x,\xi) = J(y) + \frac{1}{p\lambda^{p-1}}\,|\xi - y|^p$$

Using the assumption (J1) and the fact that $|y| \leq M(x)$, we get :

$$\begin{aligned} J_\lambda(x,\xi) &\geq ((|y|-M(x))^+)^p + \frac{1}{p\lambda^{p-1}}\Big|\,|\xi|-|y|\,\Big|^p \\ &\geq \frac{1}{p\lambda^{p-1}}\Big|\,|\xi|-|y|\,\Big|^p \\ &\geq \frac{1}{p\lambda^{p-1}}((|\xi|-M(x))^+)^p. \end{aligned}$$

Then (2.12) is simple consequence of (2.13) and the inequality :

$$\frac{1}{2^{p-1}}\,|\xi|^p \leq ((|\xi|-M(x))^+)^p + |M(x)|^p$$

Now, since $J_\lambda(x,.)$ is convex, for any $x \in \Omega$, we have

$$J_\lambda(x,\nabla u_\lambda) \leq \nabla_\xi J_\lambda(x,\nabla u_\lambda)\cdot\nabla u_\lambda, \qquad \text{for any } \lambda > 0.$$

Then

$$\begin{aligned} 0 \leq \int_\Omega J_\lambda(x,\nabla u_\lambda) &\leq \int_\Omega \nabla_\xi J_\lambda(x,\nabla u_\lambda)\cdot(\nabla u_\lambda - \nabla g_0) + \int_\Omega J_\lambda(x,\nabla g_0) \\ &\leq \int_\Omega (u_\lambda - g_0)\,d\mu + \int_\Omega J_\lambda(x,\nabla g_0) \end{aligned} \tag{2.14}$$

where g_0 is in $\mathcal{G}$. We have ∇g_0 in $D(x)$ then $\int_\Omega J_\lambda(x,\nabla g_0)$ is bounded. Moreover $u_\lambda - g_0$ in $W_0^{1,p}(\Omega)$, for $C(\Omega,p)$ the constant of Poincaré inequality and $C(\Omega)$, the constant of the continuous embedding of $L^\infty(\Omega)$ in $W^{1,p}(\Omega)$, we get

$$\begin{aligned} \int_\Omega (u_\lambda - g_0)\,d\mu &\leq \|u_\lambda - g_0\|_{L^\infty(\Omega)}\,\mu(\Omega) \\ &\leq C(\Omega)\,\|u_\lambda - g_0\|_{W^{1,p}(\Omega)}\,\mu(\Omega) \\ &\leq C(\Omega,p)C(\Omega)\,\|\nabla u_\lambda - \nabla g_0\|_{L^p}\,\mu(\Omega). \end{aligned}$$

Hence

$$\int_\Omega (u_\lambda - g_0)\, d\mu \quad \leq \quad C_p \left(\| |\nabla u_\lambda| + |\nabla g_0| \|_{L^p} \right), \tag{2.15}$$

where $C_p := C(\Omega, p)C(\Omega)\mu(\Omega)$. Using (2.13) we get

$$\frac{1}{2^{p-1}} \int_\Omega (|\nabla u_\lambda| - M(x))^{+p} \ \leq \ \int_\Omega J_\lambda(x, \nabla u_\lambda) \leq \int_\Omega (u_\lambda - g_0)\, d\mu + \int_\Omega J_\lambda(x, \nabla g_0),$$

then by using (2.15) and Young inequality we get

$$\begin{aligned} & \frac{1}{2^{p-1}} \int_\Omega (|\nabla u_\lambda| - M(x))^{+p} \\ \leq \ & C_p \left(\| |\nabla u_\lambda| + |\nabla g_0| \|_{L^p} \right) + C \\ \leq \ & C_p \left(\left\| (|\nabla u_\lambda| - M)^+ \right\|_{L^p} + \| |\nabla g_0| + M \|_{L^p} \right) + C \\ \leq \ & C_p \left(\frac{\epsilon^p}{p} \int_\Omega (|\nabla u_\lambda| - M(x))^{+p} + \frac{1}{\epsilon^{p'} p'} + \| |\nabla g_0| + M \|_{L^p} \right) + C. \end{aligned}$$

This implies that

$$\begin{aligned} \frac{1}{2^{p-1}} \int_\Omega (|\nabla u_\lambda| - M(x))^{+p} \ \leq \ & \frac{\epsilon^p C_p}{p} \int_\Omega (|\nabla u_\lambda| - M(x))^{+p} \\ & + C_p \left(\frac{1}{\epsilon^{p'} p'} + \| |\nabla g_0| + M \|_{L^p} \right) + C. \end{aligned}$$

Taking $\epsilon^p = \dfrac{p}{2^p C_p}$, we have $\epsilon^{p'} = \dfrac{1}{2} \left(\dfrac{p}{2C_p} \right)^{\frac{1}{p-1}}$ and

$$\frac{1}{2^p} \int_\Omega (|\nabla u_\lambda| - M(x))^{+p} \leq C_p \left(\frac{1}{\frac{1}{2}\left(\frac{p}{2C_p}\right)^{\frac{1}{p-1}}} \, \frac{p-1}{p} + \| |\nabla g_0| + M \|_{L^p} \right) + C.$$

$$\frac{1}{2^p} \int_\Omega (|\nabla u_\lambda| - M(x))^{+p} \leq C_p \left(\frac{(p-1) 2^{\frac{p}{p-1}} C_p^{\frac{1}{p-1}}}{p^{\frac{p}{p-1}}} + \| |\nabla g_0| + M \|_{L^p} \right) + C.$$

Since the Poincaré constant $C(\Omega, p)$ is bounded as p tend to $+\infty$ (cf. [45]) and $g_0 \in W^{1,\infty}(\Omega)$, we deduce that there exists $C = C(\Omega, p, \mu, g_0)$ bounded as $p \to \infty$, such that

$$\frac{1}{2^p}\int_\Omega (|\nabla u_\lambda| - M(x))^{+p} \leq C(\Omega, p, \mu, g_0). \tag{2.16}$$

In particular, this implies that ∇u_λ is bounded in $L^p(\Omega)^n$. Using the fact that $u_\lambda = g$, $\mathcal{L}^{n-1}-$a.e. $\partial\Omega$, we deduce that u_λ is bounded in $W^{1,p}(\Omega)$. To prove that $(w_\lambda)_{\lambda>0}$ is bounded in $L^1(\Omega)^n$, recall that, for any $\xi \in I\!R^n$ and a.e. $x \in \Omega$, we have

$$\begin{aligned} w_\lambda(x)\cdot\xi &\leq J_\lambda(x,\xi) + w_\lambda(x)\cdot\nabla u_\lambda(x) - J_\lambda(x, \nabla u_\lambda(x)) \\ &\leq J_\lambda(x,\xi) + w_\lambda(x)\cdot\nabla u_\lambda(x). \end{aligned}$$

This implies that

$$\begin{aligned} \int_\Omega w_\lambda(x)\cdot\xi\,dx &\leq \int_\Omega J_\lambda(x,\xi)\,dx + \int_\Omega w_\lambda(x)\cdot\nabla u_\lambda(x)dx \\ &\leq \int_\Omega J_\lambda(x,\xi)\,dx + \int_\Omega (u_\lambda - g_0)\,d\mu. \end{aligned}$$

In one hand, using (2.15) and ∇u_λ is bounded in $L^p(\Omega)^n$, we see that $\int_\Omega (u_\lambda - g_0)d\mu$ is bounded. On the other hand, we see that $\int_\Omega J(x,\xi)dx$ is bounded for any $\xi \in B(0,\alpha)$. This implies that $w_\lambda \cdot \xi$ is bounded in $L^1(\Omega)$, for any $\xi \in B(0,\alpha)$. Here thanks to assumption (J_3), $\alpha > 0$ is given, such that $B(0,\alpha) \subseteq D(x)$ This implies that w_λ is bounded in $L^1(\Omega)^n$. Indeed, it's enough to take $\xi = \dfrac{\alpha w_\lambda}{2\,|w_\lambda|}$. This ends up the proof. □

Lemma 2.5. *There exists $(u,\Phi) \in W^{1,p}(\Omega) \times \mathcal{M}_b(\Omega)^n$ and a subsequence that we denote again by $\lambda \to 0$, such that*

$$u_\lambda \to u, \qquad \text{in } W^{1,p}(\Omega)\text{- weak} \tag{2.17}$$

and

$$\omega_\lambda \to \Phi, \qquad \text{in } \mathcal{M}_b(\Omega)^n\text{- weak}^*. \tag{2.18}$$

Moreover, $u = g$ *in* $\partial\Omega$ *and we have*

1. *The measure* Φ *satisfies* $-\nabla \cdot \Phi = \mu, \quad \text{in } \Omega$
2. *For any* $\xi \in I\!R^n$, $\varphi \in \mathcal{D}(\Omega)$, $z \in \mathcal{C}^1(\Omega)$ *and* $\lambda_0 > 0$, *we have*

$$\begin{aligned}\int_\Omega J(x,\xi)\varphi \;\; &\geq \;\; \int_\Omega J_{\lambda_0}(x,\nabla u)\varphi + \int_\Omega \varphi\,(\xi - \nabla z) d\Phi + \int_\Omega \varphi\,(z-u)\,d\mu \\ &\quad + \int_\Omega \nabla\varphi\,(u-z) d\Phi.\end{aligned}$$

3. *Denoting the limit* (u,Φ) *by* (u_p,Φ_p), *the sequence* $(u_p,\Phi_p)_{p\geq 1}$ *satisfies*
   (a) *for any* $p \geq 1$,

$$\frac{1}{2^p}\int_\Omega (|\nabla u_p| - M(x))^{+p} \leq C(\Omega,p,\mu,g_0).,\qquad (2.19)$$

   where $C(\Omega,p,\mu,g_0)$ *is given by Lemma 2.4.*

   (b) *for any* $\xi \in D(x)$,

$$\int_\Omega \xi\, d\Phi_p \leq \int_\Omega J(x,\xi)\,dx + \int_\Omega (u_p - g_0)\,d\mu.\qquad (2.20)$$

Proof : Thanks to Lemma 2.4, there exist u in $W^{1,p}(\Omega)$, $\Phi \in \mathcal{M}_b(\Omega)^n$ and a subsequence such that (2.22) and (2.23) are fulfilled. Moreover, we see that for g_0 as in Lemma 2.4, we have $(u_\lambda - g_0)$ in $W_0^{1,p}(\Omega)$. Using the fact that $W_0^{1,p}(\Omega)$ is weakly closed in $W^{1,p}(\Omega)$ we get $u - g_0$ in $W_0^{1,p}(\Omega)$, which implies $u = g$ in $\partial\Omega$.

By using the Rellich-Kondrachov Theorem [Theorem 9.16 [41]], we get

$$u_\lambda \to u, \quad \text{in } C(\bar{\Omega}) \text{ strongly}$$

and

$$\int_\Omega u_\lambda d\mu \to \int_\Omega u\, d\mu.$$

Recall that for any $\varphi \in \mathcal{D}(\Omega)$ such that $\varphi \geq 0$, we have

$$\int_\Omega J(x,\xi)\varphi \geq \int J_\lambda(x,\nabla u_\lambda)\varphi + \int_\Omega \omega_\lambda(\xi - \nabla u_\lambda)\varphi.$$

Since, for any $(x,\xi) \in \Omega \times I\!R^n$, $(J_\lambda(x,\xi))_{\lambda\geq 0}$ is nondecreasing with respect to λ, for any $0<\lambda\leq\lambda_0$, we have :

$$\begin{aligned}\int_\Omega J_\lambda(x,\xi)\varphi &\geq \int_\Omega J_{\lambda_0}(x,\nabla u_\lambda)\varphi+\int_\Omega \omega_\lambda\cdot(\xi-\nabla u_\lambda)\varphi\\ &\geq \int_\Omega J_{\lambda_0}(x,\nabla u_\lambda)\varphi+\int_\Omega \omega_\lambda\cdot(\xi-\nabla z)\varphi+\int_\Omega \omega_\lambda\cdot\nabla(\varphi\,(u_\lambda-z))\\ &\quad+\int_\Omega \omega_\lambda\cdot\nabla\varphi\,(u_\lambda-z)\\ &\geq \int_\Omega J_{\lambda_0}(x,\nabla u_\lambda)\varphi+\int_\Omega \omega_\lambda\cdot(\xi-\nabla z)\varphi-\int_{\overline{\Omega}} \varphi\,(u_\lambda-z)df_\lambda\\ &\quad+\int_\Omega \omega_\lambda\cdot\nabla\varphi\,(u_\lambda-z). \end{aligned} \tag{2.21}$$

Moreover, since $J_{\lambda_0}(.,\xi)$ is convex, l.s.c. and nondecreasing, we have

$$\int_\Omega J_{\lambda_0}(x,\nabla u_p)\varphi\leq\liminf_{\lambda\to 0}\int_\Omega J_{\lambda_0}(x,\nabla u_\lambda)\varphi.$$

So, letting $\lambda\to 0$ in (2.21), we get (2.19). The last part of the lemma follows by using (2.16) and (2.11).

□

Lemma 2.6. *Let $n\leq q<\infty$, and $(u_p,\Phi_p)_{p\geq q}$ be the sequence given by Lemma 2.5. There exists $(u,\Phi)\in W^{1,\infty}(\Omega)\times\mathcal{M}_b(\Omega)^n$ and, a subsequence that we denote again by $p\to\infty$, such that*

$$u_p\to u,\qquad \text{in } W^{1,q}(\Omega)\text{- weak,} \tag{2.22}$$

and

$$\Phi_p\to\Phi,\qquad \text{in } \mathcal{M}_b(\Omega)^n\text{- weak}^*. \tag{2.23}$$

Moreover, we have

1. *The measure Φ satisfies $-\nabla\cdot\Phi=\mu$, in Ω*
2. *For any $\xi\in I\!R^n$, $\varphi\in\mathcal{D}(\Omega)$ and $\lambda_0>0$, we have*

$$\int_\Omega J(x,\xi)\varphi \;\geq\; \int_\Omega J_{\lambda_0}(\nabla u)\varphi+\int_\Omega \Phi_r\cdot(\xi-\nabla u)\,\varphi dx+\int_\Omega \varphi(\xi-\nabla_{|\Phi_s|}u)d\Phi_s.$$

Proof : Thanks to (2.20), it is clear that the sequence $(\Phi_p)_{p\geq q}$ is bounded in $\mathcal{M}_b(\Omega)^n$ and (2.23) holds to be true. As to the sequence $(u_p)_{p\geq q}$, using Holder inequality and (2.20) we have

$$\begin{aligned} \frac{1}{2^q}\int_\Omega (|\nabla u_p| - M(x))^{+q} &\leq \left(\frac{1}{2^p}\int_\Omega (|\nabla u_p| - M(x))^{+p}\right)^{q/p} |\Omega|^{\frac{p-q}{p}} \\ &\leq C(\Omega, p, \mu, g_0)^{q/p}|\Omega|^{\frac{p-q}{p}}. \end{aligned} \tag{2.24}$$

Using the fact that $C(\Omega, p, \mu, g_0)$ is bounded as $p \to \infty$, we deduce that $(u_p)_{p\geq q}$ is bounded in $W^{1,q}(\Omega)$ and (2.22) holds to be true. By using the Rellich-Kondrachov Theorem [Theorem 9.16 [41]], we get

$$u_p \to u, \quad \text{in } C(\bar{\Omega}) \text{ strongly}$$

and

$$\int_\Omega u_p d\mu \to \int_\Omega u\, d\mu.$$

Moreover, letting $p \to \infty$ in (2.24) we get

$$\frac{1}{2^q}\int_\Omega (|\nabla u| - M(x))^{+q} \leq |\Omega|$$

This implies that $u \in W^{1,\infty}(\Omega)$

Recall that letting $\lambda \to 0$ in (2.21), we get

$$\begin{aligned} \int_\Omega J(x,\xi)\varphi &\geq \int_\Omega J_{\lambda_0}(x, \nabla u_p)\varphi + \int_\Omega \varphi\,(\xi - \nabla z) d\Phi_p + \int_\Omega \varphi\,(z-u)\,d\mu \\ &\quad + \int_\Omega \nabla\varphi\,(u-z) d\Phi_p. \end{aligned}$$

In particular, taking $z = u_\epsilon$ where $(u_\epsilon)_{\epsilon>0}$ is a sequence of Lipschitz function which converges uniformly to u, letting $p \to \infty$ we get

$$\int_\Omega J(x,\xi)\varphi \geq \int_\Omega J_{\lambda_0}(x,\nabla u)\varphi + \int_\Omega \varphi\Phi_r\cdot(\xi - \nabla u)dx + \lim_{\epsilon\to 0}\int_\Omega \varphi\,(\xi - \nabla u_\epsilon)d\Phi_s,$$

where $\Phi = \Phi_r\mathcal{L}^n + \Phi_s$ is the Radon-Nikodym decomposition of the measure Φ. Now, since $\mu \in \mathcal{M}_b(\Omega) \subset \{\nabla\cdot\sigma, \sigma \in L^1(\Omega)^n\}$ and $-\nabla\cdot\Phi_s = \mu + \nabla\cdot\Phi_r$, we deduce that

$\Phi_r \in \mathcal{M}_{\mathcal{T}}(\Omega)$. So

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\Omega \varphi(\xi-\nabla u_\epsilon)d\Phi_s &= \int_\Omega \varphi\xi d\Phi_s - \lim_{\epsilon\to 0}\int_\Omega \varphi\frac{\Phi_s}{|\Phi_s|}\cdot\nabla u_\epsilon d|\Phi_s| \\ &= \int_\Omega \varphi\xi d\Phi_s - \lim_{\epsilon\to 0}\int_\Omega \varphi\frac{\Phi_s}{|\Phi_s|}\cdot P_{|\Phi_s|}\nabla u_\epsilon d|\Phi_s| \\ &= \int_\Omega \varphi\xi d\Phi_s - \int_\Omega \varphi\frac{\Phi_s}{|\Phi_s|}\cdot\nabla_{|\Phi_s|} u d|\Phi_s| \\ &= \int_\Omega \varphi(\xi-\nabla_{|\Phi_s|}u)d\Phi_s.\end{aligned}$$

This ends up the proof of the Lemma. □

Lemma 2.7. *Under the assumptions of Lemma 2.6, let us consider the couple $(u,\Phi)\in W^{1,\infty}(\Omega)\times\mathcal{M}_b(\Omega)^n$ given by Lemma 2.6. We have $u\in K$ and*

1. $\Phi_r(x)\in\partial J(x,\nabla u(x)) \quad \mathcal{L}^n-a.e.\ \Omega$
2. $\frac{\Phi_s}{|\Phi_s|}(x)\cdot\nabla_{|\Phi_s|}u(x) = S_{D(x)}\left(\frac{\Phi_s}{|\Phi_s|}(x)\right), \quad |\Phi_s|\text{- } a.e.\ x\in\Omega.$

Proof : Thanks to Lemma 2.6, for any $\xi\in I\!R^n$ and $\varphi\in\mathcal{D}(\Omega),\varphi\geq 0$ we have

$$\int_\Omega J(x,\xi)\varphi \geq \int_\Omega J_{\lambda_0}(x,\nabla u)\varphi + \int_\Omega \varphi\Phi_r\cdot(\xi-\nabla u)dx + \int_\Omega \varphi(\xi-\nabla_{|\Phi_s|}u)d\Phi_s. \quad (2.25)$$

This implies that

$$J(x,\xi)\geq J_{\lambda_0}(x,\nabla u(x)) + (\xi-\nabla u(x))\cdot\Phi_r \qquad \mathcal{L}^n \quad p.p. \quad \Omega.$$

Hence, for any $x\in\Omega$, $J_{\lambda_0}(x,\nabla u(x))$ is bounded in Ω with respect to λ_0. This implies that

$$\nabla u(x)\in D(x), \qquad \text{for a.e.} \quad x\in\Omega. \quad (2.26)$$

So, letting $\lambda_0\to 0$ in (2.25) and using Fatou lemma, we get

$$\int_\Omega J(x,\xi)\varphi \geq \int_\Omega J(x,\nabla u)\varphi + \int_\Omega \varphi\Phi_r\cdot(\xi-\nabla u)dx + \int_\Omega \varphi(\xi-\nabla_{|\Phi_s|}u)d\Phi_s, \quad (2.27)$$

for any $\varphi\in\mathcal{D}(\Omega),\varphi\geq 0$ and $\xi\in I\!R^n$. In one hand, this implies that, for any $\xi\in I\!R^n$, we have

$$J(x,\xi) \geq J(x,\nabla u(x)) + (\xi - \nabla u(x)) \cdot \Phi_r(x) \quad \mathcal{L}^n - a.e. \quad x \in \Omega \; ;$$

so that $\Phi_r(x) \in \partial J(x,\nabla u)$, $\mathcal{L}^n$- a.e. $x \in \Omega$. On the other hand, thanks (2.27) we see that for any $\xi \in \overline{D(x)}$

$$\left(\xi - \nabla_{|\Phi_s|} u\right) \cdot \frac{\Phi_s}{|\Phi_s|} \leq 0, \quad |\Phi_s|\text{-a.e. in } \Omega.$$

This implies that

$$\frac{\Phi_s}{|\Phi_s|}(x) \cdot \nabla_{|\Phi_s|} u(x) \geq S_{D(x)}\left(\frac{\Phi_s}{|\Phi_s|}(x)\right), \quad |\Phi_s|\text{- a.e. in } \Omega$$

Combining this with the result of Proposition 2.4, we deduce the second part of the lemma.

□

2.6 Existence of solution

Now we have the definition of weak solution of stationary problem as we expect

Definition 2.1. *The couple $(u,\Phi) \in K \times M_b(\Omega)^n$ is called the weak solution of (2) if and only if*

$$\Phi_r(x) \in \partial_\xi J(x,\nabla u(x)), \quad \mathcal{L}^n \, a.e. \; x \in \Omega,$$

$$\frac{\Phi_s}{|\Phi_s|}(x) \cdot \nabla_{|\Phi_s|} u(x) = S_{D(x)}\left(\frac{\Phi_s}{|\Phi_s|}(x)\right), \quad |\Phi_s|\text{- } a.e. \; x \in \Omega,$$

and

$$\int_\Omega \Phi_r \cdot \nabla \xi + \int_\Omega \nabla_{|\Phi_s|} \xi \, d\Phi_s = \int_\Omega \xi \, d\mu, \quad \text{for any } \xi \in \mathcal{C}_0^1(\Omega).$$

Lemma 2.8. *For any $\mu \in \mathcal{M}_b(\Omega)$, the problem*

$$\min\left\{\int_\Omega J(x,\nabla z(x))\,dx + \frac{1}{2}\int_\Omega z^2(x)\,dx - \int_\Omega z\,d\mu \; ; \; z \in \mathcal{H}_g\right\},$$

has at most one solution.

Proof :

$$\mathcal{I}(z) = \int_\Omega J(x, \nabla z(x))\, dx + \frac{1}{2}\int_\Omega z^2(x)\, dx - \int_\Omega z\, d\mu.$$

Suppose that u_1 and u_2 are two solution of minimization problem. We denote by $v = \dfrac{u_1 + u_2}{2}$ and we have :

$$\begin{aligned} \mathcal{I}(v) &= \int_\Omega J(x, \nabla \frac{u_1 + u_2}{2})\, dx + \frac{1}{2}\int_\Omega (\frac{u_1 + u_2}{2})^2(x)\, dx - \int_\Omega \frac{u_1 + u_2}{2}\, d\mu \\ &\le \frac{\mathcal{I}(u_1) + \mathcal{I}(u_2)}{2}. \end{aligned}$$

From this we get $u_1 = u_2$ a.e. □

Proof of Theorem 2.1 : First, thanks to Lemma 2.7, the problem (P_1) has a solution (u, Φ). For any $\xi \in \mathcal{H}_g$, we have

$$\begin{aligned} \int_\Omega (\xi - u) d\mu &= \int_\Omega \Phi_r \cdot \nabla(\xi - u) + \int_\Omega \nabla_{|\Phi_s|}(\xi - u) d\Phi_s \\ &\le \int_\Omega J(x, \nabla \xi) - J(x, \nabla u). \end{aligned}$$

This implies that u is solution of (P_2).

For the converse part, let v be a solution of (P_2). Let us denote by h the measure given by

$$h = \mu + v\, \mathcal{L}^n.$$

It is not difficult to see that v is also a solution of

$$\min\left\{ \int_\Omega J(x, \nabla z(x))\, dx + \frac{1}{2}\int_\Omega z^2(x)\, dx - \int_\Omega z\, dh\, ;\; z \in \mathcal{H}_g \right\}, \tag{2.28}$$

Thanks to Lemma 2.8, the problem (2.28) has at most one solution. Thus v is the unique solution. Now, following the same arguments of Section 2, let us consider the regularization J_λ and the regularization problem

$$\min\left\{ \int_\Omega J_\lambda(x, \nabla z(x))\, dx + \frac{1}{2}\int_\Omega z^2(x)\, dx - \int_\Omega z\, dh\, ;\; z \in \mathcal{H}_g \right\}. \tag{2.29}$$

Following the same arguments of Lemma 2.3, (2.29) has a solution u_λ, and $w_\lambda :=$

$\partial_\xi J(x, \nabla u_\lambda)$ satisfies

$$\begin{cases} -\text{div}\omega_\lambda = h - u_\lambda & \text{in } \Omega \\ u_\lambda = g & \text{on } \partial\Omega. \end{cases}$$

The sequence u_λ is bounded in $L^2(\Omega)$, with implies that $h - u_\lambda\, \mathcal{L}^n$ is bounded in $\mathcal{M}_b(\Omega)$. Thanks to Lemma 2.4, there exists $(u, \omega) \in W^{1,\infty}(\Omega) \times \mathcal{M}_b(\Omega)^n$ and a subsequence that we denote again by λ, such that, as $\lambda \to 0$, we have

$$u_\lambda \to u \text{ in } W^{1,p}(\Omega)\text{- weak}$$

and

$$\omega_\lambda \to w \text{ in } \mathcal{M}_b(\Omega)^n\text{- weak* .}$$

Moreover, $u \in K$ and the measure w satisfies

$$-\nabla \cdot w = h - u, \quad \text{in } \Omega.$$

And, setting $w = \Phi_r \mathcal{L}^n + \Phi_s$, we have $\Phi_r(x) \in \partial J(x, \nabla u(x))$, $\mathcal{L}^n$ − a.e. $x \in \Omega$ and $\frac{\Phi_s}{|\Phi_s|}(x) \cdot \nabla_{|\Phi_s|} u(x) = S_{D(x)} \left(\frac{\Phi_s}{|\Phi_s|}(x) \right)$, $|\Phi_s|$- a.e. in Ω. So, thanks to the first part of the proof, it follows that u is a solution of the problem (2.28). By uniqueness, we get $u = v$, so that $h - u = \mu$ and we conclude that (v, Φ) is a weak solution of (P_1). □

2.7 Dual problem and equivalences

Lemma 2.9. *For any $\mu \in \mathcal{M}_b(\Omega)$, if $(u, \Phi) \in W^{1,\infty}(\Omega) \times \mathcal{M}_b(\Omega)^n$ is a weak solution of the problem (P_1), then Φ is a solution of problem (P_3).*

Proof : Taking $(u, \Phi) \in W^{1,\infty}(\Omega) \times \mathcal{M}_b(\Omega)^n$ is a weak solution of the problem (P_1)

and $\psi \in \mathcal{S}(\mu)$. In one hand, thanks to (2.3) and (2.4), we have

$$\begin{aligned}
&\int_\Omega J^*(x,\Phi_r) + \int_\Omega J(x,\nabla u) + \int_\Omega S_{D(x)}\left(\frac{\Phi_s(x)}{|\Phi_s(x)|}\right) d|\Phi_s(x)| - T(g,\Phi) \\
= &\int_\Omega \Phi_r \cdot \nabla u + \int_\Omega \nabla_{|\Phi_s|} u d\Phi_s - T(g,\Phi) \\
= &\int_\Omega u d\mu.
\end{aligned}$$

On the other hand, since $\psi \in \mathcal{S}(\mu)$, by using Proposition 2.4, we get

$$\begin{aligned}
\int_\Omega u d\mu &= \int_\Omega \psi_r \cdot \nabla u + \int_\Omega \nabla_{|\psi_s|} u d\psi_s - T(g,\psi) \\
&\leq \int_\Omega J^*(x,\psi_r) + \int_\Omega J(x,\nabla u) + \int_\Omega S_{D(x)}\left(\frac{\psi_s(x)}{|\psi_s(x)|}\right) d|\psi_s(x)| - T(g,\psi).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\int_\Omega J^*(x,\Phi_r) + \int_\Omega S_{D(x)}\left(\frac{\Phi_s(x)}{|\Phi_s(x)|}\right) d|\Phi_s(x)| - T(g,\Phi) \\
\leq &\int_\Omega J^*(x,\psi_r) + \int_\Omega S_{D(x)}\left(\frac{\psi_s(x)}{|\psi_s(x)|}\right) d|\psi_s(x)| - T(g,\psi).
\end{aligned}$$

Since $\Phi \in \mathcal{S}(\mu)$ and $\psi \in \mathcal{S}(\mu)$ is arbitrary, we deduce that

$$\begin{aligned}
&\int_\Omega J^*(x,\Phi_r) + \int_\Omega S_{D(x)}\left(\frac{\Phi_s(x)}{|\Phi_s(x)|}\right) d|\Phi_s(x)| - T(g,\Phi) \\
= &\min_{\psi\in\mathcal{S}(\mu)} \left\{ \int_\Omega J^*(x,\psi_r) + \int_\Omega S_{D(x)}\left(\frac{\psi_s(x)}{|\psi_s(x)|}\right) d|\psi_s(x)| - T(g,\psi) \right\}.
\end{aligned}$$

□

Lemma 2.10. *For any $\mu \in \mathcal{M}_b(\Omega)$, if Φ is a solution of (P_3), then exists u such that the couple $(u,\Phi) \in W^{1,\infty}(\Omega) \times \mathcal{M}_b(\Omega)^n$ is a weak solution of (P_1).*

Proof : Let $(u, \bar{\Phi})$ be the couple given in Lemma 2.7, we have

$$
\begin{aligned}
I &:= J^*(x, \bar{\Phi}_r) + \int_\Omega J(x, \nabla u) + \int_\Omega S_{D(x)} \left(\frac{\bar{\Phi}_r(x)}{|\bar{\Phi}_r(x)|} \right) d|\bar{\Phi}_r(x)| - T(g, \bar{\Phi}) \\
&= \int_\Omega \bar{\Phi}_r \cdot \nabla u + \int_\Omega S_{D(x)} \left(\frac{\bar{\Phi}_r(x)}{|\bar{\Phi}_r(x)|} \right) d|\bar{\Phi}_r(x)| - T(g, \bar{\Phi}) \\
&= \int_\Omega \bar{\Phi}_r \cdot \nabla u + \int \nabla_{|\bar{\Phi}_r|} \, d\bar{\Phi}_r - T(g, \bar{\Phi}),
\end{aligned}
$$

so that

$$I = \int_\Omega u d\mu. \tag{2.30}$$

On the other hand, assuming that Φ is a solution of (P_3), (2.30) implies that

$$\int_\Omega J^*(x, \Phi_r) + \int_\Omega J(x, \nabla u) + \int_\Omega S_{D(x)} \left(\frac{\Phi_s(x)}{|\Phi_s(x)|} \right) d|\Phi_s(x)| - T(g, \Phi) \leq \int_\Omega u d\mu. \tag{2.31}$$

Using the fact that $\Phi \in \mathcal{S}(\mu)$, $\Phi_r \cdot \nabla u \leq J^*(x, \Phi_r) + J(x, \nabla u)$ a.e. in Ω and $\frac{\Phi_s}{|\Phi_s|}(x) \cdot \nabla_{|\Phi_s|} u(x) \leq S_{D(x)} \left(\frac{\Phi_s}{|\Phi_s|}(x) \right)$, $|\Phi_s|$- a.e. in Ω, we obtain

$$
\begin{aligned}
\int_\Omega u d\mu &= \int_\Omega \Phi_r \cdot \nabla u + \int_\Omega \nabla_{|\Phi_s|} u d\Phi_s - T(g, \Phi) \\
&\leq \int_\Omega J^*(x, \Phi_r) + \int_\Omega J(x, \nabla u) + \int_\Omega S_{D(x)} \left(\frac{\Phi_s(x)}{|\Phi_s(x)|} \right) d|\Phi_s(x)| - T(g, \Phi).
\end{aligned} \tag{2.32}
$$

Thanks to (2.31) and (2.32), we have :

$$\int_\Omega \Phi_r \cdot \nabla u = \int_\Omega J^*(x, \Phi_r) + \int_\Omega J(x, \nabla u)$$

and

$$\int_\Omega \nabla_{|\Phi_s|} u d\Phi_s = \int_\Omega S_{D(x)} \left(\frac{\Phi_s(x)}{|\Phi_s(x)|} \right) d|\Phi_s(x)|$$

□

Proof of Theorem 2.2 : Thanks to Theorem 1 there exists $(u, \Phi) \in W^{1,\infty}(\Omega) \times \mathcal{M}_b(\Omega)^n$ a weak solution of (P_1). The proof is a direct consequence of Lemma 2.9 and Lemma 2.10. □

2.8 Corollaries

Corollary 2.1. *Assume that J satisfies the assumptions (J1) and (J2). Moreover, if we assume that $J(x, \xi)$ is symmetric, then u is a solution of (P_2) if and only if $u \in K$ and there exists $\Phi \in \mathcal{M}_b(\Omega)^n$ such that (2.3), (2.5) and (2.6) are fulfilled.*

Proof of Corolarry 2.1 : If $J(x, .)$ is symetric, then we have

$$\overline{D(x)} = \overline{B(0, R(x))}, \quad \text{for any } x \in \Omega,$$

where $R : \Omega \to [0, \infty]$. There fore,

$$|\nabla u(x)| \leq R(x) \quad \mathcal{L}^n \quad \text{a.e. } x \in \Omega.$$

Using Lemma 1 of [65], there exists u_ϵ a sequence in $D(\Omega)$ such that $u_\epsilon \to u \in C(\Omega)$ and $|\nabla u_\epsilon(x)| \leq R(x)$ a.e. $x \in \Omega$. In particular, this implies that

$$\left| P_{|\Phi_s|} \nabla u_\epsilon \right| \leq R(x) \quad |\Phi_s| - a.e.$$

By using the $L^\infty(\Omega, d|\Phi_s|)$- weak* continuity of the operator $\nabla_{|\Phi_s|}$ we get

$$\left| \nabla_{|\Phi_s|} u(x) \right| \leq R(x), \quad |\Phi_s| - \text{a.e. in } \Omega. \tag{2.33}$$

This implies that $\nabla_{|\Phi_s|} u(x) \in \overline{D(x)} \qquad |\Phi_s| -$ a.e. in Ω. This ends up the proof of Lemma. □

Corollary 2.2. *For any $\mu \in \mathcal{M}_b(\Omega)$, the problem (P_1) has a weak solution (u, Φ).*

In particular, by using (2.4) we can deduce the existence of a solution for variational formulation associated with the problem (P_1) as well as its equivalence with a weak formulation and the minimization problem.

Corollary 2.3. *Under the assumptions (J1-J3), let* $\mu \in \mathcal{M}_b(\Omega)$ *and* $u \in K$ *be given. Then,* u *is a solution of* (P_2) *if and only if there exists* $\Phi \in L^1(\Omega)^n$ *such that*

$$\int_\Omega \nabla(u-\xi)\cdot\Phi\,dx \leq \int_\Omega (u-\xi)\,d\mu, \qquad \textit{for any } \xi \in K. \tag{2.34}$$

The equation (2.34) will be called the variational formulation associated with (P_1) and $(u,\Phi) \in K \times L^1(\Omega)^n$ given by Corollary 2.3 is a variational solution of (P_1).

The equivalence between the three formulations is summarized in the following Corollary

Corollary 2.4. *Under the assumptions (J1-J3), let* $\mu \in \mathcal{M}_b(\Omega)$ *and* $(u,\Phi) \in K \times M_b(\Omega)^n$ *be given. The following propositions are equivalent :*

1. (u,Φ) *is a weak solution of* (P_1).
2. (u,Φ_r) *is a variational solution of* (P_1).
3. u *is a solution of the minimization problem* (P_2).

3 Evolution Problem

We consider the following equation

$$\begin{cases} \left.\begin{array}{l} \partial_t u(t) - \nabla \cdot (\Phi(t)) = \mu(t) \\ \\ \Phi \in \partial_\xi J(x, \nabla u) \end{array}\right\} & in \quad \Omega, \text{ for } t \in (0,T), \\ \\ u = 0 & \text{on } \Sigma := (0,T) \times \Gamma, \\ \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{3.1}$$

where $J : \Omega \times I\!R^n \to [0, \infty)$; $J(x, \xi)$ is continuous with respect to x, and l.s.c. with respect to ξ, and satisfies $J(x, 0) = 0$, for any $x \in \Omega$. Moreover, we assume that J satisfies the following assumptions

(J1) There exists $M(x)$ in $L^\infty(\Omega)$ such that $D(x) \subseteq \mathcal{B}(0, M(x))$ for all x in Ω.
(J2) For any $x \in \Omega$, $J(x, .)$ is convex.
(J3) $0 \in Int(D(x))$.

In following section, we begin with our main results. In Section 3.2 and 3.3, we consider a regularization problem by considering Yosida approximation of J and we use the compactness arguments to get the existence of weak solution. In Section 3.4, we prove that the weak solution also gives us a variational solution. In Section 3.5, we use the doubling and dedoubling variables technique to get the uniqueness of variational solutions. By passage to the limit of approximate solutions, we prove a contractions principle for our solutions. Then, we give the proof of our main results.

3.1 Main results

Throughout this chapter, $0 < T < \infty$, $Q = (0,T) \times \Omega$, $\Sigma = (0,T) \times \Gamma$. We denote by

$$K = \Big\{ z \in W_0^{1,\infty}(\Omega) \; ; \; \nabla z(x) \in D(x), \text{ a.e. } x \in \Omega \Big\}$$

$$K_T = \Big\{ z \in \mathcal{C}([0,T); L^2(\Omega)) \cap L^\infty(0,T; W_0^{1,\infty}(\Omega)); z(t) \in K \text{ for any } t \in [0,T) \}.$$

So, for any $u \in K_T$ and $\mu \in L^1(0,T; w^* - \mathcal{M}_b(\Omega))$ the quantity $\displaystyle\iint_Q u \, d\mu$ is well defined.

Definition 3.1. *The couple* (u, Φ) *is called a variational solution of* (3.1) *if* $u \in K_T$, $u(0) = u_0$, $\Phi \in L^1(Q)^n$, *and for any* $\xi \in K$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u(t) - \xi|^2 + \int_\Omega \Phi(t) \cdot \nabla(u(t) - \xi) \le \int_\Omega (u(t) - \xi) \, d\mu(t) \quad \text{in } \mathcal{D}'(0,T). \tag{3.2}$$

Definition 3.2. *The couple* (u, Φ) *is called a weak solution of* (3.1) *if* $u \in K_T$, $u(0) = u_0$, $\partial_t u \in L^\infty(0,T; w^* - \mathcal{M}_b(\Omega))$, $\Phi_s \in L^\infty(0,T; w^* - \mathcal{M}_b(\Omega)^n)$ *and moreover we have*

- *for* $\mathcal{L}^1$- *a.e.* $t \in [0,T)$, $\Phi_s(t) \perp \mathcal{L}^n$,

$$\Phi_r(t) \in \partial_\xi J(., \nabla u(t)), \quad \mathcal{L}^n\text{-a.e. } \Omega \tag{3.3}$$

  and

$$\frac{\Phi_s(t)}{|\Phi_s(t)|} \cdot \nabla_{|\Phi_s|} u(t) = S_{D(x)}\left(\frac{\Phi_s(t)}{|\Phi_s(t)|}\right) \quad |\Phi_s(t)|\text{- a.e. in } \Omega. \tag{3.4}$$

- *for any* $\xi \in C_0^1(\Omega)$,

$$\frac{d}{dt}\int_\Omega u(t)\,\xi + \int_\Omega \Phi_r(t) \cdot \nabla \xi + \int_\Omega \nabla_{|\Phi_s(t)|}\xi \, d\Phi_s(t) = \int_\Omega \xi \, d\mu(t), \quad \text{in } \mathcal{D}'(0,T). \tag{3.5}$$

Theorem 3.1. *For any* $u_0 \in K$ *and* $\mu \in BV(0,T; w^* - \mathcal{M}_b(\Omega))$, *(3.1) has a variational solution* (u, Φ). *Moreover, if* (u_i, Φ_i) *is a variational solution of* (P_{μ_i}), *for*

$i \in \{1,\ 2\}$, *then*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_1(t)-u_2(t)|^2 \le \int_\Omega (u_1(t)-u_2(t))\, d(\mu_1(t)-\mu_2(t)) \quad \text{in } \mathcal{D}'(0,T). \quad (3.6)$$

Theorem 3.2. *Let* $u_0 \in K$ *and* $\mu \in BV(0,T; w^* - \mathcal{M}_b(\Omega))$. *Then,* (u, Φ_r) *is the variational solution of (3.1) if and only if there exist* $\Phi_s \in L^\infty(0,T; w^* - \mathcal{M}_b(\Omega)^n)$ $\Phi_s(t) \perp \mathcal{L}^n$ *such that* (u, Φ) *is a weak solution of* (3.1)

3.2 Regularization problem

We denoted by $W^{-1,p'}(\Omega)$ the dual space of $W_0^{1,p}(\Omega)$. When $p > n$, $\mathcal{M}_b(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$ then for any $\mu \in L^1([0,T], w^* - \mathcal{M}_b(\Omega))$, there exist $\{f_i\}_{i=0}^n$, such that for any $v \in L^p(\Omega)$, we have

$$<\mu(t), v>= \int_\Omega f_0(t) v + \sum_{i=1}^n \int_\Omega f_i(t) \frac{\partial v}{\partial x_i}$$

where $f_i(t) \in L^{p'}(\Omega)$ (see Proposition 9.20 [41]). Using that fact $\mu \in L^1([0,T], w^* - \mathcal{M}_b(\Omega))$ we get $f_i \in L^1([0,T], w^* - \mathcal{M}_b(\Omega))$. By taking

$$f_\lambda(t,x) = f_0(t,x) * \rho_\lambda - \sum_{i=1}^n \frac{\partial}{\partial x}(f_i(t,x) * \rho_\lambda),$$

where ρ_λ is a sequence of mollifiers, we have $f_\lambda \in C^\infty(Q)$ and $f_\lambda(t) \to \mu(t)$ in $\mathcal{M}_b(\Omega)$- weak* . Moreover,

$$\frac{f_\lambda(t+h) - f_\lambda(t)}{h} = \frac{(f_0(t+h) - f_0(t)) * \rho_\lambda(t)}{h} - \sum_{i=1}^n \frac{\frac{\partial}{\partial x_i}(f_i(t+h) - f_i(t)) * \rho_\lambda(t)}{h}.$$

Then we have

$$\frac{f_\lambda(t+h) - f_\lambda(t)}{h} \to \frac{\mu(t+h) - \mu(t)}{h} \text{ in } \mathcal{M}_b(\Omega)\text{- weak}^* .$$

So if $\mu \in BV(0,T, w * - \mathcal{M}_b(\Omega))$, we have $V(f_\lambda, T)$ is bounded. □

Lemma 3.1. *For any* $f_\lambda \in W^{1,1}(0,T,L^2(\Omega))$*; there exists* $u_\lambda \in L^\infty([0,T];W^{1,p}(\Omega)) \cap W^{1,\infty}([0,T]);L^2(\Omega))$ *and* $\omega_\lambda = \nabla_\xi J_\lambda(x,\nabla u_\lambda) \in L^\infty([0,T],L^2(\Omega))$ *such that* ω_λ *is a solution of :* $\partial_t u_\lambda - \nabla\cdot\omega_\lambda = f_\lambda$ *in* Q *where* $u_\lambda(0) = u_0 \in K$.
Moreover for $S \in C^1$, S *convex and* $S(0) = 0$ *we have*

$$\frac{d}{dt}\int_\Omega S(u_\lambda - v_\lambda)dx \le \int_\Omega (f_\lambda - g_\lambda)S'(u_\lambda - v_\lambda)dx. \tag{3.7}$$

In particular,

$$\frac{d}{dt}\int_\Omega |u_\lambda - v_\lambda| \;\; \le \;\; \int_\Omega |f_\lambda - g_\lambda|\, dx. \tag{3.8}$$

Proof : The first part of lemma can be concluded by using the theory of parabolic equation of divergence type (see Proposition 5.7 [21]), and the evolution equation of type Leray-Lions (cf. 7.1 [41]). Remark that, we have $\nabla_\xi J_\lambda(x,\xi) \le C\frac{1}{\lambda^{p-1}}|\xi|^{p-1}$.
We take $S_n \in C^2$ such that $S_n \to S$ and $S'_n \to S'$. We have $u_\lambda - v_\lambda \in W^{1,\infty}(0,T;L^2(\Omega))$, then take $S'_n(u_\lambda - v_\lambda) \in C^1$ as a test function of

$$\partial_t(u_\lambda - v_\lambda) - \nabla\cdot\nabla J_\lambda(x,\nabla u_\lambda) - \nabla\cdot\nabla J_\lambda(x,\nabla v_\lambda) = f_\lambda - g_\lambda.$$

we get

$$\int_\Omega \partial_t(u_\lambda - v_\lambda)S'_n(u_\lambda - v_\lambda) + \int_\Omega (\nabla J_\lambda(x,\nabla u_\lambda) - \nabla J_\lambda(x,\nabla v_\lambda))\cdot\nabla S'_n(u_\lambda - v_\lambda)$$
$$= \int_\Omega (f_\lambda - g_\lambda)S'_n(u_\lambda - v_\lambda).$$

Using the fact that

$$(\nabla J_\lambda(x,\nabla u_\lambda) - \nabla J_\lambda(x,\nabla v_\lambda))\cdot(\nabla u_\lambda - \nabla v_\lambda)S''_n(u_\lambda - v_\lambda) \ge 0$$

we get

$$\int_\Omega \partial_t(u_\lambda - v_\lambda)S'_n(u_\lambda - v_\lambda) \le \int_\Omega (f_\lambda - g_\lambda)S'_n(u_\lambda - v_\lambda).$$

Then,

$$\int_\Omega \partial_t(u_\lambda - v_\lambda)S'(u_\lambda - v_\lambda) \leq \int_\Omega (f_\lambda - g_\lambda)S'(u_\lambda - v_\lambda)$$

which implies

$$\int_\Omega \frac{d}{dt}S(u_\lambda - v_\lambda)dx \leq \int_\Omega (f_\lambda - g_\lambda)S'(u_\lambda - v_\lambda)dx.$$

The last part follows by taking $S' = H_\epsilon$, where

$$H_\epsilon(r) = \begin{cases} 1 \text{ if } r \geq \epsilon \\ \dfrac{r}{\epsilon} \text{ if } -\epsilon \leq r \leq \epsilon \\ -1 \text{ if } -\epsilon \geq r \end{cases}$$

and letting $\epsilon \to 0$. □

Lemma 3.2. *We suppose that* $\mu \in BV(0, T, w^* - \mathcal{M}_b(\Omega))$ *and* $\nabla \cdot \nabla J(x, \nabla u_0) \in L^1(\Omega)$, *if* u_λ *is the solution in Lemma 3.1, then* $\partial_t u_\lambda$ *is bounded in* $L^\infty(0, T; w^* - \mathcal{M}_b(\Omega))$.

Proof : We follow the same idea as Lemma 3.4 [68]. We see that $u_\lambda(t + h)$ is the solution of $-\nabla \cdot \nabla J_\lambda(x, \nabla u_\lambda(\cdot + h)) = f_\lambda(\cdot + h) - \partial_t u_\lambda(\cdot + h)$ in $(0, T - h) \times \Omega$ and $u_0 = u_\lambda(h)$. Then we apply (3.8), we have

$$\int_\Omega |u_\lambda(t) - u_\lambda(t + h)| \leq \int_\Omega |u_0 - u_\lambda(h)| + \int_0^t \int_\Omega |f_\lambda(s) - f_\lambda(s + h)|\, ds.$$

We have u_0 is the solution of $-\nabla \cdot \nabla J_\lambda(x, \nabla u_\lambda(t)) = -\nabla \cdot \nabla J_\lambda(x, \nabla u_0) - \partial_t u_\lambda$. By applying again (3.8) we have :

$$\begin{aligned} \int_\Omega |u_0 - u_\lambda(h)| &\leq \int_0^h \int_\Omega |\nabla \cdot \nabla J_\lambda(x, \nabla u_0) - f_\lambda(t)| \\ &\leq \int_0^h \int_\Omega |f_\lambda(t)| + h \left\| \nabla \cdot \nabla J_\lambda(x, \nabla u_0) \right\|_{L^1(\Omega)}. \end{aligned}$$

Then we get :

$$\int_\Omega |u_\lambda(t) - u_\lambda(t+h)| \;\; \leq \;\; \int_0^h \int_\Omega |f_\lambda(t)| + h \left\| \nabla \cdot \nabla J_\lambda(x, \nabla u_0) \right\|_{L^1(\Omega)} + \int_0^T \int_\Omega |f_\lambda(s) - f_\lambda(s+h)|\, ds.$$

Dividing by h and letting $h \to 0$, we obtain

$$\|\partial_t u_\lambda\|_{L^\infty(0,T;w^* - \mathcal{M}_b(\Omega))} \leq \|f_\lambda(0)\|_{L^\infty(\Omega)} + \|\nabla \cdot \nabla J_\lambda(x, \nabla u_0)\|_{L^1(\Omega)} + V(f_\lambda, T) \tag{3.9}$$

Using the fact that $V(f_\lambda, T)$ is bounded. We conclude that $\partial_t u_\lambda$ is bounded by C and this constant not depends on p. □

Lemma 3.3. *Under the assumption of Lemma 3.2,* $(u_\lambda)_{\lambda \geq 0}$ *is bounded in* $L^\infty(0,T;W^{1,p}(\Omega))$ *and* $(w_\lambda)_{\lambda \geq 0}$ *is bounded in* $L^\infty(0,T;L^1(\Omega)^n)$.

Proof : For a.e. t in $[0,T]$, using Sobolev embedding and inequalities as Lemma 2.4, we have

$$\int_\Omega u_\lambda(t) f_\lambda(t) dx \;\; \leq \;\; C_p \|\nabla u_\lambda(t)\|_{L^p(\Omega)} \|f_\lambda(t)\|_{L^\infty(\Omega)} \tag{3.10}$$

and

$$\int_\Omega u_\lambda(t) d(\partial_t u_\lambda(t)) \;\; \leq \;\; C_p \|\nabla u_\lambda(t)\|_{L^p(\Omega)} \|\partial_t u_\lambda(t)\|_{\mathcal{M}_b(\Omega)} \tag{3.11}$$

where $C_p := C(\Omega) C(\Omega, p)$. Moreover, we have

$$\frac{1}{2^{p-1}} \int_\Omega (|\nabla u_\lambda(t)| - M(x))^{+p} \;\; \leq \;\; \int_\Omega J_\lambda(x, \nabla u_\lambda(t)) \leq \int_\Omega u_\lambda(t) f_\lambda(t) - \int_\Omega u_\lambda(t) d\partial_t u_\lambda(t),$$

then by using (3.10) and (3.11) we get

$$\begin{aligned} \frac{1}{2^{p-1}} \int_\Omega (|\nabla u_\lambda| - M(x))^{+p} \;\; &\leq \;\; C_p \|\nabla u_\lambda(t)\|_{L^p} \left(\|f_\lambda(t)\|_{L^\infty(\Omega)} + \|\partial_t u_\lambda(t)\|_{\mathcal{M}_b(\Omega)} \right) \\ &\leq \;\; C_p C \left\| (|\nabla u_\lambda| - M)^+ \right\|_{L^p} + \|M\|_\infty |\Omega|^{\frac{1}{p}} \\ &\leq \;\; C_p C \left(\frac{\epsilon^p}{p} \int_\Omega (|\nabla u_\lambda| - M(x))^{+p} + \frac{1}{\epsilon^{p'} p'} + \|M\|_\infty |\Omega|^{\frac{1}{p}} \right) \end{aligned}$$

where $C = \|f_\lambda\|_{L^\infty(Q)} + \|\partial_t u_\lambda\|_{L^\infty(0,T;w^*-\mathcal{M}_b(\Omega))}$. This constant is bounded, not depend on t, p. This implies that

$$\begin{aligned}\frac{1}{2^{p-1}}\int_\Omega (|\nabla u_\lambda(t)| - M(x))^{+p} &\leq \frac{\epsilon^p C_p C}{p}\int_\Omega (|\nabla u_\lambda(t)| - M(x))^{+p} \\ &+ C_p C\left(\frac{1}{\epsilon^{p'} p'} + \|M\|_\infty |\Omega|^{\frac{1}{p}}\right).\end{aligned}$$

Taking $\epsilon^p = \dfrac{p}{2^p C_p C}$, $\epsilon^{p'} = \dfrac{1}{2}\left(\dfrac{p}{2C_p C}\right)^{\frac{1}{p-1}}$ and we have

$$\begin{aligned}&\left(\frac{1}{2^{p-1}} - \frac{1}{2^p}\right)\int_\Omega (|\nabla u_\lambda(t)| - M(x))^{+p} \\ &\leq C_p C\left(\frac{1}{\frac{1}{2}\left(\frac{p}{2C_p C}\right)^{\frac{1}{p-1}}}\,\frac{p-1}{p} + \|M\|_\infty |\Omega|^{\frac{1}{p}}\right).\end{aligned}$$

Since the Poincaré constant $C(\Omega, p)$ is bounded as p tend to $+\infty$ (cf. [45]) we deduce that there exists $C = C(\Omega, p, \mu)$ bounded as $p \to \infty$, such that

$$\frac{1}{2^p}\int_\Omega (|\nabla u_\lambda(t)| - M(x))^{+p} \leq C(\Omega, p, \mu). \tag{3.12}$$

In particular, this implies that $\nabla u_\lambda(t)$ is bounded in $L^\infty(\Omega)^n$. Then $u_\lambda(t)$ is bounded in $W_0^{1,\infty}(\Omega)$. Hence u_λ is bounded in $L^\infty([0,T]; W_0^{1,\infty}(\Omega))$. Recall that, for any $\xi \in I\!R^n$, and a.e. $x \in \Omega$ we have

$$\begin{aligned}w_\lambda(x,t)\cdot\xi &\leq J_\lambda(x,\xi) + w_\lambda(x,t)\cdot\nabla u_\lambda(x,t) - J_\lambda(x, \nabla u_\lambda(x,t)) \\ &\leq J_\lambda(x,\xi) + w_\lambda(x,t)\cdot\nabla u_\lambda(x,t).\end{aligned}$$

So for a.e. t we have

$$\begin{aligned}\int_\Omega w_\lambda(x,t)\cdot\xi\,dx &\le \int_\Omega J_\lambda(x,\xi)\,dx+\int_\Omega w_\lambda(x,t)\cdot\nabla u_\lambda(x,t)dx\\ &\le \int_\Omega J_\lambda(x,\xi)\,dx+\int_\Omega u_\lambda(t)\,(\partial_t u_\lambda(t)-f_\lambda(t))dx.\end{aligned}$$

Using the same arguments as Lemma 2.4, we obtain that $(w_\lambda)_{\lambda\ge 0}$ is bounded in $L^\infty(0,T;L^1(\Omega)^n)$. □

3.3 Existence of weak solution

Lemma 3.4. *For any $z\in K_T$, there exists $(z_\varepsilon)_{\varepsilon>0}$ a sequence in $L^\infty(0,T;\mathcal{C}^1_0(\Omega))\cap K_T$ such that, as $\varepsilon\to 0$, for any $q\ge 1$,*

$$z_\varepsilon\to z \quad \text{in } L^q(0,T;W^{1,q}_0(\Omega))-\text{weak}$$

and

$$z_\varepsilon\to z \quad \text{uniformly in} \quad Q.$$

Proof : The proof follows exactly the same ideas of the proof of Proposition 2.4. □

Lemma 3.5. *Let $u_0\in K$. For any $f\in BV(0,T;w^*-\mathcal{M}_b(\Omega))$ there exists $(u,\Phi_r,\Phi_s)\in K_T\times L^1(Q)^n\times L^\infty(0,T;w^*-\mathcal{M}_b(\Omega)^n))$, such that :*

$$u_\lambda\to u \text{ in } L^\infty(0,T;W^{1,p}(\Omega))\text{- weak}$$

$$\omega_\lambda(t)\to\Phi(t)=\Phi_s(t)\mathcal{L}^n+\Phi_s(t) \text{ in } w^*\text{ - }\mathcal{M}_b(\Omega)^n \text{ for } \mathcal{L}^1-a.e.\ t\in[0,T).$$

Moreover (u,Φ) is a weak solution of (3.1)

Proof. By using the definition of ω_λ, for any $\varphi\in\mathcal{D}(Q)$ such that $\varphi\ge 0$, we have

$$\int_Q J(x,\xi)\varphi\ge\int_Q J_\lambda(x,\nabla u_\lambda)\varphi+\int_Q \omega_\lambda\cdot(\xi-\nabla u_\lambda)\varphi.$$

Since for any $\xi \in I\!R^n$, $x \in \Omega$, $(J_\lambda(x,\xi))_{\lambda \geq 0}$ is nondecreasing, we get :

$$\begin{aligned}\int_Q J(x,\xi)\varphi &\geq \int_Q J_{\lambda_0}(x,\nabla u_\lambda)\varphi + \int_Q \omega_\lambda \cdot (\xi - \nabla u_\lambda)\varphi \\ &\geq \int_Q J_{\lambda_0}(x,\nabla u_\lambda)\varphi + \int_Q \omega_\lambda \cdot (\xi - \nabla z)\varphi + \int_Q \omega_\lambda \cdot \nabla(\varphi\,(u_\lambda - z)) \\ &\qquad + \int_Q \omega_\lambda \cdot \nabla\varphi\,(u_\lambda - z) \\ &\geq \int_Q J_{\lambda_0}(x,\nabla u_\lambda)\varphi + \int_Q \omega_\lambda(\xi - \nabla z)\varphi - \int_Q (f_\lambda - \partial_t u_\lambda)\varphi\,(u_\lambda - z) \\ &\qquad + \int_Q \omega_\lambda \cdot \nabla\varphi\,(u_\lambda - z).\end{aligned}$$

Thank to Lemma 3.3, by using Rellich-Kondrachov Theorem we get

$$u_\lambda \to u \text{ in } L^\infty([0,T];C_0(\Omega))\text{- weak .}$$

$$\omega_\lambda(t) \to \Phi(t) = \Phi_s(t)\mathcal{L}^n + \Phi_s(t) \text{ in } \text{ w* - } \mathcal{M}_b(\Omega)^n \text{ for } \mathcal{L}^1 - \text{a.e. } t \in [0,T).$$

where $\Phi(t) = \Phi_r(t)\mathcal{L}^n + \Phi_s(t)$ is the Radon-Nikodym decomposition of the measure $\Phi(t)$ for a.e. $t \in [0,T)$.
Moreover, we have $u \in L^\infty([0,T];W_0^{1,\infty}(\Omega))$.
Since

$$\partial_t u_\lambda \to \partial_t u \text{ in } L^\infty(0,T;w^* - \mathcal{M}_b(\Omega))\text{- weak *}$$

and

$$f_\lambda \to f \text{ in } L^\infty(0,T;w^* - \mathcal{M}_b(\Omega))\text{- weak * ,}$$

passing to the limit in the same way as Lemma 2.6 and Lemma 2.7, we obtain

$$\begin{aligned}\int_Q J(x,\xi)\varphi \;\; &\geq \;\; \int_Q J_{\lambda_0}(x,\nabla u)\varphi + \int_Q \varphi\Phi_r \cdot (\xi - \nabla u) \\ &\qquad + \lim_{\epsilon\to 0}\int_0^T \int_\Omega \varphi(\xi - \nabla u_\epsilon(t))d\Phi_s(t)dt\end{aligned}$$

Then for any $\xi \in D(x)$ and $\varphi \in \mathcal{D}(Q)$ we have

$$\int_Q J(x,\xi)\varphi \geq \int_Q J(x,\nabla u)\varphi + \int_Q \varphi\Phi_r\cdot(\xi-\nabla u) + \int_0^T\int_\Omega \varphi(\xi-\nabla_{|\Phi_s(t)|}u(t))d\Phi_s(t)dt.$$

This implies first that,

$$J(x,\xi) \geq J(x,\nabla u(t,x)) + (\xi-\nabla u(t,x))\cdot\Phi_r(t,x) \quad a.e. \quad \text{in } Q;$$

hence

$$\Phi \in \partial_\xi J(.,\nabla u) \qquad \mathcal{L}^{n+1} - a.e.\text{in } Q.$$

On the other hand, for any $\xi \in \overline{D(x)}$

$$\big(\xi - \nabla_{|\Phi_s(t)|}u(t)\big)\cdot\frac{\Phi_s(t)}{|\Phi_s(t)|} \leq 0, \quad |\Phi_s(t)|\text{-a.e. in } \Omega,$$

this implies that

$$\frac{\Phi_s(t)}{|\Phi_s(t)|}\cdot\nabla_{|\Phi_s(t)|}u(t) = S_{D(x)}\left(\frac{\Phi_s(t)}{|\Phi_s(t)|}\right) \qquad |\Phi_s(t)|\text{- a.e. in } \Omega.$$

As to the equation (3.5), this follows by taking ξ as a test function in $\partial_t u_\lambda(t) - \nabla\cdot\omega_\lambda(t) = f_\lambda(t)$ then passage to the limit $\lambda \to 0$.
We have $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$ and the fact that if $u \in L^q(0,T;W_0^{1,p}(\Omega))$ be such that $\dfrac{\partial u}{\partial t} \in L^q(0,T;W^{-1,p'}(\Omega))$. Then u belongs to $C([0,T];L^2(\Omega))$ (cf. Theorem 1 page 473 [48]). □

3.4 Variational solution

Lemma 3.6. *If* (u,Φ_r,Φ_s) *is a weak solution of problem* (3.1) *then* (u,Φ_r) *is a variational solution.*

Proof. For any $h > 1$ and $\epsilon > 0$, let us consider

$$u_\epsilon^h(t,x) = \frac{1}{2h}\int_{t-h}^{t+h} u_\epsilon(s,x)ds \text{ for any } (t,x) \in Q.$$

where u_ϵ is a sequence given by Lemma 3.4. It is not difficult to see that $u_\epsilon^h \in W^{1,q}(0,T;W_0^{1,\infty}(\Omega))$ and, for any $t \in (0,T)$

$$u_\epsilon^h(t) \to \frac{1}{2h}\int_{t-h}^{t+h} u(\tau)d\tau =: u_h(t), \qquad \text{uniformly in} \quad \Omega, \quad as \quad \epsilon \to 0.$$

Taking σu_ϵ^h as a test function we get :

$$\begin{aligned}\iint_Q \sigma \nabla u_\epsilon^h d\Phi_s + \iint_Q \sigma \nabla u_\epsilon^h \cdot \Phi_r \quad &= \quad \int_0^T \int_\Omega \sigma_t u u_\epsilon^h + \int_0^T \int_\Omega \sigma(t)u(t)\frac{u_\epsilon(t+h)-u_\epsilon(t-h)}{2h} \\ &\quad + \int_0^T \sigma(t)\int_\Omega u_\epsilon^h(t)d\mu(t)dt.\end{aligned}$$

Since $u \in C([0,T);L^2(\Omega))$ we get :

$$\begin{aligned} &\lim_{h\to 0}\lim_{\epsilon\to 0}\int_0^T \int_\Omega \sigma(t)u(t)\frac{u_\epsilon(t+h)-u_\epsilon(t-h)}{2h} \\ = \quad &\lim_{h\to 0}\int_0^T \int_\Omega \sigma(t)u(t)\frac{u(t+h)-u(t-h)}{2h} \\ = \quad &-\lim_{h\to 0}\int_0^T \int_\Omega u(t)u(t)\frac{\sigma(t+h)-\sigma(t-h)}{2h} \\ = \quad &-\frac{1}{2}\int_0^T \int_\Omega u^2\sigma_t.\end{aligned}$$

Then we have

$$\lim_{h\to 0}\lim_{\epsilon\to 0}\iint_Q \sigma\nabla u_\epsilon^h d\Phi_s = \iint_Q \frac{1}{2}\sigma_t u^2 + \iint_Q \sigma u d\mu - \iint_Q \sigma \nabla u \cdot \Phi_r.$$

Using the definition of $S_{D(x)}$, we have

$$\lim_{h\to 0}\lim_{\epsilon\to 0}\iint_Q \sigma(t)\nabla u_\epsilon^h(t)d\Phi_s(t)dt \le \iint_Q \sigma(t)S_{D(x)}\left(\frac{\Phi_s(t)}{|\Phi_s(t)|}\right)d\left|\Phi_s(t)\right|dt.$$

Using the fact that u satisfying (3.4), for a.e. $t \in [0,T)$,

$$\frac{\Phi_s(t)}{|\Phi_s(t)|}\cdot \nabla_{|\Phi_s(t)|}u(t) = S_{D(x)}\left(\frac{\Phi_s(t)}{|\Phi_s(t)|}\right) \qquad |\Phi_s(t)|\text{- a.e. in } \Omega$$

We deduce

$$\iint_Q \sigma(t) S_{D(x)}\left(\frac{\Phi_s(t)}{|\Phi_s(t)|}\right) = \iint_Q \frac{1}{2}\sigma_t u^2 + \iint_Q \sigma u d\mu - \iint_Q \sigma \nabla u \cdot \Phi_r. \tag{3.13}$$

Another hand, let $\xi \in K$, we can take ξ_ϵ as in Lemma 3.4, and $\sigma \in D(0,T)$. Then taking $\sigma\xi_\epsilon$ as a test function in (3.5) we get

$$\begin{aligned}\iint_Q \sigma_t u\xi + \iint_Q \sigma\xi d\mu - \iint_Q \sigma\nabla\xi \cdot \Phi_r &= \iint_Q \sigma \nabla_{|\Phi_s|}\xi d\Phi_s(t)dt \\ &\leq \iint_Q \sigma(t) S_{D(x)}\left(\frac{\Phi_s(t)}{|\Phi_s(t)|}\right) d|\Phi_s(t)|dt.\end{aligned}$$

Using (3.13) we get

$$\iint_Q \sigma_t u\xi + \iint_Q \sigma\xi d\mu - \iint_Q \sigma\nabla\xi \cdot \Phi_r \leq \iint_Q \frac{1}{2}\sigma_t u^2 + \iint_Q \sigma u d\mu - \iint_Q \sigma\nabla u \cdot \Phi_r.$$

This ends up the proof. □

3.5 Contraction

Lemma 3.7. *If (u_1, Φ_1) and (u_2, Φ_2) is the variational solution of problem (P_μ) then $u_1 = u_2$*

Proof : We use the doubling and dedoubling variable technique. Let $\sigma = \sigma(t_1, t_2) \in D(0,T)^2$ and $u_1 = u_1(t_1), u_2 = u_2(t_2)$ we have :

$$\begin{aligned}&-\frac{1}{2}\int_0^T \int_\Omega \sigma_{t_1}(t_1,t_2)|u_1(t_1) - u_2(t_2)|^2 dxdt_1 \\ &\quad + \int_0^T \int_\Omega \sigma(t_1,t_2)\Phi_1(t_1)(\nabla u_1(t_1) - \nabla u_2(t_2))dxdt_1 \\ &\qquad \leq \int_0^T \int_\Omega \sigma(t_1,t_2)(u_1(t_1) - u_2(t_2))d\mu(t_1)dt_1\end{aligned}$$

and

$$
\begin{aligned}
-\frac{1}{2}\int_0^T \int_\Omega \sigma_{t_2}(t_1,t_2)|u_1(t_1)-u_2(t_2)|^2 dxdt_2 \\
+\int_0^T \int_\Omega \sigma(t_1,t_2)\Phi_2(t_2)(\nabla u_1(t_1)-\nabla u_2(t_2))dxdt_2 \\
\leq \int_0^T \int_\Omega \sigma(t_1,t_2)(u_1(t_1)-u_2(t_2))d\mu(t_2)dt_2.
\end{aligned}
$$

Integrating the first equality w.r.t t_2, the second one w.r.t. t_1, adding them together, and using the fact that

$$
(\Phi_1(t_1)-\Phi_2(t_2))\,(\nabla u_1(t_1)-\nabla u_2(t_2)) \geq 0, \qquad \text{a.e. in } \Omega
$$

we get

$$
\begin{aligned}
-\frac{1}{2}\int_0^T\int_0^T \int_\Omega (\sigma_{t_1}(t_1,t_2)+\sigma_{t_2}(t_1,t_2))|u_1(t_1)-u_2(t_2)|^2 dxdt_1dt_2 \\
\leq \int_0^T\int_0^T \int_\Omega \sigma(t_1,t_2)(u_1(t_1)-u_2(t_2))(d\mu(t_1)-d\mu(t_2))dt_1dt_2.
\end{aligned}
$$

Let $\xi \in D(0,T)$, $\xi \geq 0$ and ρ_ϵ be a modifier function in $I\!R$. We take

$$
\sigma(t_1,t_2)=\rho_\epsilon\left(\frac{t_1-t_2}{2}\right)\xi\left(\frac{t_1+t_2}{2}\right)
$$

and we have

$$
\sigma_{t_1}(t_1,t_2)+\sigma_{t_2}(t_1,t_2)=\rho_\epsilon\left(\frac{t_1-t_2}{2}\right)\xi'\left(\frac{t_1+t_2}{2}\right).
$$

We get

$$
\begin{aligned}
-\frac{1}{2}\int_0^T\int_0^T \int_\Omega \rho_\epsilon\left(\frac{t_1-t_2}{2}\right)\xi'\left(\frac{t_1+t_2}{2}\right)|u_1(t_1)-u_2(t_2)|^2 dxdt_1dt_2 \qquad (3.14)\\
\leq \int_0^T\int_0^T \int_\Omega \rho_\epsilon\left(\frac{t_1-t_2}{2}\right)\xi\left(\frac{t_1+t_2}{2}\right)(u_1(t_1)-u_2(t_2))(d\mu(t_1)-d\mu(t_2))dt_1dt_2.
\end{aligned}
$$

Let $\epsilon \to 0$ first term of (3.14) converge to

$$-\frac{1}{2}\int_0^T\int_\Omega |u_1(t)-u_2(t)|^2\xi_t dt.$$

Now, let us prove the last one converge to 0. To that aim, we set

$$g(t,s)=\int_\Omega u_1(t)d\mu(s); \qquad f(t,s)=\int_\Omega u_2(t)d\mu(s).$$

Then we get

$$I=\int_0^T\int_0^T \rho_\epsilon\left(\frac{t_1-t_2}{2}\right)\xi\left(\frac{t_1+t_2}{2}\right)[g(t_1,t_1)-g(t_1,t_2)-f(t_2,t_1)+f(t_2,t_2)]dt_1dt_2$$

Setting $z=\dfrac{t_1-t_2}{2}$, we have

$$\begin{aligned}
I &= C\int_0^T(\int_0^T \rho_\epsilon(z)\xi(t_1-z)[-g(t_1,t_1)+g(t_1,t_1-2z) \\
&\qquad +f(t_1-2z,t_1)-f(t_1-2z,t_1-2z)]dz)dt_1 \\
&= -C\int_0^T g(t_1,t_1)[\int_0^T \rho_\epsilon(z)\xi(t_1-z)dz]dt_1 \\
&\quad +C\int_0^T[\int_0^T \rho_\epsilon(z)\xi(t_1-z)g(t_1,t_1-2z)dz]dt_1 \\
&\quad +C\int_0^T[\int_0^T \rho_\epsilon(z)\xi(t_1-z)f(t_1-2z,t_1)dz]dt_1 \\
&\quad -C\int_0^T[\int_0^T \rho_\epsilon(z)\xi(t_1-z)f(t_1-2z,t_1-2z)dz]dt_1.
\end{aligned}$$

When $\epsilon \to 0$ we have the right hand side I converge to 0. Then we can conclude that

$$-\frac{1}{2}\int_0^T\int_\Omega |u_1(t)-u_2(t)|^2\xi_t dt \le 0.$$

This implies that

$$\frac{d}{dt}\int_\Omega |u_1(t)-u_2(t)|^2 \le 0 \text{ in } D'(0,T).$$

Since $u\in C([0,T);L^2(\Omega))$ and $u_1(0)=u_2(0)$, then $u_1=u_2$, $\mathcal{L}^{n+1}-a.e$ in Q. □

Lemma 3.8. *If* (u_1, Φ_1) *and* (u_2, Φ_2) *are the variational solutions of the problem* (P_{μ_1}) *and* (P_{μ_2}) *respectively, then*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_1(t) - u_2(t)|^2 \leq \int_\Omega (u_1(t) - u_2(t))\, d(\mu_1(t) - \mu_2(t)) \quad \textit{in } \mathcal{D}'(0,T). \quad (3.15)$$

Proof : Using the uniqueness of u_1 and u_2 (see Lemma 3.7), it is enough to prove for u_1^λ and u_2^λ instead, and pass to the limit by letting $\lambda \to 0$ in (3.7). We get

$$-\int_0^T \int_\Omega S(u_1 - u_2)\varphi_t \leq \int_0^T \int_\Omega S'(u_1 - u_2) d(\mu_1 - \mu_2)\varphi(t).$$

We denote by $T_k(r) = max(-k, min(k, r))$ and taking $S'(r) = T_k(r)$ and letting $k \to \infty$ we get

$$-\int_0^T \int_\Omega \frac{1}{2}|u_1 - u_2|^2 \varphi_t \leq \int_0^T \int_\Omega (u_1 - u_2)\, d(\mu_1 - \mu_2)\varphi,$$

which imply (3.15).

□

3.6 Proof of theorems

Proof of Theorem 3.1 : The proof is a direct consequence of Lemma 3.5, Lemma 3.6 and Lemma 3.8 □

Proof of Theorem 3.2 : Thanks to Lemma 3.5 there exists $(u, \Phi_r, \Phi_s) \in K_T \times L^1(Q)^n \times L^\infty(0, T; w^* - \mathcal{M}_b(\Omega)^n))$ such that (u, Φ) is a weak solution of (3.1). The proof is a direct consequence of Lemma 3.6 and Lemma 3.7. □

4 Discrete Collapsing Sandpile Model

The dynamics of granular materials has been studied quite intensively due to their importance in various naturally occurring phenomena such as landslides, rock-falls, desert dunes evolution, sediment transport in rivers, ... and engineering transportation applications. The description of such flows still represents a major challenge for the theory. In the last decade, several studies have been devoted to the mathematical and numerical studies of granular system. Different models have been proposed using kinetic approach (cf. [23, 24]), cellular automata (cf. [50, 55, 64, 78]) or partial differential equations (cf. [8, 10, 16, 19, 20, 29, 46, 51, 52, 54, 59, 62, 66, 77]).

Granular materials are complex objects and it is important to understand their behavior by using simple prototypes. Actually, it is known that one of the approach that may be relevant for their study is based on modeling the dynamics of pile of cubes. That is, to imagine that the matter at the microscopic level consists of particles similar to cubes (in some cases, a particle can be linked to a certain volume of material) arranged on a regular grid. The principle after consists in establishing simple rules across the unit cell and repeat until the interplay between cells occurs by itself coherent structures or organized forms at the macroscopic scale. Of course, the elementary constituents of a material are so numerous that the study at the microscopic level needs probabilistic methods. However, appropriate scaling of time enables a transition to deterministic models of nonlocal type (see for instance [55] and [67]). These rescaling takes into account rigorously the fact that there is a very large number of particles and there is a significant gap between the time scales of microscopic and macroscopic.

A typical example is the growing pile of cubes (cf. [55]) which corresponds to the evolution of stack of unit cubes resting on the plane when new cubes are being added to the pile. In [55], Evans and Rezakhanlou introduce a stochastic description of the dynamics and proved that, if we randomly add more and more, smaller and smaller cubes, we obtain an interesting continuum limit, which is an evolution governed by the sub differential of a convex functional that is very connected to Prigozhin model for sandpile [77]. To that aim, they introduce an intermediate nonlinear discrete dynamics of nonlocal type at the level of cubes. By using Partial Integro-Differential Equation, N. Igbida shows in [67] that this discrete nonlocal equation gives a right deterministic description of the dynamics of a growing pile of granular structure when the component are not very small. Our aim here is to show how to use this kind of discrete equation to model the collapsing of an unstable pile of cubes.

The chapter is organized as follows : in the next section we establish our discrete model and study the existence and uniqueness of the solution. In section 3 we develop a numerical study of the model based on duality argument. At last, we give numerical simulations showing the stabilization of unstable discrete structures.

4.1 The discrete model for the collapse of a pile

It is well known by now, that the collapsing phenomena in granular materials can be described by nonlinear evolution equations governed by nondecreasing critical angles. In the continuous case, recall that combining the continuity equation of fluid dynamics and phenomenological equation N. Igbida introduce in [66] (see also [54] and [52]) a sub-gradient flow for variational problems with time dependent gradient constraints. The gradient constraints are interpreted as critical angle of sandpile. In particular, the continuous model [66] produces an evolution in time of avalanches in a drying of a sandpile, rather than instantaneous collapse. Our aim here is to introduce a discrete non local model that we can associate with such phenomena.

4.1.1 The discrete model.

We consider the surface of the pile be divided into cubes of integer point $i \in \mathbb{Z}^n, n = 1$ or 2. So, a stack of cubes can be described by an application $u : \mathbb{Z}^n \to \mathbb{R}$, where $u(i)$ describes the density of cubes at the position i.

The collapse is produced when the slope of the surface exceeds an angle of stability. In the discrete case the stability condition for a profile u reads (cf. [55] and

[67])

$$|u(i) - u(j)| \leq 1 \quad \text{for} \quad i \sim j, \tag{4.1}$$

where we use $i \sim j$ to describe $|i - j| \leq 1$. Assume that, we start with un unstable configuration represented by $u_0 : \mathbb{Z}^n \to \mathbb{R}$ such that

$$|u_0(i) - u_0(j)| > 1 \quad \text{for some} \quad i \sim j.$$

To reach a stable configuration, we assume a suitable of various avalanches are produced, so as to stabilize the pile. More precisely, we assume that the pile tends to stabilizes itself by taking a continuous sequence of intermediate profile characterized by

$$|u(i) - u(j)| \leq c(t) \quad \text{for} \quad i \sim j, \tag{4.2}$$

where $c : [0, T) \to \mathbb{R}^+$ is a given non increasing function satisfying

$$\lim_{t \to T} c(t) = 1.$$

Here, the stability constraint, forces the pile to rearrange itself to reach a stable profile. So, a suitable of various unstable configurations are produced with non increasing angle of stability that converges to 1, as $t \to T \leq \infty$.

The dynamics of the height $u(t, i)$ of the pile at a fixed point $i \in \mathbb{Z}^n$, can be derived as follows. For a small time Δt, the evolution of u is given by :

$$u(t + \Delta t, i) \simeq u(t, i) + \Delta t\, Q(t, i),$$

where $Q(t, i)$ is the rate of material arriving at the position i. We can express Q as follows

$$Q(t, i) = I(t, i) - O(t, i),$$

where, $I(t, i)$ records the material arriving to the position i from the neighborhood positions and $O(t, i)$ records the material leaving the position i towards neighborhood positions. We have

$$I(t, i) = \sum_{j: j \sim i} \alpha(t, j, i) \qquad \text{and} \qquad O(t, i) = \sum_{j: j \sim i} \alpha(t, i, j),$$

where $\alpha(t,i,j)$ records the material arriving to the position j from the neighborhood positions i. This implies that

$$\frac{u(t+\Delta t,i)-u(t,i)}{\Delta t}+\sum_{j:j\sim i}(\alpha(t,i,j)-\alpha(t,j,i))\simeq 0.$$

At each time $t>0$, we put

$$\sigma(t,i,j)=\alpha(t,i,j)-\alpha(t,j,i).$$

Obviously, σ is anti-symmetric, i.e

$$\sigma(t,i,j)=-\sigma(t,j,i).$$

Letting $\Delta t\to 0$, we obtain

$$\partial_t u(t,i)+\sum_{j:j\sim i}\sigma(t,i,j)=0.$$

To complete the model we have to give the connection between σ and u. Since the dynamics is induced by the discrete constraint (4.2), we can assume that the cubes move only when the limiting condition is turning to be exceeded. So, the dynamics in turn is concentrated on the set $X_{c(t)}(u(t))$, where

$$X_r(v):=\{(i,j)\in\mathbb{Z}^n\times\mathbb{Z}^n : |v(i)-v(j)|=r \text{ and } i\sim j\}$$

for a given $r>0$ and a given application $v\,:\,\mathbb{Z}^n\to\mathbb{R}$, so that

$$\mathit{support}(\sigma(t,.,.))\subseteq X_{c(t)}(u(t)).$$

Finally, our model is the following system :

$$(DM) \quad \begin{cases} \partial_t u(t,i) + \displaystyle\sum_{j:j\sim i} \sigma(t,i,j) = 0, \ t > 0, \ i \in \mathbb{Z}^n, \\ \\ |u(t,i) - u(t,j)| \leq c(t) \text{ for } i \sim j, \\ \\ \sigma(t,i,j) = -\sigma(t,j,i) \ \text{ and } \ support(\sigma(t,.,.)) \subseteq X_{c(t)}(u(t)). \end{cases}$$

In the case where $c(t) = 1$, for any $t \in [0,T)$, and the equation is subject to a non null source term. This model describes a growing sandpile with respect to an external source of cube. Indeed, in this case the system (DM) is the discrete model that we can associate with the continuous nonlocal model for discrete structures in $\mathbb{R}^2$ (see [67] for more details).

To study this problem, we recall the infinite-dimensional ℓ^p spaces defined by

$$\ell^p(\mathbb{Z}^n) = \begin{cases} \Big\{\eta : \ \mathbb{Z}^n \longrightarrow \mathbb{R} \ ; \ \|\eta\|_p := \Big(\displaystyle\sum_{i\in\mathbb{Z}^n} |\eta(i)|^p\Big)^{1/p} < \infty\Big\}, & \text{for } 1 \leq p < \infty \\ \\ \Big\{\eta : : \mathbb{Z}^n \longrightarrow \mathbb{R} \ ; \ \|\eta\|_\infty := \displaystyle\max_{i\in\mathbb{Z}^n} |\eta(i)| < \infty\Big\}, & \text{for } p = \infty \ . \end{cases}$$

For a given $r > 0$, we introduce the convex set

$$K(r) = \{z \in \ell^2(\mathbb{Z}^n) : |z(i) - z(j)| \leq r \quad \text{for} \quad i \sim j\}.$$

4.1.2 Existence and uniqueness of a solution.

For $\lambda > 0$, we consider $\Big(t_l\Big)_{l=1,...n}$ a $\lambda-$ discretization of $[0,T)$, that is $t_0 = 0 < t_1 < ... < t_{n-1} < T = t_n$. For any $\lambda > 0$, we say that u_λ is a $\lambda-$approximate solution of (DM), if there exists $\Big(t_l\Big)_{l=1,...n}$ a $\lambda-$ discretization of $[0,T)$, such that

$$u_\lambda(t) = \begin{cases} u_0 & \text{for } t \in \ [0,t_1], \\ \\ u_l & \text{for } t \in \ ]t_{l-1}, t_l], \ l = 2, ...n \end{cases} \tag{4.3}$$

and u_l solves the Euler implicit time discretization of (DM)

$$\left. \begin{cases} u_l(i) + \sum\limits_{j:j\sim i} \sigma_l(i,j) = u_{l-1}(i), & i \in \mathbb{Z}^n \\ \\ u_l \in K(c(t_l)), \;\; \sigma_l(i,j) = -\sigma_l(j,i), & i,j \in \mathbb{Z}^n \\ \\ \text{and} \;\; support(\sigma_l) \subseteq X_{c(t_l)}(u_l), \end{cases} \right\} \quad l = 1, ..., n. \tag{4.4}$$

See that the generic problem is given by

$$(DSP) \qquad \begin{cases} v(i) + \sum\limits_{j:j\sim i} \sigma(i,j) = g(i) & \text{for any } i \in \mathbb{Z}^n, \\ \\ v \in K(r), \;\; \sigma(i,j) = -\sigma(j,i) & \text{for any } (i,j) \in \mathbb{Z}^n \times \mathbb{Z}^n, \\ \\ \text{and} \;\; support(\sigma) \subseteq X_r(v), \end{cases}$$

where $r \geq 1$ is a given constant and $g : \mathbb{Z}^2 \to \mathbb{R}$ is a given application. In this chapter, we prove

Theorem 4.1. *Let $g \in \ell^2(\mathbb{Z}^n)$ and $v \in K(r)$. Then $v = \mathbb{P}_{K(r)}(g)$ if and only if, there exists $\sigma \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)$, such that the couple (v, σ) satisfies (DSP).*

In the Theorem $\mathbb{P}_{K(r)}(g)$ denote the standard projection onto the convex set $K(r)$. Remember that $v = \mathbb{P}_{K(r)}(g)$ if and only if $v \in K(r)$ and

$$J(v) = \frac{1}{2}\|v - g\|^2_{\ell^2(\mathbb{Z}^n)} = \min_{z \in K(r)} J(z). \tag{4.5}$$

Now, let us consider $\mathbb{I}_{K(r)}$ the convex indicator function of $K(r)$ given as

$$\mathbb{I}_{K(r)}(z) = \begin{cases} 0 & \text{if } z \in K(r) \\ \\ +\infty & \text{otherwise.} \end{cases}$$

As a consequence of Theorem 4.1, the characterization of $\partial \mathbb{I}_K$ in terms of a discrete equation is given by the following Corollary :

Corollary 4.1. *Let $g \in \ell^2(\mathbb{Z}^n)$ and $v \in K(r)$. Then, $g \in \partial I\!I_{K(r)}(v)$ if and only if there exists $\sigma \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)$, such that the couple (v, σ) satisfies*

$$\begin{cases} \sum_{j:j\sim i} \sigma(i,j) = g(i) & \text{for any } i \in \mathbb{Z}^n, \\ \\ v \in K(r), \quad \sigma(i,j) = -\sigma(j,i) & \text{for } (i,j) \in \mathbb{Z}^n \times \mathbb{Z}^n, \\ \\ \text{and } \ support(\sigma) \subseteq X_r(v). \end{cases}$$

In particular, this corollary gives the connexion between the evolution problem (DM) and the nonlinear dynamics

$$\begin{cases} u_t(t) + \partial I\!I_{K(c(t))}(u(t)) \ni 0 \text{ for } t \in (0,T) \\ \\ u(0) = u_0. \end{cases} \tag{4.6}$$

Again, for any $\lambda > 0$, we say that u_λ is a $\lambda-$approximate solution of (4.6), if there exists $\Big(t_l\Big)_{l=1,\dots n}$ a $\lambda-$ discretization of $[0,T)$, such that

$$u_\lambda(t) = \begin{cases} u_0 & \text{for } t \in \ [0, t_1], \\ \\ u_l & \text{for } t \in \,]t_{l-1}, t_l], \ l = 2, \dots n \end{cases}$$

and u_l solves the Euler implicit time discretization scheme of (4.6), that is

$$u_l = I\!P_{K(c(t_l))} u_{l-1}, \qquad \text{for } l = 1, \dots n.$$

It is clear that this problem is a particular case of the stationary problem

$$v + \partial I\!I_{K(r)}(v) \ni g \qquad\qquad \text{i.e.} \sum_{i \in \mathbb{Z}^n} (g-v)\, v = \max_{\xi \in K(r)} \sum_{i \in \mathbb{Z}^n} (g-v)\, \xi. \tag{4.7}$$

Applying Corollary 4.1, for any $g \in \ell^2(\mathbb{Z}^n)$, there exists a solution u of the problem

(DSP) in the sense that $u \in K(r)$ and

$$\sum_i \Big(g(i) - u(i)\Big)\Big(u(i) - \xi(i)\Big) \geq 0 \quad \text{for any } \xi \in K(r).$$

For the existence and uniqueness of the solution of (DM), we prove the following results

Theorem 4.2. *Assume that $c \in W^{1,\infty}(0,\ T)$, $u_0 \in K(c(0))$ and $0 < T < \infty$. Then the problem (DM) has a unique solution $u \in W^{1,1}\Big(0, T;\ \ell^2(\mathbb{Z}^n)\Big)$ and u satisfies*

$$\begin{cases} u_t(t) + \partial\mathbb{I}_{K(c(t))}(u(t)) \ni 0 \ \textit{for}\ t \in (0,T) \\ \\ u(0) = u_0. \end{cases} \tag{4.8}$$

Moreover, if u_λ is a λ- approximate solution, then

$$u_\lambda \longrightarrow u \quad \textit{in} \quad \mathcal{C}([0,T];\ell^2(\mathbb{Z}^n)) \qquad \textit{as}\ \ \lambda \longrightarrow 0.$$

Proof : It is not difficult to see that u is a solution of (4.8) if and only if $v(t) := u(t)/c(t)$ is a solution of

$$\begin{cases} v_t(t) + \partial\mathbb{I}_{K(1)}(v(t)) + F(t, v(t)) \ni 0 \text{ for } t \in (0,T) \\ \\ v(0) = u_0/c(0), \end{cases} \tag{4.9}$$

where $F(t,r) = \dfrac{c'(t)}{c(t)} r$. It is clear that, F is measurable in t and Lipschitz continuous with respect to r. Since $\dfrac{c'(t)}{c(t)} \in L^\infty(0,\ T)$, thanks to Proposition 3.13 of [40], the problem (4.9) has a unique solution $v \in W^{1,\infty}\Big(0, T;\ \ell^2(\mathbb{Z}^n)\Big)$. Then, using similar arguments of [66] combining v and u_λ, one can prove that the $\lambda-$approximate solution converges to u and this ends up the proof of the theorem. □

Proposition 4.1. *Assume that, there exists (σ, v) such that $(\sigma, v) \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n) \times$*

$K(r)$ satisfying

$$\begin{cases} v(i) + \sum_{j:j\sim i} \sigma(i,j) = g(i) & \text{for } i \in \mathbb{Z}^n, \\ \sigma(i,j) = -\sigma(j,i) & \text{for } (i,j) \in \mathbb{Z}^n \times \mathbb{Z}^n, \\ \text{and} \quad \sigma(i,j) \neq 0 \Rightarrow |v(i) - v(j)| = r & \text{for } (i,j) \in \mathbb{Z}^n \times \mathbb{Z}^n, \end{cases} \tag{4.10}$$

then v is solution of the problem (4.7).

Proof : Let $z \in K(r)$. First we see that

$$\begin{aligned} I &= \sum_i (g(i) - v(i))(v(i) - z(i)) \\ &= \sum_i \sum_{j:j\sim i} \sigma(i,j)(v(i) - z(i)) \\ &= \sum_i \sum_{j:j\sim i} \sigma(i,j)(v(i) - v(j) + v(j) - z(j) + z(j) - z(i)) \\ &= \sum_i \sum_{j:j\sim i} \sigma(i,j)(v(i) - v(j) + z(j) - z(i)) + \sum_i \sum_{j:j\sim i} \sigma(i,j)(v(j) - z(j)) \\ &= \sum_i \sum_{j:j\sim i} \sigma(i,j)(v(i) - v(j) + z(j) - z(i)) - \sum_i \sum_{j:j\sim i} \sigma(j,i)(v(j) - z(j)). \end{aligned}$$

Since $(\sigma, v) \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n) \times K(r)$ and using Fubini's theorem, we have

$$\sum_i \sum_{j:j\sim i} \sigma(j,i)(v(j) - z(j)) = \sum_j \sum_{i:i\sim j} \sigma(j,i)(v(j) - z(j)).$$

Then

$$\begin{aligned} I &= \sum_i \sum_{j:j\sim i} \sigma(i,j)(v(i) - v(j) + z(j) - z(i)) - \sum_j \sum_{i:i\sim j} \sigma(j,i)(v(j) - z(j)) \\ &= \sum_i \sum_{j:j\sim i} \sigma(i,j)(v(i) - v(j) + z(j) - z(i)) - I, \end{aligned}$$

which implies that

$$2I = \sum_i \sum_{j:j\sim i} \sigma(i,j)(v(i) - v(j) + z(j) - z(i)).$$

Now, using the condition $\sigma(i,j) \neq 0 \Rightarrow |v(i) - v(j)| = r$, then we obtain

- If $v(i) - v(j) = r$ we get $v(i) - v(j) + z(j) - z(i) \geq 0 \quad$ and $\sigma(i,j) > 0$, then

$$\sigma(i,j)(v(i) - v(j) + z(j) - z(i)) \geq 0.$$

- If $v(i) - v(j) = -r$ we have $v(i) - v(j) + z(j) - z(i) \leq 0 \quad$ and $\quad \sigma(i,j) < 0$, then

$$\sigma(i,j)(v(i) - v(j) + z(j) - z(i)) \geq 0.$$

Consequently, $I \geq 0$ and this completes the proof of Proposition. □

This proposition gives a first part of the proof of Theorem 4.1. The second and final part of the proof is given at the end of Section 4.2.1.

4.2 Numerical study

Now, our aim is to study numerical approximation of $P_{K(r)}g$. Thanks to theorem 4.1, the problem (DSP) has a unique solution v satisfying

$$v + \partial \mathbb{I}_{K(r)}(v) \ni g. \tag{4.11}$$

To give a numerical method for the approximation of the solution of (4.7), we use dual arguments. Thanks to Theorem 4.1, a solution of (DSP) is given by

$$v = \mathbb{P}_{K(r)}g, \tag{4.12}$$

4.2.1 Dual formulation

To study the numerical approximation of $P_{K(r)}g$, we use dual arguments. To this aim, we introduce the set of anti-symmetric bounded sequence defined on $\mathbb{Z}^n \times \mathbb{Z}^n$ by :

$$\ell^1_{as}(\mathbb{Z}^n \times \mathbb{Z}^n) = \left\{ \hat{\mu} \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n) \,; \hat{\mu}(i,j) = -\hat{\mu}(j,i), \text{ for any } (i,j) \in \mathbb{Z}^n \times \mathbb{Z}^n \right\}$$

and the set $\mathcal{S}_{as}(\mathbb{Z}^n \times \mathbb{Z}^n)$ of sequences of $\ell^1_{as}(\mathbb{Z}^n \times \mathbb{Z}^n)$ concentrated on the set $\Big\{(i,j) \in \mathbb{Z}^n \times \mathbb{Z}^n \ ; \ : i \sim j\Big\}$; i.e.

$$\mathcal{S}_{as} = \Big\{\hat{\mu} \in \ell^1_{as}(\mathbb{Z}^n \times \mathbb{Z}^n) \ ; \hat{\mu}(i,j) = 0 \text{ for } |i-j| > 1\Big\}.$$

Considering the operator $\Lambda : \ell^2(\mathbb{Z}^n) \longrightarrow \mathcal{C}_0(\mathbb{Z}^n \times \mathbb{Z}^n)$ defined by

$$\begin{aligned} \Lambda z : \quad & \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{R}^+ \\ & (i,j) \longrightarrow \Lambda z(i,j) = \begin{cases} z(i) - z(j) & \text{if } i \sim j \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

the problem (4.5) can be rewritten as

$$\min_{z \in \ell^2(\mathbb{Z}^n)} \Big\{J(z) + H(\Lambda z)\Big\}, \tag{4.13}$$

where the function $H : \mathcal{C}_0(\mathbb{Z}^n \times \mathbb{Z}^n) \longrightarrow \mathbb{R}^+$ is given by

$$H(\Lambda z) = \begin{cases} 0 & \text{if } \|\Lambda z\|_\infty \leq r \\ +\infty & \text{otherwise .} \end{cases}$$

Using standard duality argument (cf. [58]), we compute the dual problem associated to (4.5). This is the aim of the following proposition :

Proposition 4.2. *Let $g \in \ell^2(\mathbb{Z}^n)$. Then, the dual problem of (4.5) is given by*

$$\max_{\eta \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)} G(\eta),$$

where $G : \ \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n) \longrightarrow \mathbb{R}$ is defined by

$$G(\eta) = -\frac{1}{2}\sum_i \Big(\sum_{j:j\sim i} \eta(i,j) - \eta(j,i)\Big)^2 - \sum_i \Big(\sum_{j:j\sim i} \eta(i,j) - \eta(j,i)\Big) g(i) - r\sum_{i,j} |\eta(i,j)|. \tag{4.14}$$

Proof : Thanks to Theorem 4.2 of [58], the dual problem of (4.13) can be written as :

$$\max_{\eta^* \in \big(\mathcal{C}_0(\mathbb{Z}^n \times \mathbb{Z}^n)\big)^*} \Big\{ - J^*(\Lambda^* \eta^*) - H^*(-\eta^*)\Big\}, \tag{4.15}$$

where J^*, H^* and Λ^* are the conjugate of J, H and Λ, respectively. Recall that $\Big(\mathcal{C}_0(\mathbb{Z}^n \times \mathbb{Z}^n)\Big)^* = \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)$. First, we see that, for any $\eta \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)$, we have

$$\begin{aligned}
\langle \Lambda^*\eta, z\rangle &= \langle \eta, \Lambda z\rangle \\
&= \sum_i \sum_j \eta(i,j)\Lambda z(i,j) \\
&= \sum_i \sum_{j:j\sim i} \eta(i,j)\Big((z(i) - z(j)\Big). \\
&= \sum_i \sum_{j:j\sim i} \eta(i,j)z(i) - \sum_j \sum_{i:i\sim j} \eta(j,i)z(i) \\
&= \sum_i \sum_{j:j\sim i} \Big(\eta(i,j) - \eta(j,i)\Big)z(i),
\end{aligned}$$

where we have used the fact that $(i,j) \longrightarrow \eta(i,j)z(j)$ is in $\ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)$. This implies that

$$\begin{aligned}
\Lambda^* : \quad \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n) &\longrightarrow \quad \ell^2(\mathbb{Z}^n) \\
\eta &\longrightarrow \quad (\Lambda^*\eta)(i) \quad = \sum_{j:j\sim i} \Big(\eta(i,j) - \eta(j,i)\Big).
\end{aligned}$$

On the other hand, for any $z^* \in \ell^2(\mathbb{Z}^n)$, we have

$$\begin{aligned}
J^*(z^*) &= \sup_{z\in\ell^2(\mathbb{Z}^n)} \sum_i z^*(i)z(i) - \frac{1}{2}\sum_i |z(i) - g(i)|^2 \\
&= \sup_{z\in\ell^2(\mathbb{Z}^n)} \sum_i z^*(i)(z(i) - g(i)) - \frac{1}{2}\sum_i |z(i) - g(i)|^2 + \sum_i z^*(i)g(i) \\
&= \sup_{y\in\ell^2(\mathbb{Z}^n)} \sum_i z^*(i)y(i) - \frac{1}{2}\sum_i |y(i)|^2 + \sum_i z^*(i)g(i) \\
&= \frac{1}{2}\sum_i |z^*(i)|^2 + \sum_i z^*(i)g(i).
\end{aligned}$$

At last, for any $p^* \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)$, we have

$$\begin{aligned}
H^*(p^*) &= \sup_{p\in\ell^1(\mathbb{Z}^n\times\mathbb{Z}^n)} \sum_{i,j} p^*(i,j)p(i,j) - H(p) \\
&= \sup_{p\in\ell^1(\mathbb{Z}^n\times\mathbb{Z}^n)} \sum_{i,j} p^*(i,j)p(i,j) \text{ if } ||p||_\infty \leq r \\
&= r\sum_{i,j} |p^*(i,j)|,
\end{aligned}$$

and the proof is complete. □

Moreover, we have

Lemma 4.1. *Assume that, there exists $w \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)$ such that*

$$G(w) = \max_{\eta \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)} G(\eta),$$

then $w \in \mathcal{S}_{as}$.

Proof : We assume, by contradiction, that the maximum w of G satisfies $w(i_0, j_0) \neq 0$ at a point $(i_0 \nsim j_0)$, and take

$$\bar{w} = \begin{cases} w & \text{if } (i,j) \neq (i_0, j_0) \\ 0 & \text{if } (i_0, j_0). \end{cases}$$

A simple calculation, gives

$$\begin{aligned} G(w) &= -\frac{1}{2} \sum_i \Big(\sum_{j:j\sim i} w(i,j) - w(j,i) \Big)^2 - \sum_i \Big(\sum_{j:j\sim i} w(i,j) - w(j,i) \Big) g(i) - r \sum_{i,j} |w(i,j)| \\ &= G(\bar{w}) - r|w(i_0, j_0)|, \end{aligned}$$

then $G(\bar{w}) > G(w)$ and we get the contradiction with the maximality of G at w. Now, taking $\tilde{w} \in \mathcal{S}_{as}$ as the following :

$$\begin{aligned} \tilde{w}(i,j) &= \frac{1}{2}\Big(w(i,j) - w(j,i) \Big) \\ \tilde{w}(j,i) &= \frac{1}{2}\Big(w(j,i) - w(i,j) \Big), \end{aligned}$$

we see that $\tilde{w}(i,j) - \tilde{w}(j,i) = w(i,j) - w(j,i)$. On the other hand, we have

$$|\tilde{w}(i,j)| \leq \frac{1}{2}|w(i,j)| + \frac{1}{2}|w(j,i)|,$$

then $-\sum |\tilde{w}(i,j)| \geq -\sum |w(i,j)|$ and we deduce that

$$G(\tilde{w}) \geq G(w).$$

From this, we get

$$\max_{\eta \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)} G(\eta) = \max_{\eta \in \mathcal{S}_{as}} G(\eta).$$

This completes the proof of Lemma. □

As consequence of Proposition 4.2 and Lemma 4.1, we have the following result :

Theorem 4.3. *Let $g \in \ell^2(\mathbb{Z}^n)$ and $v := \mathbb{P}_{K(r)}(g)$. Then, there exists $w \in \mathcal{S}_{as}$ and $v \in K(r)$ such that*

$$G(w) = \max_{\eta \in \mathcal{S}_{as}} G(\eta) = \min_{z \in K(r)} J(z) = J(v).$$

Moreover, for any $i \in \mathbb{Z}^n$,

$$v(i) = g(i) + \sum_{j : j \sim i} \Big(w(i,j) - w(j,i) \Big).$$

Proof : Thanks to proposition 4.2, we have

$$J(v) = \min_{z \in K(r)} J(z) = \max_{\eta \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)} G(\eta).$$

Using lemma 4.1, we obtain

$$\max_{\eta \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n)} G(\eta) = \max_{\eta \in \mathcal{S}_{as}} G(\eta) = G(w).$$

Thanks to the extremality relation between v and w, we have

$$(\Lambda^* w, -w) \in \Big(\partial J(v), \partial H(\Lambda v) \Big). \tag{4.16}$$

Since $\Lambda^* w \in \partial J(v)$, then we have

$$\sum_{j : j \sim i} w(i,j) - w(j,i) = v(i) - g(i), \quad \text{for any } i \in \mathbb{Z}^n.$$

We deduce, that

$$v(i) = g(i) + \sum_{j : j \sim i} \Big(w(i,j) - w(j,i) \Big), \quad \text{for any } i \in \mathbb{Z}^n.$$

and the proof is finished. □

Proof of Theorem 4.1 finished : Thanks to Proposition 4.1, we know that, if there exists (σ, v) such that $(\sigma, v) \in \ell^1(\mathbb{Z}^n \times \mathbb{Z}^n) \times K(r)$ satisfies (4.10), then $v = \mathbb{P}_{K(r)}(g)$. Now, taking $g \in \ell^2(\mathbb{Z}^n)$ and $v \in K(r)$ satisfies $v + \partial \mathbb{I}_{K(r)}(v) \ni g$. Thanks to Theorem 4.3, we have

$$v(i) + \sum_{j:j\sim i} \sigma(i,j) = g(i), \quad \text{for } i \in \mathbb{Z}^n,$$

where $\sigma(i,j) = w(i,j) - w(j,i) = 2\, w(i,j)$. Now we prove that, if $\sigma(i,j) \neq 0$ then $|v(i) - v(j)| = r$ for any $(i,j) \in \mathbb{Z}^n \times \mathbb{Z}^n$. Indeed, thanks to (4.16), we have $-w \in \partial H(\Lambda v)$, then we get

$$H(\Lambda v) + H^*(-w) =< -w, \Lambda v >$$

which implies that

$$r \sum_{i,j} |w(i,j)| = -\sum_{i} \sum_{j:j\sim i} w(i,j) \Big(v(i) - v(j) \Big),$$

and therefore, we have $w(i,j) = 0$ for $i \nsim j$ and

$$r|w(i,j)| = -w(i,j) \Big(v(i) - v(j) \Big) \quad \text{for } i \sim j.$$

Consequently

$$\sigma(i,j) \neq 0 \Rightarrow |v(i) - v(j)| = r \text{ for } (i,j) \in \mathbb{Z}^n \times \mathbb{Z}^n$$

and the proof is finished. □

4.2.2 Numerical results and simulations

To compute numerically the solution of the problem (4.6), we attempt to discretize it by the Euler implicit scheme. Let us denote by Δt the time step and $u^n(i)$ the approximate solution at time $t = n\Delta t$ for $n \in \mathbb{N}$. Then the study of the problem (4.6) becomes to solve a sequence of stationary equations. Starting with u^0, we need to compute u^{n+1} satisfying :

$$u^{n+1} + \partial \mathbb{I}_{K(r)}(u^{n+1}) \ni u^n + \Delta t f^n := g^n \tag{4.17}$$

where $f^n(i) = f(n\Delta t)$ for $n = 0, 1, 2, ..., K$; where $K \in \mathbb{N}$ is given. The problem (4.17) can be rewritten as

$$J(u^{n+1}) = \min_{z \in K(r)} J(z). \tag{4.18}$$

where $J(z) = \dfrac{1}{2}\|z - g^n\|^2_{\ell^2(\mathbb{Z}^2)}$. Thanks to Theorem 4.3, it is clear that the problem is equivalent to find a numerical method to minimize the functional $\tilde{G} : \mathcal{S}_{as} \longrightarrow \mathbb{R}$ defined by

$$\tilde{G}(\eta) = \frac{1}{2}\sum_{i \in A_N}\Big(\sum_{j:j\sim i}\eta(i,j)-\eta(j,i)\Big)^2+\sum_{i \in A_N}\Big(\sum_{j:j\sim i}\eta(i,j)-\eta(j,i)\Big)g^n(i)+r\sum_{(i,j)\in A_N\times A_N}|\eta(i,j)|$$

for a given $g^n(i)$, where $A_N := \Big\{ i = (i_1, i_2) \in \mathbb{Z}^2 \quad \text{such that} \; -N \le i_1, i_2 \le N \Big\}$ with N is a given large (enough) integer.
Thanks to the fact that $\sigma \in \mathcal{S}_{as}$ the functional can be rewritten as

$$\begin{aligned}
\tilde{G}(\eta) \; &= 2\sum_{-N\le i_1,i_2\le N}\Big(\sum_{k\in\{-1,1\}}\eta(i_1,i_2,i_1+k,i_2)+\sum_{l\in\{-1,1\}}\sigma(i_1,i_2,i_1,i_2+l)\Big)^2 \\
&+2\sum_{-N\le i_1,i_2\le N}\Big(\sum_{k\in\{-1,1\}}\eta(i_1,i_2,i_1+k,i_2)+\sum_{l\in\{-1,1\}}\sigma(i_1,i_2,i_1,i_2+l)\Big)g^n(i_1,i_2) \\
&+r\sum_{-N\le i_1,i_2\le N}\Big(\sum_{k\in\{-1,1\}}|\eta(i_1,i_2,i_1+k,i_2)|+\sum_{l\in\{-1,1\}}|\sigma(i_1,i_2,i_1,i_2+l)|\Big)
\end{aligned}$$

and for convenience taking $\eta(i_1, i_2, j_1, j_2) = 0$ for $\max\{j_1, j_2\} > N$ or $\min\{j_1, j_2\} < -N$.

Since, the functional $\tilde{G}$ is non-differentiable, we use a relaxation algorithm (cf. [61]). Denoting the cartesian basis vectors by e_j for $j = 1, ..., M$ with $M = 8\,N^2$; the algorithm can be written as follows :

1. Initiate the algorithm with w^0, set $k = 0$

2. For $j = 1, ..., M$, we solve the one-dimensional subproblems $\min\limits_{\xi \in I\!R} \Psi_{jk}(\xi)$ where Ψ_{jk} is defined as :

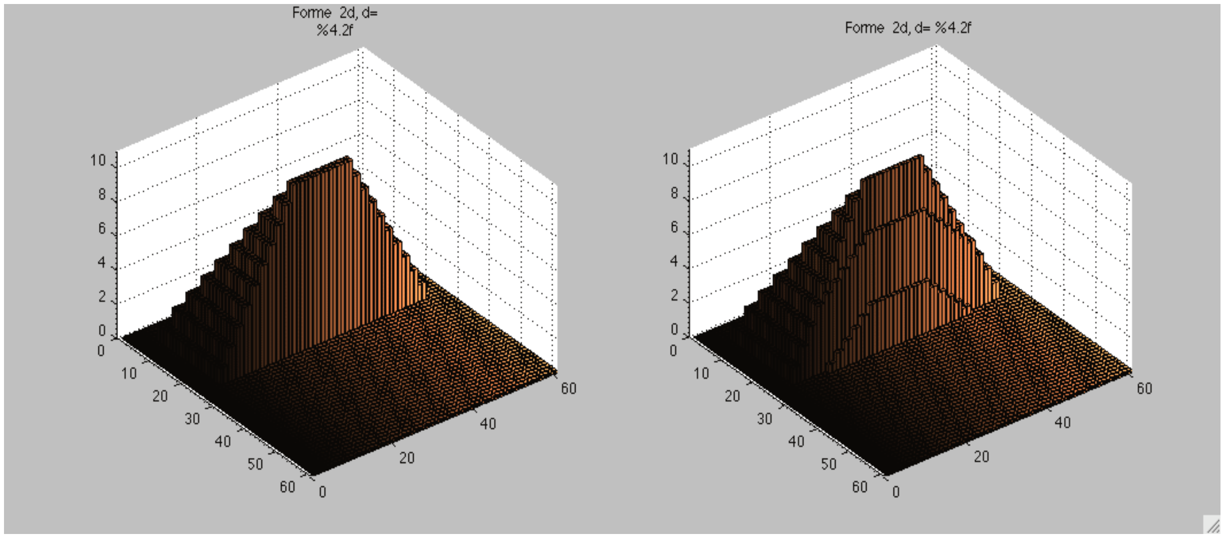
$$\Psi_{jk} : \quad I\!R \longrightarrow I\!R$$

$$\xi \longmapsto \tilde{G}(w^k + \sum_{i<j} \xi_{ik} + \xi e_j).$$

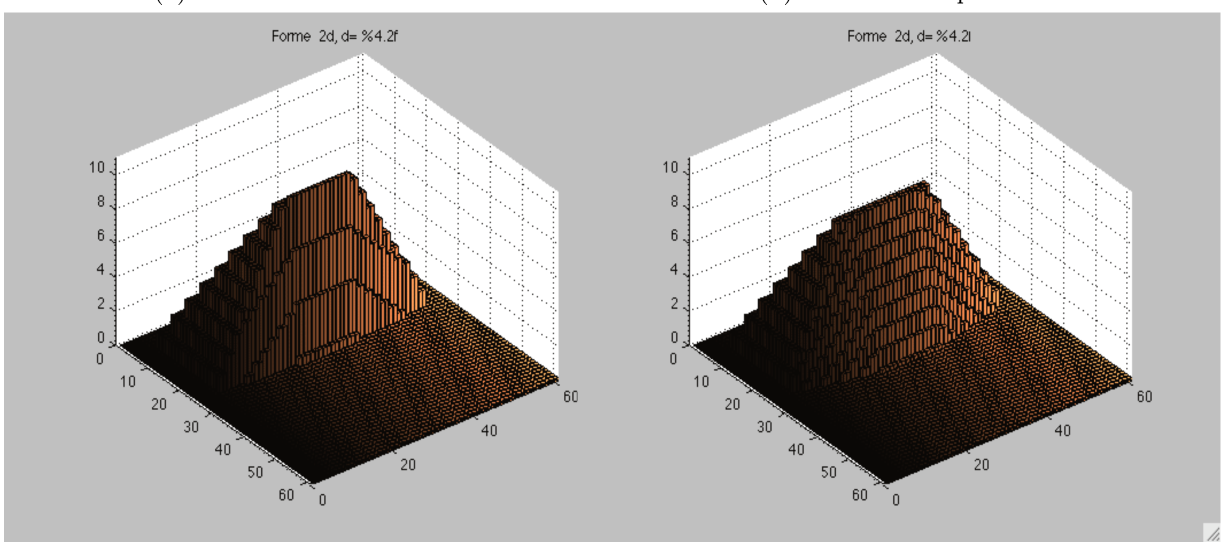
   Since Ψ_{jk} is the sum of a polynomial of degree two and an absolute value, we are used a Newton algorithm to find ξ_k^* when Ψ_{jk} is differentiable, and computing directly ξ_k^* otherwise. After, taking $\xi_{jk} = \xi_k^* \, e_j$.

3. Take $w^{k+1} = w^k + \lambda \sum\limits_{j=1,...,M} \xi_{jk}$, where $\lambda \in (0, 2)$ is an over-relaxation parameter.

4. As stopping criterion we use : $||w^k - w^{k+1}||_{\ell^2(I\!R^M)} < tol$, for a given convergence tolerance tol.

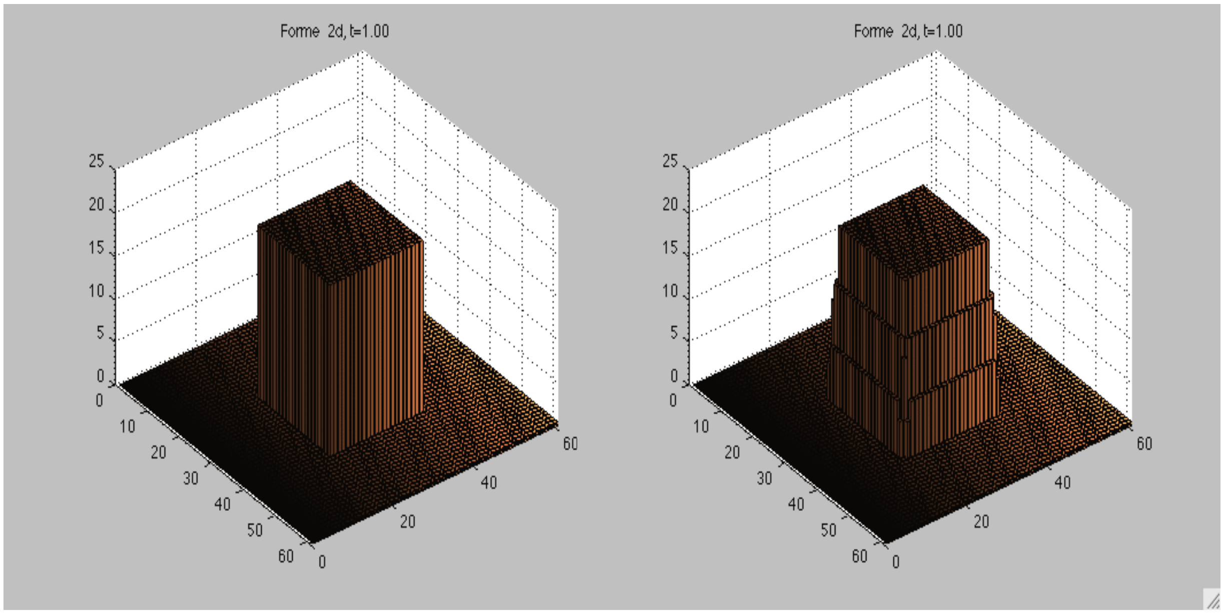


(a) Initial unstable structure (b) intermediate profile

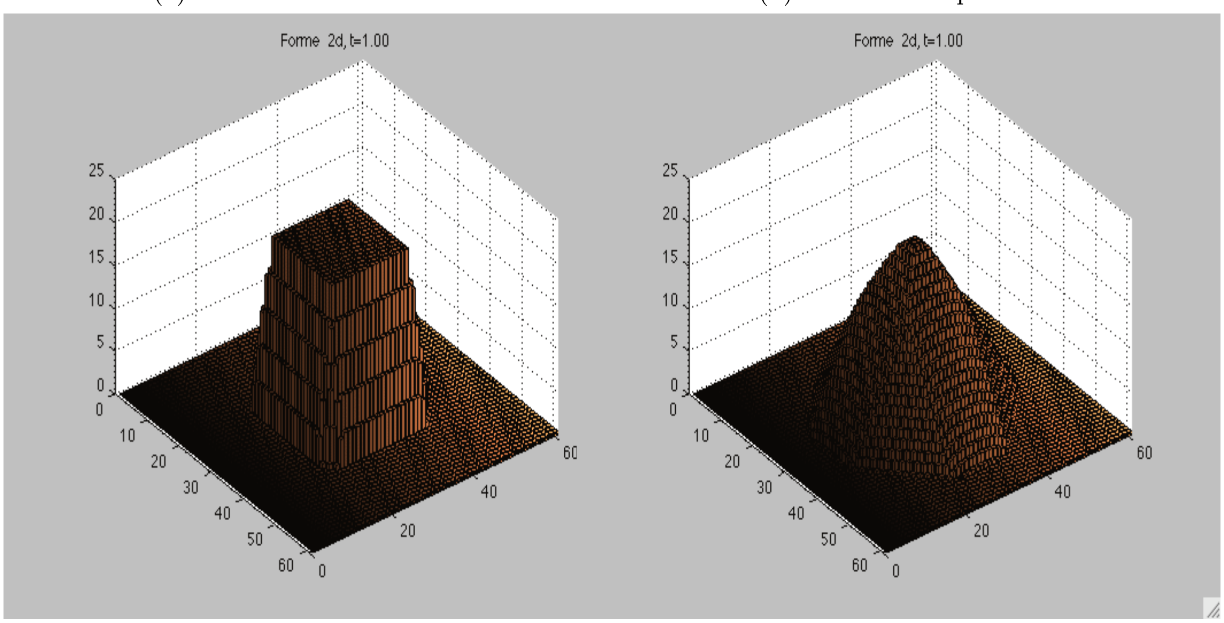


(c) intermediate profile (d) Final stable structure

FIGURE 4.1 – Stabilization of unstable discrete structure.



(a) Initial unstable structure　　(b) intermediate profile



(c) intermediate profile　　(d) Final stable structure

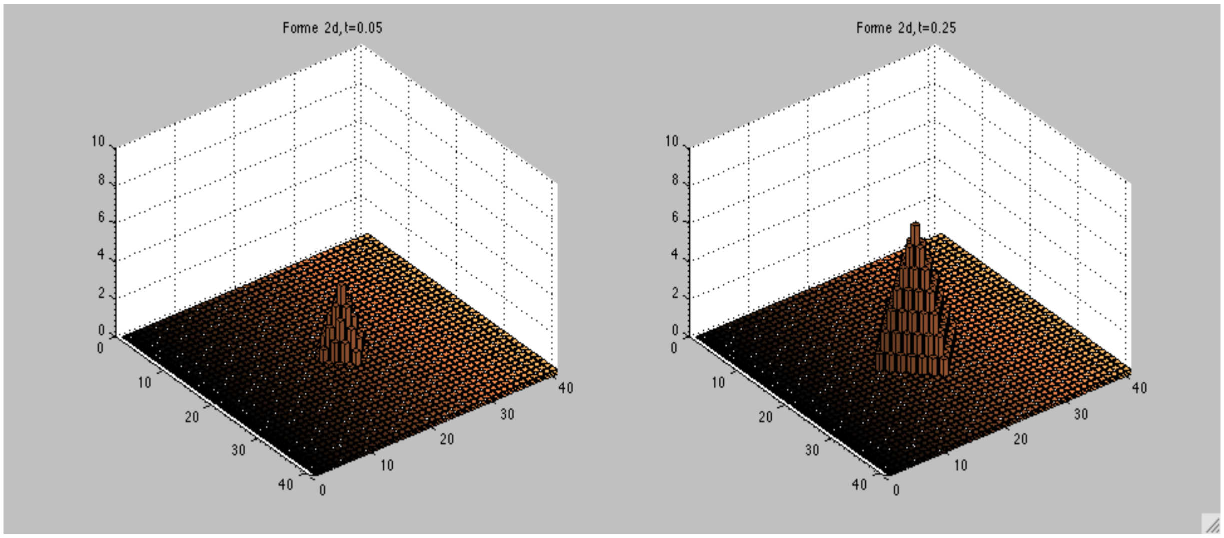
FIGURE 4.2 – Stabilization of unstable discrete structure.

Our numerical algorithm enables us also to simulate the growing of a discrete structure when we have a source of distribution of materials. Recall that in this case, the problem may be written as :

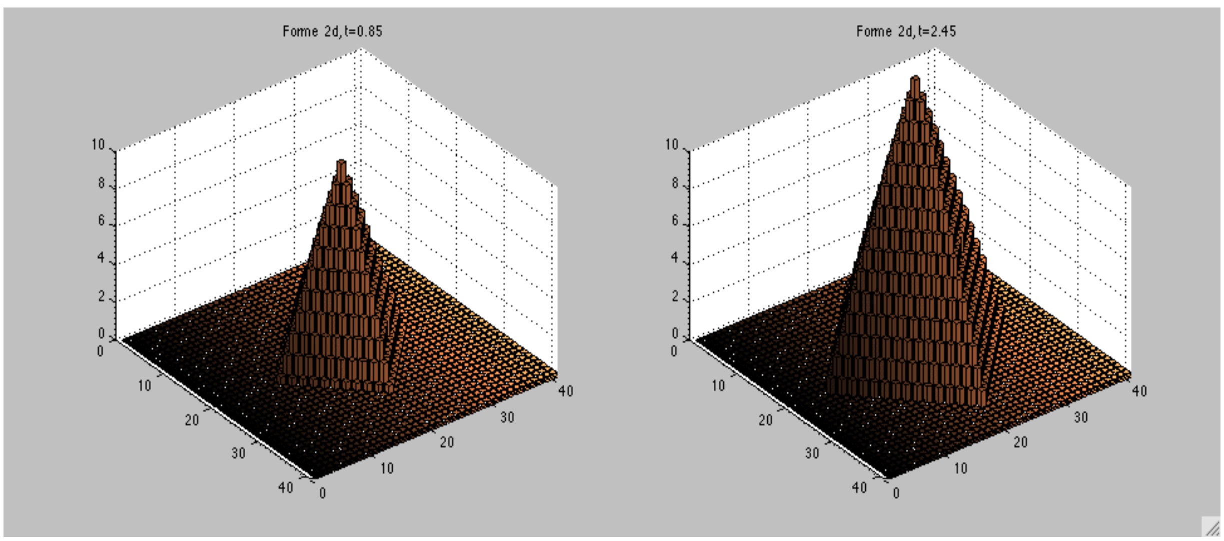
$$\begin{cases} u_t(t) + \partial \mathbb{I}_{K(1)}(u(t)) \ni f \text{ for } t \in (0,T) \\ \\ u(0) = u_0, \end{cases}$$

where f modeling the source term.
The idea is to keep the stability condition; i.e. $c \equiv 1$ and using the time

discretization of the problem, we compute successive projection of terms including the material that the source add per time. We present some numerical experiments of a growing pile of cubes. In all the simulation below, we have chosen relaxation parameter $\lambda = 1.2$, and convergence tolerance $tol = 10^{-6}$. The first case is devoted to the constant source term $f(t,i)$ distributed on the subdomain.

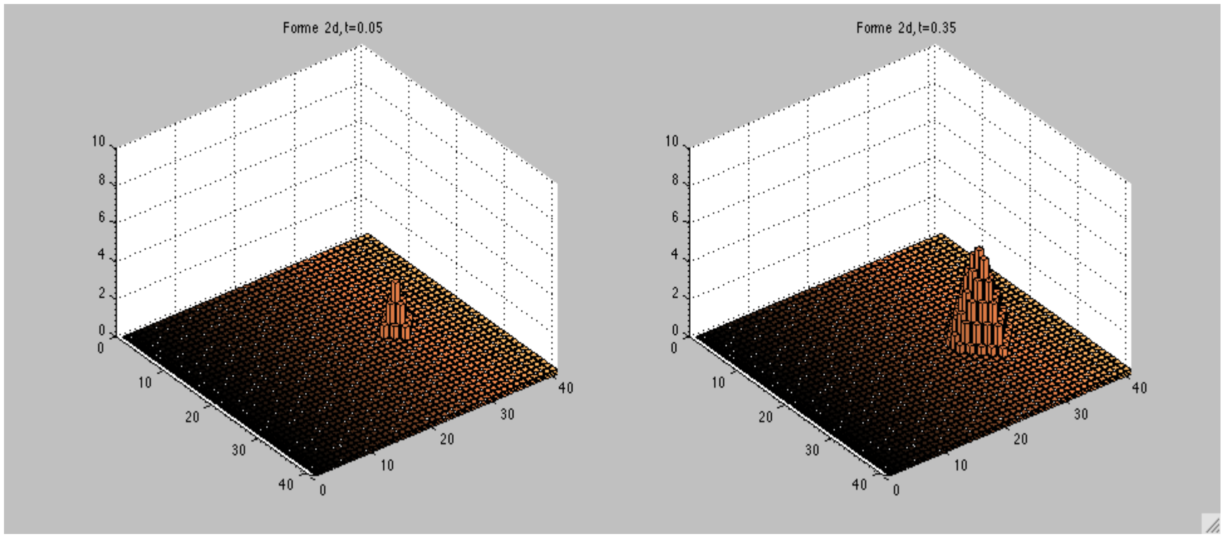


(a) Initial structure (b) intermediate profile

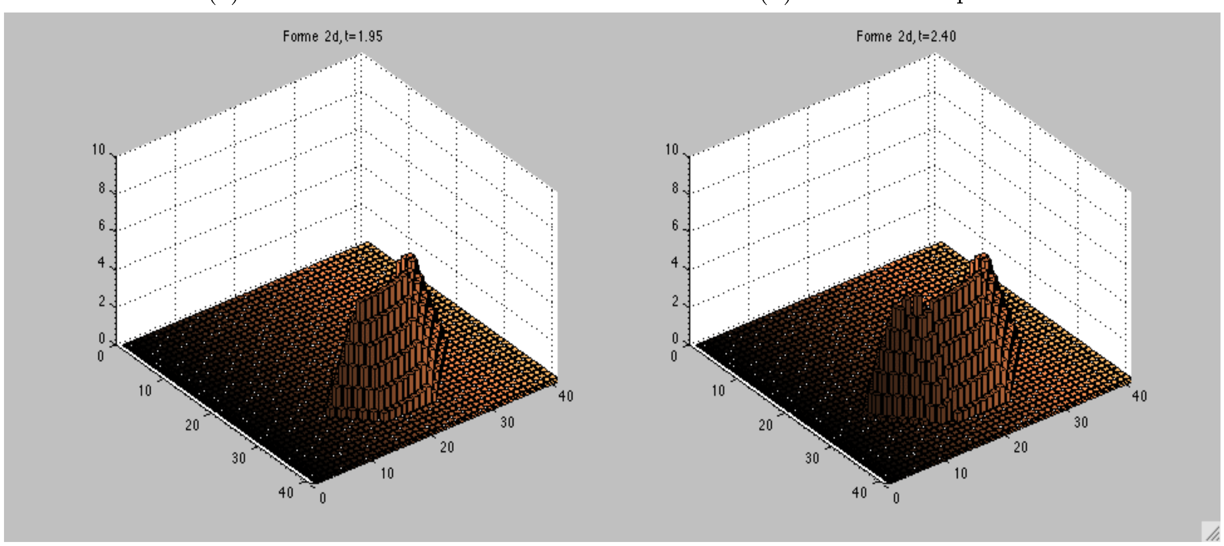


(c) intermediate profile (d) Final structure

FIGURE 4.3 – Growing discrete structure with central source.



(a) Initial structure (b) intermediate profile



(c) intermediate profile (d) Final structure

FIGURE 4.4 – Growing discrete structure with moving source.

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