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ANALYSE DES DYNAMIQUES INERTIELLES ET ALGORITHMES ASSOCIÉS POUR L'OPTIMISATION DU PREMIER ORDRE

Thèse dirigée par **Samir ADLY** et **Van Ngai HUYNH**

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University of Limoges
FACULTY OF SCIENCES AND TECHNOLOGIES

Analysis of inertial dynamics and associated algorithms for first-order optimization

by

Van Nam VO

Thesis supervisors :

Prof. Samir ADLY & **Prof. Van Ngai HUYNH**

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To my family,

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Introduction and mathematical background

1.1 Overall introduction

Age of digital revolution

Over the last two decades, we have witnessed a digital revolution in which data is at the centre of models. We are dealing with extracting meaningful information from large digital data bases utilizing appropriate deep-learning algorithms. Since the data is so huge, we refer to it as big data that directly impacts the formulation of models in high-dimensional spaces. Mathematics, statistics and computer science are pioneering the introduction of a modern generation of algorithms capable of addressing these issues with the required accuracy and in an acceptable time frame. Moreover, Machine Learning has recently been developed and widely used in various sectors with tangible applications. The optimization problem of a function is critical in the many phases of data processing. An optimization issue of this type reflects the minimizing of errors, that is, the difference between the analyzed data and the mathematical model that describes these data. They have broad usefulness in diverse applications, including signal processing, imaging sciences, machine learning, communication systems and astronomy. As a result, it is not an exaggeration to claim that we are in the age of the digital revolution.

First-order methods

First-order methods have occupied the forefront of research and became popular recently because of their usefulness in solving large-scale optimization problems in Machine Learning and Data Science by using just the gradient of the function. In particular, thanks to its simplicity, the Gradient Descent Method (GDM) is widely used in data science, image processing to minimize a function in the context of the explosion of digital information. Nevertheless, one of the disadvantages of that method is its slowness.

In 1964, B. Polyak suggested an improvement to the (GDM), adding a momentum term associated to a gradient descent step [69]. The heavy ball with friction (HBF) is best known as the continuous Ordinary Differential Equation (ODE) model of the Polyak momentum. That is an inertial dynamical system with a constant viscous damping coefficient. It might be viewed mechanically as the movement of a material point subject to viscous friction dampening and conservative potential forces. The heavy ball with friction is a second-order dissipative system in which the existence of inertia helps the system to overcome some of the (GDM)'s acknowledged disadvantages and accelerates convergence. But the (HBF) is not a descent method. The convergence behavior of the trajectories towards a critical point of the potential is well-known as long as various assumptions, for instance, convexity or analyticity of the potential term are satisfied. For a strongly convex function, (HBF) provides convergence at an exponential rate whenever the viscous damping coefficient is suitably chosen ; while for a general convex function, this rate is only $\mathcal{O}(\frac{1}{t})$ (in the worst case). This, however, is not better than the steepest descent.

Later, in 1983, Nesterov [60] first introduced another momentum method known as Nesterov Accelerated Gradient (NAG). The continuous ODE associating the Nesterov Accelerated Gradient algorithm was pointed out by Su-Boyd-Candès [71] after Su et al. introduced an Asymptotic Vanishing Damping (AVD) coefficient of the form $\frac{\alpha}{t}$, with $\alpha > 0$ and the time variable $t > 0$. Namely, for f being a general convex function, the condition $\alpha > 3$ ensures not only the asymptotic convergence rate of the values with a rate of order $o\left(1/t^2\right)$ but also the weak convergence of the solution trajectories towards optimal points.

Based on Fast Iterative Shrinkage-Thresholding Algorithm (FISTA) method of Beck and Teboulle [38], bit by bit, this topic has been enriched by the contributions of many authors. Many papers have been rapidly devoted to extending these results to the inertial proximal gradient algorithms solving additively structured optimization problems involving smooth and nonsmooth parts by splitting methods. For more detail, we refer the reader to [14], [19], [46], [73] and references therein. The presence of the Hessian-driven damping in [10] allows damping of the transversal oscillations which might arise with the model (HBF). Recent research consecrated to inertial dynamics have combined asymptotic vanishing damping with Hessian-driven damping. The comparable algorithms, in fact, include a correction term in the Nesterov method, which reduces the oscillatory aspects [30, 15, 71].

In 1961, Gelfand and Tsetlin [48] introduced the Ravine method, which is closely correlated to the Nesterov method. Both Ravine's and Nesterov's approaches have the same dynamic interpretation and enjoy comparable fast convergence properties, as illustrated in [20]. The low-resolution ODE (in the sense of [70]) of the Ravine method and Nesterov Accelerated Gradient is given by the Su-Boyd-Candès dynamic. Additionally, a more precise dynamic interpretation of these approaches is provided by the high-resolution ODE of the accelerated gradient methods proposed by Nesterov and Ravine, which shows the Hessian-driven damping. While Alesca, Laszlo, and Pinta discussed the implicit form in [7], the explicit model of the Hessian-driven damping was first introduced in [15] and [70].

Structured monotone problems

Equations with potential and nonpotential terms are used in a variety of scenarios originating primarily from physics, biology, and decision sciences. It derives from the occurrence of both cooperative and noncooperative aspects in decision sciences and game theory, for example. This is the case in physics when the processes of diffusion and convection coexist. Similar structures emerge from the Lagrangian methodology to linear constrained optimization problems. As a result, the study of additively structured monotone problems involving the sum of potential and nonpotential operators is more likely to be as significant as first-order approaches.

The purpose of this thesis is to examine the convergence properties of the trajectories generated by damped inertial dynamics driven by the sum of a potential (typically being the gradient of a continuously differentiable convex function) and a nonpotential monotone operator. Our approach is apparently in accordance with the Lyapunov analysis combined with an appropriate adjustment of the parameters involved in the dynamics. The explicit and implicit Newton-type damping will be discussed in more detail throughout Chapters 2 - 4.

Composition convex optimization

As we well know, the class composition convex optimization problems is presented in many applications, especially in image processing and machine learning. The accelerated gradient method initiated by Nesterov in 1983 ([61], [62]) is truly a prior step to designing powerful first-order methods for solving smooth convex optimization problems. Based on this acceleration scheme, the amount of algorithms were extensively developed for solving composition convex optimization of the form

$$\min\{f(x) + \Phi(x) : x \in \mathbb{R}^n\}, \tag{1.1}$$

in which the objective function is given by the sum of two convex functions including a smooth and a nonsmooth one. Especially, by combining the forward-backward method with Nesterov's acceleration scheme, Beck-Teboulle ([38]) have proposed the *fast iterative shrinkage-thresholding* algorithm (FISTA) for solving (1.1) which has many applications in image processing. Later, in [29] (see also [28]), Attouch-Peypouquet have shown that the convergence rate of the *accelerated forward-backward* method is actually $o(1/k^2)$ rather than $\mathcal{O}(1/k^2)$. Despite the uniform smoothness condition playing a central role in the development and analysis of first-order methods, there are variety applications where the objective function does not have this property, though being convex and differentiable [50]. Therefore, we aim to investigate the algorithms introduced in [66] in case of f is relative smooth and propose a method that employs the Bregman distance of the reference function instead of Euclidean distance. These results will be discussed in Chapter 5.

1.2 Outline of the dissertation

The fundamental goal of my thesis is to model, mathematically study, and numerically simulate inertial dynamics for first-order optimization. As a result of the growth of various applications in physics, biology, human sciences, and other fields, many problems have arisen that include equations with both potential and nonpotential components. The research on the convergence of damped inertial dynamics involving by maximally monotone operators enables to link between dynamic systems and numerical optimization. The tools used are coming from optimization, variational and set-valued analysis, Lyapunov stability theory and differential inclusions. For each model, we will concentrate on the existence and uniqueness of solution and the asymptotic characteristics of trajectories.

A part from Chapter 1 and Chapter 6 that contains the introduction and conclusions, the dissertation consists of four pivot chapters whose outline is organized as follows.

In Chapter 2, we introduce the following second-order differential equation which will form the basis of our analysis :

$$\ddot{x}(t) + \gamma\dot{x}(t) + \nabla f(x(t)) + B(x(t)) + \beta_f \nabla^2 f(x(t))\dot{x}(t) + \beta_b B'(x(t))\dot{x}(t) = 0, \ t \geq t_0. \tag{DINAM}$$

We show that the Cauchy problem is well-posed (in the sense of existence and uniqueness of solutions) using the first-order equivalent formulation of (DINAM), and we analyze the asymptotic convergence properties of the trajectories generated by (DINAM). Using appropriate Lyapunov functions, we indicate that any trajectory of (DINAM) converges weakly as $t \to +\infty$, and that its limit belongs to $S = (\nabla f + B)^{-1}(0)$. Also in that chapter, an application to the LASSO problem with a nonpotential operator and a coupled system in dynamical games will be studied.
Next, in Chapter 3, we study the convergence properties of the sequences generated by an inertial proximal algorithm obtained by implicit discretization of the continuous dynamics (DINAM). We highlight the interplay between the damping parameters β_f, β_b, γ and the cocoercivity parameter λ, which plays a significant role in our Lyapunov analysis. We analyze an inertial proximal-gradient splitting algorithm which makes use of the gradient of f and the resolvent of B. We also examine a variant of this proximal-gradient algorithm and the effect of errors, perturbations where the role of the operators is reversed.

Furthermore, Chapter 4 is devoted to the study of second-order evolution equation

$$\ddot{x}(t) + \gamma\dot{x}(t) + \nabla f\Big(x(t) + \beta_f\dot{x}(t)\Big) + B\Big(x(t) + \beta_b\dot{x}(t)\Big) = 0. \tag{iDINAM}$$

Similarly, we show that the Cauchy problem is well-posed and analyze the asymptotic convergence properties of the trajectories generated by (iDINAM). We study the convergence properties of the sequences generated by an inertial proximal algorithm obtained by discretization of the continuous dynamics (iDINAM).

Lastly, in Chapter 5, we focus on the convergence properties of the generalized Nesterov's algorithm and accelerated forward-backward algorithm for *composition convex* optimization problem of the form

$$\min\{f(x) + \Phi(x) : x \in \mathbb{R}^n\}, \tag{1.2}$$

in which $\Phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper, lower-semicontinuous, convex function and $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable, convex function whose gradient is L-Lipschitz continuous on dom Φ. We also highlight the convergent rate of our scheme by setting appropriate parameters and the smoothness, convexity of function f.

1.3 Mathematical background

In this section, we introduce the mathematical background on Hilbert space, convex analysis, caculus, and some technical lemmas. These tools will be used in the whole thesis. The material which follows is mainly taken from [43].

1.3.1 Hilbert space basics

Definition 1.3.1 *Let $\mathcal{H}$ be a complex vector space. An inner product on $\mathcal{H}$ is a function, $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, such that*

(i) $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$ *;*

(ii) $\langle \bar{x, y}\rangle = \langle y, x\rangle$ *;*

(iii) $\|x\|^2 \geq 0$ *and* $\|x\|^2 = 0$ *if and only if* $x = 0$.

We will often find the following formula useful :

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y\rangle \\ &= \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle \\ &= \|x\|^2 + \|y\|^2 + 2\mathrm{Re}\langle x, y\rangle. \end{aligned}$$

Theorem 1.3.1 (Schwartz Inequality) *Let $\mathcal{H}$ be an inner product space, then for all $x, y \in \mathcal{H}$*

$$|\langle x, y\rangle| \leq \|x\|\|y\|$$

and equality holds if and only if x and y are linearly dependent.

Corollary 1.3.1 *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be an inner product space and $\|x\| := \sqrt{\langle x, x\rangle}$. Then $\|\cdot\|$ is a norm on $\mathcal{H}$. Moreover $\langle\cdot,\cdot\rangle$ is continuous on $\mathcal{H}\times\mathcal{H}$, where $\mathcal{H}$ is viewed as the normed space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$.*

Definition 1.3.2 (Weak convergence) *A sequence of points (x_k) in a Hilbert space $\mathcal{H}$ is said to converge weakly to a point x in $\mathcal{H}$ provided that*

$$\langle x_n, y\rangle \to \langle x, y\rangle, \text{ as } n \to +\infty$$

for all $y \in \mathcal{H}$. We write $x_n \rightharpoonup x$, as $n \to +\infty$.

1.3.2 Convexity

Definition 1.3.3 *A subset C of $\mathbb{R}^n$ is called convex if for each $x, y \in C$ and for each $\lambda \in [0, 1]$ we have*

$$\lambda x + (1-\lambda)y \in C,$$

i.e.the closed line seqment $[x, y] \subset C$ whenever $x, y \in C$.

Definition 1.3.4 *Let $f : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be an extended real-valued function. The effective domain of f is defined by*

$$\mathrm{dom}(f) = \{x \in \mathbb{R}^n : f(x) < +\infty\}.$$

The function f is said to be proper if its effective domain is non-empty.

Definition 1.3.5 *Let $f : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be an extended real-valued function. The function f is said to be lower semi-continous at $x_0 \in \mathbb{R}^n$ if for every $\lambda < f(x_0)$ there exists $r > 0$ such that*

$$\lambda < f(x) \text{ for all } x \in x_0 + r\mathbb{B}.$$

The function f is said to be lower semi-continous if it is lower semi-continous at each point.

Definition 1.3.6 *Let $S \subset \mathbb{R}^n$. The affine hull or affine span of a set S in Euclidean space $\mathbb{R}^n$ is the smallest affine set containing S, or equivalently, the intersection of all affine sets containing S.*

Definition 1.3.7 *The relative interior of a set S denoted $\mathrm{rint}(S)$ is defined as its interior* *within the affine hull of S.*

Definition 1.3.8 *A proper function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be convex if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)y,$$

for every $\lambda \in [0,1]$ and every $x, y \in \mathrm{dom}(f)$.
If the inequation holds with strictly equality, then f is called strictly convex.
Give a function $f : \mathbb{R}^n \to \mathbb{R}$ and a point $x \in \mathbb{R}^n$, we will denote by $\nabla f(x)$ the gradient of f at x (whenever it exists).

Definition 1.3.9 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. We say that $p \in \mathbb{R}^n$ is a subgradient of f at a point $x_0 \in \mathrm{dom}(f)$ if*

$$\langle p, x - x_0 \rangle \leq f(x) - f(x_0) \text{ for each } x \in \mathbb{R}^n.$$

The set of all such p denoted by $\partial f(x_0)$ is called the subgradient of f at x_0.

Proposition 1.3.1 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differential function on an open convex set $\Omega \subset \mathbb{R}^n$. Then, f is convex on Ω if and only if*

$$f(x) \geq f(x_0) + \langle \nabla f(x_0), x - x_0 \rangle,$$

for all $x, x_0 \in \Omega$.
f is strictly convex on Ω if and only if

$$f(x) > f(x_0) + \langle \nabla f(x_0), x - x_0 \rangle,$$

for all $x, x_0 \in \Omega$ with $x \neq x_0$.

1.3.3 Monotone Operators

Definition 1.3.10 *A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be*

(i) monotone if

$$\langle F(x) - F(y), x - y \rangle \geq 0 \text{ for all } x, y \in \mathbb{R}^n.$$

(ii) strictly monotone if

$$\langle F(x) - F(y), x - y \rangle > 0 \text{ for all } x, y \in \mathbb{R}^n \text{ with } x \neq y.$$

(iii) maximally monotone if it is monotone and its graph is maximal in the sense of

inclusion, i.e., the graph of F is not properly contained in the graph of any other monotone operator.

Proposition 1.3.2 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function. Then f is (strictly) convex if and only if ∇f is (strictly) monotone.*

Definition 1.3.11 *Let $\mathcal{H}$ be a real Hilbert space. The operator $T : \mathcal{H} \to \mathcal{H}$ is said to be λ-cocoercive for some $\lambda > 0$ if*

$$\langle Ty - Tx, y - x\rangle \geq \lambda \|Ty - Tx\|^2, \ \forall x, y \in \mathcal{H}.$$

The operator $T : \mathcal{H} \to \mathcal{H}$ is said to be L-Lipschitz for some $L > 0$ if

$$\|Ty - Tx\| \leq L\|y - x\|, \ \forall x, y \in \mathcal{H}.$$

Let us quickly show that the sum of two cocoercive operators is still cocoercive. For further properties concerning cocoercive operators see [37].

Lemma 1.3.1 *Let $T_1, T_2 : \mathcal{H} \to \mathcal{H}$ be two cocoercive operators with respective cocoercivity coefficients $\lambda_1, \lambda_2 > 0$. Then $T := T_1 + T_2 : \mathcal{H} \to \mathcal{H}$ is $\lambda-$cocoercive with $\lambda = \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}$.*

Proof. According to the cocoercivity assumptions of T_1 and T_2, we have

$$\langle T_1y - T_1x, y - x\rangle \geq \lambda_1 \|T_1y - T_1x\|^2, \ \forall x, y \in \mathcal{H},$$

$$\langle T_2y - T_2x, y - x\rangle \geq \lambda_2 \|T_2y - T_2x\|^2, \ \forall x, y \in \mathcal{H}.$$

The following lemmas are useful in future. Firstly, let us show that the sum $T = T_1 + T_2$ is still cocoercive. As a result of elementary computation in Hilbert spaces, for all $x, y \in \mathcal{H}$, we have

$$\begin{aligned}
\|Ty - Tx\|^2 &= \|T_1y - T_1x + T_2y - T_2x\|^2 \\
&= \|T_1y - T_1x\|^2 + \|T_2y - T_2x\|^2 + 2\langle T_1y - T_1x, T_2y - T_2x\rangle \\
&\leq \|T_1y - T_1x\|^2 + \|T_2y - T_2x\|^2 + \frac{\lambda_1}{\lambda_2}\|T_1y - T_1x\|^2 + \frac{\lambda_2}{\lambda_1}\|T_2y - T_2x\|^2 \\
&= \left(\lambda_1^{-1} + \lambda_2^{-1}\right)\left(\lambda_1\|T_1y - T_1x\|^2 + \lambda_2\|T_2y - T_2x\|^2\right).
\end{aligned}$$

Since T_1 and T_2 are cocoercive, we deduce that

$$\begin{aligned}
\|Ty - Tx\|^2 &\leq \left(\lambda_1^{-1} + \lambda_2^{-1}\right)\left(\langle T_1y - T_1x, y - x\rangle + \langle T_2y - T_2x, y - x\rangle\right) \\
&= \left(\lambda_1^{-1} + \lambda_2^{-1}\right)\langle Ty - Tx, y - x\rangle.
\end{aligned}$$

Equivalently,

$$\langle Ty - Tx, y - x\rangle \geq \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\|Ty - Tx\|^2, \ \forall x, y \in \mathcal{H}.$$

So, T is still $\lambda-$cocoercive with $\lambda = \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2} > 0$.
Let us indicate that this estimate is sharp. Take $T_1 : \mathcal{H} \to \mathcal{H}, x \mapsto \lambda_1^{-1}x$ and $T_2 : \mathcal{H} \to \mathcal{H}, x \mapsto \lambda_2^{-1}x$. It is clear that T_1, T_2 are two cocoercive operators with cocoercivity coefficients λ_1, λ_2 respectively. Then their sum operator is equal to $Tx = \left(\lambda_1^{-1} + \lambda_2^{-1}\right) x = \lambda^{-1}x$ with $\lambda = \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}$, and hence is λ cocoercive. This shows that we cannot obtain a better estimate. ■

1.3.4 Technical lemmas

These lemmas will be useful in our thesis and be applied in our arguments multiple times. For that purpose, we first need to recall the the first pillar, which is the following well-known and fundamental property for a smooth function in the class $\mathcal{C}^{1,1}$; see, e.g., [40, 67].

Lemma 1.3.2 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function with Lipschitz continuous gradient and Lipschitz constant $L(f)$. Then, for any $L \geq L(f)$,*

$$f(x) \leq f(y) + \langle x - y, \nabla f(y)\rangle + \frac{L}{2}\|x - y\|^2 \text{ for every } x, y \in \mathbb{R}^n. \tag{1.3}$$

The following lemma is a classic result from integration theory, often called Barlabat's theorem in control theory.

Lemma 1.3.3 *Let $1 \leq p < \infty$ and $1 \leq r \leq \infty$. Suppose that $u \in L^p([0, \infty[; \mathbb{R})$ is a locally absolutely continuous nonnegative function, $g \in L^r([0, \infty[; \mathbb{R})$ and*

$$\dot{u}(t) \leq g(t)$$

for almost every $t > 0$. Then $\lim_{t\to\infty} u(t) = 0$.

The following lemma will play a key role in the proof of our convergence theorems. The proof can be found in [8, 24].

Lemma 1.3.4 *([8]) If $w \in C^2([0, +\infty[, \mathbb{R})$ is bounded from below and satisfies the following inequality*

$$\ddot{w}(t) + \gamma\dot{w}(t) \leq g(t),$$

where γ is a positive constant and $g \in L^1([0, +\infty[, \mathbb{R})$, then $w(t)$ converges as $t \to +\infty$.

Lemma 1.3.5 *Let a, b and c be three real numbers. The quadratic form $q : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$*

$$q(X, Y) := a\|X\|^2 + 2b\langle X, Y\rangle + c\|Y\|^2$$

is said to be positive definite if and only if $ac - b^2 > 0$ *and* $a > 0$. *Moreover,*

$$q(X, Y) \geq \mu(\|X\|^2 + \|Y\|^2) \quad \text{for all} \;\; X, Y \in \mathcal{H},$$

where $\mu := \frac{1}{2}\left(a + c - \sqrt{(a-c)^2 + 4b^2}\right)$ *is the smallest eigenvalue of the positive symmetric matrix associated with* q.

The next result is so-called the continuous version of the Opial lemma (see, for example, [68], [2, Lemma 1.10], [1, Lemma 5.3]).

Lemma 1.3.6 *Let* $S \subseteq \mathcal{H}$ *be a nonempty set and* $x : [0, +\infty[$ *a given map. Assume that*

(i) *for every* $x^* \in S$, *the limit* $\lim \lim_{t \to +\infty} \|x(t) - x^*\|$ *exists;*

(ii) *every weak sequential cluster point of the map* x *belongs to* S. *Then there exists* $x_\infty \in S$ *such that* $x(t)$ *converges weakly to* x_∞ *as* $t \to +\infty$.

The following is discrete version of the Gronwall Lemma, see [19, Lemma A.9.] for another proof.

Lemma 1.3.7 *Let* a *be a positive real and* $(y_k), (g_k)$ *be nonnegative sequences such that for all* $k \geq 0$, *we have*

$$\frac{1}{2}y_k^2 \leq \frac{1}{2}a^2 + \sum_{0 \leq i < k} g_i y_i.$$

Then, the succeeding inequality holds for all $k \geq 0$: $\quad y_k \leq a + \sum_{0 \leq i < k} g_i.$

Proof. For any $\varepsilon > 0$, let us define the sequence $(z_k(\varepsilon))$ given by

$$z_k(\varepsilon) = \frac{1}{2}(a + \varepsilon)^2 + \sum_{0 \leq i < k} g_i y_i.$$

We have $z_{k+1}(\varepsilon) - z_k(\varepsilon) = g_k y_k$ and $\frac{1}{2}y_k^2 \leq z_k(\varepsilon)$ for $k \geq 0$. Thus,

$$z_{k+1}(\varepsilon) - z_k(\varepsilon) \leq \sqrt{2} g_k \sqrt{z_k(\varepsilon)}. \tag{1.4}$$

Moreover, by the definition of $(z_k(\varepsilon))$, we deduce that $(z_k(\varepsilon))$ is a nondecreasing sequence as well.

Hence,

$$\sqrt{z_{k+1}(\varepsilon)} - \sqrt{z_k(\varepsilon)} = \frac{z_{k+1}(\varepsilon) - z_k(\varepsilon)}{\sqrt{z_{k+1}(\varepsilon)} + \sqrt{z_k(\varepsilon)}} \leq \frac{z_{k+1}(\varepsilon) - z_k(\varepsilon)}{2\sqrt{z_k(\varepsilon)}}. \tag{1.5}$$

From (1.4) and (1.5), we obtain

$$\sqrt{z_{k+1}(\varepsilon)} - \sqrt{z_k(\varepsilon)} \leq \frac{1}{\sqrt{2}} g_k. \tag{1.6}$$

That implies

$$\sqrt{z_k(\varepsilon)} \leq \sqrt{z_0(\varepsilon)} + \frac{1}{\sqrt{2}} \sum_{0 \leq i < k} g_i,$$

for all $k \geq 0$. Then,

$$y_k \leq \sqrt{2z_k(\varepsilon)} \leq \sqrt{2z_0(\varepsilon)} + \sum_{0 \leq i < k} g_i = a + \varepsilon + \sum_{0 \leq i < k} g_i.$$

Taking $\varepsilon \to 0$, we obtain

$$y_k \leq a + \sum_{0 \leq i < k} g_i.$$

This completes the proof. ■

2

Asymptotic behaviour of Newton-like inertial dynamics involving the sum of potential and nonpotential terms

As you know, a massive number of situations which come from physics, biology, human sciences, etc. involve equations containing both potential and nonpotential terms. For example, in human sciences, this comes from the presence of both cooperative and noncooperative aspects. In physics, this comes from the joint presence of terms diffusion and convection. To describe such phenomena, we are often led to solving additively structured monotone problems of the type

$$\text{Find } x \in \mathcal{H} \ : \ \underbrace{\nabla f(x)}_{potential} + \underbrace{B(x)}_{nonpotential} = 0,$$

where $\mathcal{H}$ is a Hilbert space. The presence of the nonpotential term, namely B, makes it impossible to apply classical mathematical analysis methods in this context and consequently, the common tools for simulation of these problems are not appropriate.

Our ambition in this chapter is to investigate mathematically a dynamic inertial Newton method which aims at solving additively structured monotone equations involving the sum of both potential and nonpotential terms. Roughly speaking, we are looking for the zeros of the operator $A = \nabla f + B$. In which ∇f is the gradient of a continuously differentiable convex function f and B denotes the nonpotential monotone and cocoercive operator.

Apart from a viscous friction term, the dynamic includes geometric damping terms which are regulated respectively by the Hessian of the potential f and a Newton-type correction term attached to nonpotential monotone and cocoercive operator B. Thanks to a fixed point argument, we claim the well-posedness of the Cauchy problem and the weak convergence as $t \to +\infty$ of the generated trajectories towards the zeros of $\nabla f + B$.

The convergence analysis relies on the appropriate setting of the viscous parameter γ and geometric damping parameters β_b, β_f. These geometrical dampings enable us to control and attenuate the known classical oscillations of inertial methods with the viscous damping. Transforming the second-order evolution equation into a first-order dynamical system, on the other hand, allows such analysis to be extended to nonsmooth convex potentials. Due to the introduction of the nonpotential term, the proofs and techniques are original and different from the classical ones.

These brand-new results open the door or propose some new first-order accelerated algorithms in optimization taking into account the specific properties of potential and nonpotential terms.

This chapter constitutes the subject of the published paper [4] in collaboration with S. Adly and H. Attouch.

2.1 Introduction and preliminary results

Throughout this chapter, let $\mathcal{H}$ be a real Hilbert space endowed with the scalar product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. We will concentrate on solving problems in the additively structured monotone equations of the form

$$\text{Find } x \in \mathcal{H} \ : \ \nabla f(x) + B(x) = 0. \tag{2.1}$$

In the preceding equation, we recall that ∇f is the gradient of a convex continuously differentiable function $f : \mathcal{H} \to \mathbb{R}$ (that plays as the potential part), and $B : \mathcal{H} \to \mathcal{H}$ is a nonpotential operator, i.e., B is not supposed to be equal to the gradient of a given function which is assumed to be monotone and cocoercive. To reach this end, we consider continuous inertial dynamics whose solution trajectories converge as $t \to +\infty$ to solutions of (2.1). Lining with the active research stream, we work on the close relationship between continuous dissipative dynamical systems and optimization algorithms obtained by taking their temporal discretization. The main objective is to analyze the continuous dynamic. The algorithmic part and its correlation with first-order numerical optimization will be discussed in the successive companion chapter. From this viewpoint, damped inertial dynamics are a logical way to accelerate these systems. Serving as the core feature of our study, we will introduce the dynamic geometric dampings respectively driven by the Hessian for the potential component and the corresponding Newton term for the nonpotential one. Furthermore, these terms not only improve the convergence rate but also considerably reduce the oscillatory behaviour of the trajectories.
We will give special consideration to the minimal hypothesis which ensure convergence of the trajectories, and emphasize the asymmetric role of the two operators involved in the dynamic. As we shall see, a lot of statements can be enhanced to the case where $f : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is a convex proper lower semicontinuous function that helps expand the scope of application.

2.1.1 Dynamical inertial Newton method for additively structured monotone problems

Let us introduce the following second-order differential equation which will form the basis of our analysis :

$$\ddot{x}(t) + \gamma \dot{x}(t) + \nabla f(x(t)) + B(x(t)) + \beta_f \nabla^2 f(x(t)) \dot{x}(t) + \beta_b B'(x(t)) \dot{x}(t) = 0, \ t \geq t_0. \tag{DINAM}$$

We briefly use (DINAM) as an abbreviation for the dynamical inertial Newton method for additively structured monotone problems. We call $t_0 \in \mathbb{R}$ the beginning of time. Since the systems are autonomous, we can take any real number for t_0. For simplicity and without loss of generality, we set $t_0 = 0$.
To examine the corresponding Cauchy problem, we add the initial conditions : $x(0) = x_0 \in \mathcal{H}$ and $\dot{x}(0) = x_1 \in \mathcal{H}$. The term $B'(x(t))\dot{x}(t)$ is interpreted as $\frac{d}{dt}(B(x(t)))$ taken in the distribution sense. Likewise, the term $\nabla^2 f(x(t))\dot{x}(t)$ is undertood as $\frac{d}{dt}(\nabla f(x(t)))$ taken in the distribution sense as well. Because of the following assumptions, these terms are measurable functions and bounded on the bounded time intervals. So, we will only investigate strong solutions of the above equation (DINAM).
The chapter is organized as follows. Section 2.1 introduces (DINAM) with some historical perspective. In section 2.2, based on the first-order equivalent formulation of (DINAM), we show that the Cauchy problem is well-posed (in the sense of existence and uniqueness of solutions). In section 2.3, we analyze the asymptotic convergence properties of the trajectories generated by (DINAM). Using appropriate Lyapunov functions, we show that any trajectory of (DINAM) converges weakly as $t \to +\infty$, and that its limit belongs to $S = (\nabla f + B)^{-1}(0)$. The interplay between the damping parameters β_f, β_b, γ and the cocoercivity parameter λ will play a significant role in our Lyapunov analysis. In Section 2.4, we perform numerical experiments that show the well-known oscillations in the case of the heavy ball with friction are damped with the introduction of the geometric (Hessian-like) damping terms. Also in that section, an application to the LASSO problem with a nonpotential operator and a coupled system in dynamical games are studied. Section 2.5 deals with the extension of the work to the nonsmooth and convex case. Section 2.6 contains some concluding remarks and perspectives.
Before starting, we make throughout this part these following standing assumptions :

$$\begin{cases} \text{(A1)} \;\; f : \mathcal{H} \to \mathbb{R} \;\text{ is convex, of class } \mathcal{C}^1,\, \nabla f \;\text{ is Lipschitz continuous on the bounded sets;} \\ \text{(A2)} \;\; B : \mathcal{H} \to \mathcal{H} \;\text{ is } \lambda\text{-cocoercive for some } \lambda > 0; \\ \text{(A3)} \;\; \gamma > 0,\; \beta_f > 0,\; \beta_b \geq 0 \;\text{ are given real damping parameters;} \\ \text{(A4)} \;\;\; \text{the solution set } \; S := (\nabla f + B)^{-1}(0) = \{p \in \mathcal{H} : \;\nabla f(p) + B(p) = 0\} \neq \emptyset. \end{cases}$$

We highlight the fact that we do not suppose ∇f to be globally Lipschitz continuous. Our analysis is conducted without using any boundedness of ∇f is a key to further extending the theory to the nonsmooth case. As a specific feature, the inertial system (DINAM) includes two different types of driving forces associated respectively with the potential operator ∇f and the nonpotential operator B. It also involves three different types of friction, namely :

(a) The term $\gamma \dot{x}(t)$ stands for viscous damping with a positive coefficient $\gamma > 0$.

(b) The term $\beta_f \nabla^2 f(x(t))\dot{x}(t)$ is referred to Hessian driven damping, which attenuates the oscillations that naturally occur in the inertial gradient dynamics.

(c) The term $\beta_b B'(x(t))\dot{x}(t)$ is the nonpotential variant of the Hessian driven damping. It acts as a Newton-type correction term.

Note that each driving force term enters (DINAM) with its temporal derivative. In fact, we have

$$\nabla^2 f(x(t))\dot{x}(t) = \frac{d}{dt}\left(\nabla f(x(t))\right) \text{ and } B'(x(t))\dot{x}(t) = \frac{d}{dt}\left(B(x(t))\right).$$

This is crucial to observe (DINAM) to be a first-order system in time, and space ; then the corresponding Cauchy problem is well-posed. This will be proved later (see subsection 2.2.1 for more details). The assumption of the cocoercivity on the operator B is essential in the analysis of (DINAM) in terms of ensuring the existence of solutions and analysing their asymptotic behaviour as time $t \to +\infty$.
Note that the operator $B : \mathcal{H} \to \mathcal{H}$ is said to be λ-cocoercive for some $\lambda > 0$ if

$$\langle By - Bx, y - x\rangle \geq \lambda \|By - Bx\|^2, \ \forall x, y \in \mathcal{H}.$$

It is clear that B is λ-cocoercive implies B is $1/\lambda$-Lipschitz continuous. The reverse implication holds true when the operator is the gradient of a convex and differentiable function. Indeed, according to Baillon-Haddad's theorem [34], ∇f is L-Lipschitz continuous implies that ∇f is a $1/L$-cocoercive operator (we refer to [37, Corollary 18.16] for more details).

2.1.2 Historical aspects of the inertial systems with Hessian-driven damping

The timeline of studying the inertial systems with Hessian-driven damping was marked by Alvarez, Attouch, Peypouquet, and Redont. Alvarez et al. in their papers (see [10]) first considered the inertial system with Hessian-driven damping in the form

$$\ddot{x}(t) + \gamma \dot{x}(t) + \beta \nabla^2 f(x(t))\dot{x}(t) + \nabla f(x(t)) = 0.$$

Later, based on the continuous interpretation by Su, Boyd, and Candès [71] of Nesterov's accelerated gradient method, Attouch et al. [30] replaced the fixed viscous damping parameter γ and studied the new one, an asymptotic vanishing damping parameter $\frac{\alpha}{t}$, with $\alpha > 0$. At first sight, the presence of the Hessian might seem to entail numerical difficulties, however, this is not true in this case since the Hessian intervenes in the above original differential equations (ODE) in the form $\nabla^2 f(x(t))\dot{x}(t)$, which is the derivative concerning the time of $\nabla f(x(t))$. Hence, the temporal discretization of these dynamics gives first-order algorithms of the form

$$\begin{cases} y_k = x_k + \alpha_k(x_k - x_{k-1}) - \beta_k \left(\nabla f(x_k) - \nabla f(x_{k-1})\right) \\ x_{k+1} = y_k - s\nabla f(y_k). \end{cases}$$

In addition, unlike traditional accelerated gradient methods, these algorithms include a correction term that is the difference of the gradient at two successive steps. They give fast convergence to zero of the gradients and eliminate the oscillatory aspects while conserving the convergence properties of the accelerated gradient method. There have been several recent studies on that subject, for example, Attouch, Chbani, Fadili, and Riahi [15], Boţ, Csetnek, and László [42], Kim [52], Lin, and Jordan [55], Shi, Du, Jordan, and Su [70], and Alesca, Lazlo, and Pinta [7] for an implicit version of the Hessian driven damping. Recently, Castera, Bolte, Févotte, Pauwels [45] have developed the range of application to deep learning.

Additionally, in [3], Adly and Attouch studied the finite convergence of proximal-gradient inertial algorithms combining both dry friction and Hessian-driven damping. Namely, the authors considered temporal discretization of the differential inclusion

$$\ddot{x}(t) + \gamma(t)\dot{x}(t) + \partial\phi(\dot{x}(t)) + \nabla f(x(t)) \ni 0, t \in [t_0, +\infty[. \tag{2.2}$$

Then, the sequence (x_k) generated by Inertial Proximal Gradient algorithm with Dry Friction (IPGDF) :

$$\begin{cases} x_0, x_1 \in \mathcal{H}, \\ x_{k+1} = x_k + h\,\mathrm{prox}_{\frac{h}{1+h\gamma}\phi}\left(\frac{1}{h(1+h\gamma)}(x_k - x_{k-1}) - \frac{h}{1+h\gamma}\nabla f(x_k)\right), \end{cases}$$

converges to x_∞ satisfying $0 \in \partial\phi(0) + \nabla f(x_\infty)$. For more details, see Theorem 2.1 in [3].

2.1.3 Inertial dynamics involving cocoercive operators

Let us now turn to the transposition of these techniques to the case of maximally monotone operators. Álvarez and Attouch [9] and Attouch and Maingé [24] studied the equation

$$\ddot{x}(t) + \gamma\dot{x}(t) + A(x(t)) = 0, \tag{2.3}$$

when $A : \mathcal{H} \to \mathcal{H}$ is cocoercive and hence maximally monotone ; see also [41]. The cocoercivity assumption is the pivot in the study of (2.3) not only does guarantee the well-posedness of solutions also analyze their long-term behaviour. Assuming that the damping coefficient γ and the cocoercivity parameter λ fulfill the inequality $\lambda\gamma^2 > 1$, Attouch and Maingé [24] showed that each trajectory of (2.3) converges weakly to a zero of A, i.e. $x(t) \rightharpoonup x_\infty \in A^{-1}(0)$ as $t \to +\infty$. Furthermore, the condition $\lambda\gamma^2 > 1$ is sharp. Regarding general maximally monotone operators this property has been further exploited by Attouch and Peypouquet [27], and by Attouch and Laszlo [22, 23]. The key property is that for $\lambda > 0$, the Yosida approximation A_λ of A is λ-cocoercive and $A_\lambda^{-1}(0) = A^{-1}(0)$. Thus, the idea is to replace the operator A with its Yosida approximation, and adjust the Yosida regularization parameter. Another remarkable work has been done by Attouch and Maingé [24]. The authors first considered the asymptotic behavior of the second-order dissipative evolution equation with $f : \mathcal{H} \to \mathbb{R}$ convex and $B : \mathcal{H} \to \mathcal{H}$ cocoercive

$$\ddot{x}(t) + \gamma\dot{x}(t) + \nabla f(x(t)) + B(x(t)) = 0, \tag{2.4}$$

combining potential with nonpotential effects. The properties of the trajectory solution is stated as follows.

Theorem 2.1.1 ([24]) *Let us suppose that $f : \mathcal{H} \to \mathbb{R}$ is a convex differentiable function whose gradient ∇f is Lipschitz continuous on the bounded subsets of $\mathcal{H}$. Suppose that $B : \mathcal{H} \to \mathcal{H}$ is maximal monotone and λ-cocoercive for some $\lambda > 0$. Assume that the set $S = (\nabla f + B)^{-1} \neq \emptyset$ and the cocoercive parameter λ and the damping parameter γ satisfy*

$$\lambda\gamma^2 > 1.$$

Then, for each initial data x_0 and $\dot{x}_0$ in $\mathcal{H}$, the unique solution $x \in C^2([0, +\infty); \mathcal{H})$ of (2.4) satisfies :

(i) There exists $x_\infty \in S$ such that $x(t) \rightharpoonup x_\infty$ weakly in $\mathcal{H}$ as $t \to +\infty$.

(ii) $\dot{x} \in L^2(0, +\infty; \mathcal{H})$; $\lim_{t\to+\infty} |\dot{x}(t)| = 0$;

(iii) $\ddot{x} + \nabla f(x) + Bx \in L^2(0, +\infty; \mathcal{H})$ whenever $p \in S$;

(iv) for every $p \in S$, $\lim_{t \to +\infty} |x(t) - p|$ exists.

2.1.4 Link with Newton-like methods for solving monotone inclusions

Based on such historical aspects that we presented in the preceding sections, we therefore, consist initially in introducing the Hessian term and the Newton-type correcting term into this dynamic.
Let us clarify the relationship between our research and Newton's method for solving (2.1). The Newton's method for solving $\nabla f(x) = 0$ generates the following sequence (x_k) given by

$$\begin{cases} x_0 \in \mathcal{H}, \\ x_{k+1} = x_k - [\nabla^2 f(x_k)]^{-1} \nabla f(x_k), \; k \geq 0. \end{cases}$$

Equivalently, we deal with

$$\nabla^2 f(x_k)(x_{k+1} - x_k) = \nabla f(x_k). \tag{2.5}$$

The continuous dynamic associated to (2.5) is

$$\nabla^2 f(x(t))\dot{x}(t) + \nabla f(x(t)) = 0. \tag{2.6}$$

We can see that (2.6) is ill-posed since $\nabla^2 f(x)$ is only positive semi-definite (if f is only convex).
In order to overcome the ill-posedness of the continuous Newton method for a general maximally monotone operator A, Attouch and Svaiter [32] studied the first-order evolution system shown below :

$$\begin{cases} v(t) \in A(x(t)) \\ \gamma(t)\dot{x}(t) + \beta \dot{v}(t) + v(t) = 0. \end{cases} \tag{2.7}$$

This system (2.7) can be played as a continuous version of the Levenberg-Marquardt method and as a regularization of Newton's method [23]. We notice that under a general hypothesis on $\gamma(t)$, the system (2.7) is well-posed and its generated trajectories converge weakly to equilibria (zeros of A). In parallel, we obtained the results for the associated proximal algorithms by considering its implicit temporal discretization, see [2], [26], [31] for more details. This system is written formally as

$$\gamma(t)\dot{x}(t) + \beta\frac{d}{dt}\left(A(x(t))\right) + A(x(t)) = 0.$$

Thus, (DINAM) can be considered as an inertial version of this dynamical system for structured monotone operator $A = \nabla f + B$. Our work is also linked to the recent works by Attouch and Laszlo [22, 23], however, contrasting with [22, 23], due to the cocoercivity of B, instead of utilizing the Yosida regularization, we only present minimal hypotheses regarding the nonpotential component.

2.2 Well-posedness of the Cauchy-Lipschitz problem

We first show the existence and the uniqueness of the solution trajectory for the Cauchy problem associated with (DINAM) for any given initial condition data $(x_0, x_1) \in \mathcal{H} \times \mathcal{H}$.

2.2.1 First-order in time and space equivalent formulation

The following first-order equivalent formulation of (DINAM) was first considered by Alvarez, Attouch, Bolte, and Redont [10] and Attouch, Peypouquet, and Redont [30] in the framework of convex minimization. In particular, in our context, the following equivalence results from a simple differential and algebraic calculation.

Proposition 2.2.1 *Suppose that $\beta_f > 0$. Then the two problems as follows are equivalent :* $(i) \iff (ii)$

$$(i) \quad \ddot{x}(t) + \gamma\dot{x}(t) + \nabla f(x(t)) + B(x(t)) + \beta_f\nabla^2 f(x(t))\dot{x}(t) + \beta_b B'(x(t))\dot{x}(t) = 0.$$

$$(ii) \quad \begin{cases} \dot{x}(t) + \beta_f\nabla f(x(t)) + \beta_b B(x(t)) + \left(\gamma - \dfrac{1}{\beta_f}\right)x(t) + y(t) = 0; \\ \dot{y}(t) - \left(1 - \dfrac{\beta_b}{\beta_f}\right)B(x(t)) + \dfrac{1}{\beta_f}\left(\gamma - \dfrac{1}{\beta_f}\right)x(t) + \dfrac{1}{\beta_f}y(t) = 0. \end{cases}$$

Proof. $(i) \implies (ii)$. For $t \geq 0$, set

$$y(t) := -\dot{x}(t) - \beta_f\nabla f(x(t)) - \beta_b B(x(t)) - \left(\gamma - \frac{1}{\beta_f}\right)x(t), \tag{2.8}$$

which gives the first equation of (ii). By differentiating $y(\cdot)$ and using (i), we get

$$\begin{aligned}\dot{y}(t) &= -\ddot{x}(t) - \beta_f \nabla^2 f(x(t))\dot{x}(t) - \beta_b B'(x(t))\dot{x}(t) - \left(\gamma - \frac{1}{\beta_f}\right)\dot{x}(t)\\ &= \gamma\dot{x}(t) + \nabla f(x(t)) + B(x(t)) - \left(\gamma - \frac{1}{\beta_f}\right)\dot{x}(t)\\ &= \nabla f(x(t)) + B(x(t)) + \frac{1}{\beta_f}\dot{x}(t). \end{aligned} \tag{2.9}$$

By combining (2.8) and (2.9), we obtain

$$\dot{y}(t) + \frac{1}{\beta_f}y(t) = \left(1 - \frac{\beta_b}{\beta_f}\right)B(x(t)) - \frac{1}{\beta_f}\left(\gamma - \frac{1}{\beta_f}\right)x(t). \tag{2.10}$$

This indicates the second equation of (ii).

$(ii) \implies (i)$. By differentiating the first equation of (ii), we obtain

$$\ddot{x}(t) + \beta_f\nabla^2 f(x(t))\dot{x}(t) + \beta_b B'(x(t))\dot{x}(t) + \left(\gamma - \frac{1}{\beta_f}\right)\dot{x}(t) + \dot{y}(t) = 0. \tag{2.11}$$

In order to obtain an equation involving only x, we will eliminate y from this equation by using a simple trick. For this reason, we successively use the second equation in (ii), then the first equation in (ii) to obtain

$$\begin{aligned}\dot{y}(t) &= \left(1 - \frac{\beta_b}{\beta_f}\right)B(x(t)) - \frac{1}{\beta_f}\left(\gamma - \frac{1}{\beta_f}\right)x(t) - \frac{1}{\beta_f}y(t)\\ &= \left(1 - \frac{\beta_b}{\beta_f}\right)B(x(t)) - \frac{1}{\beta_f}\left(\gamma - \frac{1}{\beta_f}\right)x(t) + \frac{1}{\beta_f}\dot{x}(t)\\ &\quad + \nabla f(x(t)) + \frac{\beta_b}{\beta_f}B(x(t)) + \frac{1}{\beta_f}\left(\gamma - \frac{1}{\beta_f}\right)x(t).\end{aligned}$$

Therefore,

$$\dot{y}(t) = \nabla f(x(t)) + B(x(t)) + \frac{1}{\beta_f}\dot{x}(t). \tag{2.12}$$

From (2.11) and (2.12), we obtain (i). ■

2.2.2 Well-posedness of the evolution equation (DINAM)

The first step towrad our existence and uniqueness result obtained in the present section concerns the definition of a strong global solution of the dynamical system (DINAM).

Definition 2.2.1 *We call the function $x : [0, +\infty) \to \mathcal{H}$ a strong global solution of the* *dynamical system* (DINAM) *if satisfies the following properties :*

(i) $x, \dot{x} : [t_0, +\infty) \to \mathcal{H}$ *are locally absolutely continuous ;*

(ii) $\ddot{x}(t) + \gamma\dot{x}(t) + \nabla f(x(t)) + B(x(t)) + \beta_f \nabla^2 f(x(t))\dot{x}(t) + \beta_b B'(x(t))\dot{x}(t) = 0$ *for almost every* $t \geq t_0$ *;*

(iii) $x(t_0) = x_0$ *and* $\dot{x}(t_0) = x_1$.

For brevity reasons, we call that a mapping $x : [t_0, +\infty) \to \mathcal{H}$ is called locally absolutely continuous if it is absolutely continuous on every compact interval $[t_0, T]$, where $T > t_0$. Further, we have the following equivalently characterizations for an absolutely continuous function $x : [t_0, +\infty) \to \mathcal{H}$, (see, for instance [2, 12]) :

(a) there exists $y : [t_0, +\infty) \to \mathcal{H}$ an integrable function, such that

$$x(t) = x(t_0) + \int_{t_0}^{t} y(s)ds, \forall t \in [t_0, T];$$

(b) x is continuous function and its distributional derivative is Lebesgue integrable on the interval $[t_0, T]$;

(c) for every $\epsilon > 0$, there exists $\eta > 0$ such that for every finite family $I_k = (a_k, b_k)$ from $[t_0, T]$, the following implication is valid :

$$\left[I_k \cap I_j = \emptyset, \forall k \neq j \text{ and } \sum_k |b_k - a_k| < \eta \right] \Rightarrow \left[\sum_k \|x(b_k) - x(a_k)\| < \epsilon \right].$$

The following theorem indicates the well-posedness of the Cauchy problem for (DINAM).

Theorem 2.2.1 *Assume that* $\beta_f > 0$ *and* $\beta_b \geq 0$. *Then, for any* $(x_0, x_1) \in \mathcal{H} \times \mathcal{H}$, *there exists uniquely a strong global solution* $x : [0, +\infty) \to \mathcal{H}$ *of the continuous dynamic* (DINAM) *which satisfies the Cauchy data* $x(0) = x_0$, $\dot{x}(0) = x_1$.

Proof. System (ii) in Proposition 2.2.1 can be read equivalently as

$$\dot{Z}(t) + F(Z(t)) = 0, \quad Z(0) = (x_0, y_0),$$

where $Z(t) = (x(t), y(t)) \in \mathcal{H} \times \mathcal{H}$ and

$$\begin{aligned} F(x,y) &= \beta_f(\nabla f(x), 0) \\ &\quad + \Big(\beta_b B(x) + \left(\gamma - \frac{1}{\beta_f}\right) x + y, -\left(1 - \frac{\beta_b}{\beta_f}\right) B(x) + \frac{1}{\beta_f}\left(\gamma - \frac{1}{\beta_f}\right) x + \frac{1}{\beta_f} y\Big), \\ y_0 &= -x_1 - \beta_f \nabla f(x_0) - \beta_b B(x_0) - \left(\gamma - \frac{1}{\beta_f}\right) x_0. \end{aligned}$$

Therefore, $F = \nabla\Phi + G$, where $\Phi : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is the convex differentiable function

$$\Phi(x, y) := \beta_f f(x)$$

and $G : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$

$$G(x, y) := \left(\beta_b B(x) + \left(\gamma - \frac{1}{\beta_f}\right) x + y, -\left(1 - \frac{\beta_b}{\beta_f}\right) B(x) + \frac{1}{\beta_f}\left(\gamma - \frac{1}{\beta_f}\right) x + \frac{1}{\beta_f} y\right)$$

is a Lipschitz continuous map as a direct consequence of the Lipschitz continuity of B. Moreover, the existence of a classical solution to

$$\dot{Z}(t) + \nabla\Phi(Z(t)) + G(Z(t)) = 0, \quad Z(0) = (x_0, y_0)$$

follows from Brézis [43, Proposition 3.12]. In fact, the proof of this statement relies on a fixed point argument. It entails finding a fixed point of the mapping $u \in \mathcal{C}([0, T], \mathcal{H}) \mapsto K(u) \in \mathcal{C}([0, T], \mathcal{H})$, where $K(u) = w$ is defined as the solution of

$$\dot{w}(t) + \nabla\Phi(w(t)) = -G(u(t)), \quad w(0) = (x_0, y_0).$$

It is shown that the sequence of iterates (w_n) formed by the corresponding Picard iteration

$$\dot{w}_{n+1}(t) + \nabla\Phi(w_{n+1}(t)) = -G(w_n(t)), \quad w_{n+1}(0) = (x_0, y_0),$$

converges uniformly on $[0, T]$ to a fixed point of K. When returning to (DINAM), that is, equation (i) of Proposition 2.2.1, we recover a strong solution. In particular, $\dot{x}$ is Lipschitz continuous on the bounded time intervals, and $\ddot{x}$ taken in the distribution sense is locally essentially bounded. ■

Remark 2.2.1 It should be noted that when ∇f is assumed to be globally Lipschitz continuous, the proof might be notably simplified by applying the classical Cauchy-Lipschitz theorem.

2.3 Asymptotic convergence properties of (DINAM)

In this section, we investigate the asymptotic behaviour of the solution trajectories of (DINAM). For each solution trajectory $t \mapsto x(t)$ of (DINAM), we will show that the weak limit $\text{w-}\lim_{t\to+\infty} x(t) = x_\infty$ exists and fulfills $x_\infty \in S$, where

$$S := \{p \in \mathcal{H} : \ \nabla f(p) + B(p) = 0\}.$$

Before stating our main result, notice that $B(p)$ is uniquely defined for $p \in S$.

Lemma 2.3.1 *$B(p)$ is uniquely defined for $p \in S$, i.e.,*

$$p_1 \in S, \ p_2 \in S \Longrightarrow B(p_1) = B(p_2).$$

Proof. Since $p_1 \in S$, $p_2 \in S$ we have

$$\nabla f(p_1) + B(p_1) = \nabla f(p_2) + B(p_2) = 0.$$

By the monotonicity of ∇f we have

$$\langle \nabla f(p_2) - \nabla f(p_1), p_2 - p_1 \rangle \geq 0.$$

Replacing $\nabla f(p_1)$ with $-B(p_1)$ and $\nabla f(p_2)$ with $-B(p_2)$, we get

$$\langle B(p_2) - B(p_1), p_2 - p_1 \rangle \leq 0,$$

which by cocoercivity of B gives $\lambda \|B(p_2) - B(p_1)\|^2 \leq 0$, and hence $B(p_2) = B(p_1)$. ■

Next, we divide our problems into two cases : general case ($\beta_b \neq \beta_f$) and $\beta_b = \beta_f$.

2.3.1 General case

The general line of the proof is close to that given by Attouch and Laszlo in [22, 23]. The first significant difference with the approach developed in [22, 23] is that in our context, due to the hypothesis of cocoercivity on the nonpotential part, we do not go through the Yosida regularization of the operators. The second one is that we treat the potential and nonpotential operators differently. The Yosida regularization's computation of such sum is often beyond numerical ability, and these points, therefore, are meaningful for applications to numerical algorithms.

The asymptotic convergence properties of (DINAM) will be stated in the following theorem.

Theorem 2.3.1 *Let $B : \mathcal{H} \to \mathcal{H}$ be a λ-cocoercive operator and $f : \mathcal{H} \to \mathbb{R}$ be a $\mathcal{C}^1$ convex function whose gradient is Lipschitz continuous on the bounded sets. Suppose that $S = (\nabla f + B)^{-1}(0) \neq \emptyset$, and that the parameters involved in the evolution equation*

(DINAM) *fulfill the following conditions :* $\beta_f > 0$ *and*

$$4\lambda\gamma > \frac{(\beta_b - \beta_f)^2}{\beta_f} + 2\left(\beta_b + \frac{1}{\gamma}\right) + 2\sqrt{\left(\beta_b + \frac{1}{\gamma}\right)^2 + \frac{(\beta_b - \beta_f)^2}{\gamma\beta_f}}. \tag{2.13}$$

Then, for any solution trajectory $x : [0, +\infty[\to \mathcal{H}$ *of* (DINAM)*, the following properties are satisfied :*

(i) *(convergence)* $x(t)$ *converges weakly, as* $t \to +\infty$*, to an element of* S*.*

(ii) *(integral estimates) Set* $A := B + \nabla f$ *and* $p \in S$*. Then*

$$\int_0^{+\infty} \|\dot{x}(t)\|^2 dt < +\infty, \int_0^{+\infty} \|\ddot{x}(t)\|^2 dt < +\infty,$$
$$\int_0^{+\infty} \|B(x(t)) - B(p)\|^2 dt < +\infty, \int_0^{+\infty} \left\|\frac{d}{dt}B(x(t))\right\|^2 dt < +\infty,$$
$$\int_0^{+\infty} \|A(x(t))\|^2 dt < +\infty, \text{ and } \int_0^{+\infty} \left\|\frac{d}{dt}A(x(t))\right\|^2 dt < +\infty.$$

(iii) *(pointwise estimates)*

$$\lim_{t\to+\infty} \|\dot{x}(t)\| = 0, \lim_{t\to+\infty} \|B(x(t)) - B(p)\| = 0, \lim_{t\to+\infty} \|A(x(t))\| = 0,$$

where $B(p)$ *is uniquely defined for* $p \in S$*.*

Proof. **Lyapunov analysis.** Set $A := B + \nabla f$ and $A_\beta := \beta_b B + \beta_f \nabla f$. Take $p \in S$. Consider the function $t \in [0, +\infty[\mapsto \mathcal{V}_p(t) \in \mathbb{R}_+$ defined by

$$\mathcal{V}_p(t) := \frac{1}{2}\|x(t) - p + c\Big(\dot{x}(t) + A_\beta(x(t)) - A_\beta(p)\Big)\|^2 + \frac{\delta}{2}\|x(t) - p\|^2, \tag{2.14}$$

where c and δ are coefficients to adjust. Using the differentiation chain rule for absolutely continuous functions (see [44, Corollary VIII.10]) and (DINAM), we get

$$\dot{\mathcal{V}}_p(t) = \Big\langle \dot{x}(t) - c\left(\gamma\dot{x} + A(x(t))\right), x(t) - p + c\Big(\dot{x}(t) + A_\beta(x(t)) - A_\beta(p)\Big)\Big\rangle + \delta\langle \dot{x}(t), x(t) - p\rangle. \tag{2.15}$$

Setting $\delta := c\gamma - 1 > 0$, from (2.15) we obtain

$$\dot{\mathcal{V}}_p(t) = \langle -cA(x(t)), x(t) - p\rangle + c\langle (1 - c\gamma)\dot{x}(t) - cA(x(t)), \dot{x}(t) + A_\beta(x(t)) - A_\beta(p)\rangle. \tag{2.16}$$

We have

$$\begin{aligned}
&c\langle(1-c\gamma)\dot{x}(t)-cA(x(t)),\dot{x}(t)+A_\beta(x(t))-A_\beta(p)\rangle\\
&=c(1-c\gamma)\|\dot{x}(t)\|^2+c(1-c\gamma)\langle\dot{x}(t),A_\beta(x(t))-A_\beta(p)\rangle\\
&\quad -c^2\langle A(x(t)),\dot{x}(t)\rangle-c^2\langle A(x(t)),A_\beta(x(t))-A_\beta(p)\rangle,\\
&=c(1-c\gamma)\|\dot{x}(t)\|^2-c^2\beta_b\|B(x(t))-B(p)\|^2-c^2\beta_f\|\nabla f(x(t))-\nabla f(p)\|^2\\
&\quad +c[(1-c\gamma)\beta_b-c]\langle\dot{x}(t),B(x(t))-B(p)\rangle+c[(1-c\gamma)\beta_f-c]\langle\dot{x}(t),\nabla f(x(t))-\nabla f(p)\rangle\\
&\quad -c^2(\beta_b+\beta_f)\langle B(x(t))-B(p),\nabla f(x(t))-\nabla f(p)\rangle. \qquad (2.17)
\end{aligned}$$

Using the fact that $p\in S$, ∇f is monotone, and B is λ-cocoercive, we have

$$\begin{aligned}
-c\langle A(x(t)),x(t)-p\rangle &= -c\langle A(x(t))-A(p),x(t)-p\rangle\\
&= -c\langle\nabla f(x(t))-\nabla f(p),x(t)-p\rangle-c\langle B(x(t))-B(p),x(t)-p\rangle\\
&\leq -c\lambda\|B(x(t))-B(p)\|^2. \qquad (2.18)
\end{aligned}$$

From (2.16)-(2.18), we deduce that

$$\begin{aligned}
\dot{\mathcal{V}}_p(t)\leq{}& -c\delta\|\dot{x}(t)\|^2-[c^2\beta_b+c\lambda]\|B(x(t))-B(p)\|^2-c^2\beta_f\|\nabla f(x(t))-\nabla f(p)\|^2\\
&-[c\delta\beta_b+c^2]\langle\dot{x}(t),B(x(t))-B(p)\rangle-[c\delta\beta_f+c^2]\langle\dot{x}(t),\nabla f(x(t))-\nabla f(p)\rangle\\
&-c^2(\beta_b+\beta_f)\left\langle B(x(t))-B(p),\nabla f(x(t))-\nabla f(p)\right\rangle. \qquad (2.19)
\end{aligned}$$

Let $\Gamma:[0,+\infty[\to\mathbb{R}$ be the function defined by

$$\Gamma(t):=f(x(t))-f(p)-\left\langle\nabla f(p),x(t)-p\right\rangle,$$

and $\mathcal{E}_p:[0,+\infty[\to\mathbb{R}$ be the energy function given by

$$\mathcal{E}_p(t):=\mathcal{V}_p(t)+[c\delta\beta_f+c^2]\Gamma(t).$$

Since f is convex, we have $\Gamma(t)\geq 0$, for all $t\geq 0$. This implies $\mathcal{E}_p(t)\geq 0$ for all $t\geq 0$ as well. We have

$$\dot{\Gamma}(t)=\langle\dot{x}(t),\nabla f(x(t))-\nabla f(p)\rangle, \qquad (2.20)$$

$$\dot{\mathcal{E}}_p(t)=\dot{\mathcal{V}}_p(t)+[c\delta\beta_f+c^2]\dot{\Gamma}(t). \qquad (2.21)$$

By using (2.20) and (2.21), equation (2.19) can be rewritten as

$$\dot{\mathcal{E}}_p(t) + c\delta\|\dot{x}(t)\|^2 + [c^2\beta_b + c\lambda]\|B(x(t)) - B(p)\|^2 + c^2\beta_f\|\nabla f(x(t)) - \nabla f(p)\|^2 \tag{2.22}$$
$$+[c\delta\beta_b + c^2]\langle\dot{x}(t), B(x(t)) - B(p)\rangle + c^2(\beta_b + \beta_f)\,\langle B(x(t)) - B(p), \nabla f(x(t)) - \nabla f(p)\rangle \leq 0.$$

Let us eliminate the term $\nabla f(x(t)) - \nabla f(p)$ from this relation by using the elementary algebraic inequality

$$c^2\beta_f\|\nabla f(x(t)) - \nabla f(p)\|^2 + c^2(\beta_b + \beta_f)\,\langle B(x(t)) - B(p), \nabla f(x(t)) - \nabla f(p)\rangle$$
$$\geq -\frac{c^2(\beta_b + \beta_f)^2}{4\beta_f}\|B(x(t)) - B(p)\|^2.$$

We obtain

$$\dot{\mathcal{E}}_p(t) + c\delta\|\dot{x}(t)\|^2 + [c^2\beta_b + c\lambda - \frac{c^2(\beta_b + \beta_f)^2}{4\beta_f}]\|B(x(t)) - B(p)\|^2$$
$$+[c\delta\beta_b + c^2]\langle\dot{x}(t), B(x(t)) - B(p)\rangle \leq 0.$$

Equivalently

$$\dot{\mathcal{E}}_p(t) + c\mathcal{S}(t) \leq 0, \tag{2.23}$$

where

$$\mathcal{S}(t) := \delta\|\dot{x}(t)\|^2 + [\delta\beta_b + c]\langle\dot{x}(t), B(x(t)) - B(p)\rangle + [c\beta_b + \lambda - \frac{c(\beta_b + \beta_f)^2}{4\beta_f}]\|B(x(t)) - B(p)\|^2.$$

Set $X(t) = \dot{x}(t)$ and $Y(t) = B(x(t)) - B(p)$. We have $\mathcal{S}(t) = q(X(t), Y(t))$, where $q : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is the quadratic form

$$q(X, Y) := a\|X\|^2 + b\langle X, Y\rangle + g\|Y\|^2$$

with $a = \delta$, $b = \delta\beta_b + c$, and $g = c\beta_b + \lambda - \frac{c(\beta_b + \beta_f)^2}{4\beta_f} = \lambda - \frac{c(\beta_b - \beta_f)^2}{4\beta_f}$.
According to Lemma 1.3.5, and since $a = \delta = c\gamma - 1 > 0$, we have that q is positive definite if and only if $4ag - b^2 > 0$. Equivalently

$$4\delta\left(\lambda - \frac{c(\beta_b - \beta_f)^2}{4\beta_f}\right) - [\delta\beta_b + c]^2 > 0. \tag{2.24}$$

Our goal is to find c such that $c\gamma - 1 > 0$ and such that (2.24) is fulfilled. Take $\delta := c\gamma - 1 > 0$

as a new variable. Equivalently, we must find $\delta > 0$ such that

$$4\delta\left(\lambda - \frac{\delta+1}{\gamma}.\frac{(\beta_b-\beta_f)^2}{4\beta_f}\right) - \left(\delta\beta_b + \frac{\delta+1}{\gamma}\right)^2 > 0.$$

After the development and simplification of the preceding inequation, we obtain

$$4\lambda > \Big[\frac{(\beta_b-\beta_f)^2}{\gamma\beta_f} + \frac{2}{\gamma}\left(\beta_b + \frac{1}{\gamma}\right)\Big] + \frac{1}{\gamma^2\delta} + \Big[\left(\beta_b + \frac{1}{\gamma}\right)^2 + \frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}\Big]\delta.$$

Thus, our requirement is reduced to

$$4\lambda > \Big[\frac{(\beta_b-\beta_f)^2}{\gamma\beta_f} + \frac{2}{\gamma}\left(\beta_b + \frac{1}{\gamma}\right)\Big] + \inf_{\delta>0}\Big(\frac{1}{\gamma^2\delta} + \Big[\left(\beta_b + \frac{1}{\gamma}\right)^2 + \frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}\Big]\delta\Big).$$

Elementary optimization argument gives that

$$\inf_{\delta>0}\Big(\frac{C}{\delta} + D\delta\Big) = 2\sqrt{CD}.$$

Therefore, we end up with the condition

$$4\lambda > \Big[\frac{(\beta_b-\beta_f)^2}{\gamma\beta_f} + \frac{2}{\gamma}\left(\beta_b + \frac{1}{\gamma}\right)\Big] + \frac{2}{\gamma}\sqrt{\left(\beta_b + \frac{1}{\gamma}\right)^2 + \frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}}.$$

Equivalently,

$$4\lambda\gamma > \Big[\frac{(\beta_b-\beta_f)^2}{\beta_f} + 2\left(\beta_b + \frac{1}{\gamma}\right)\Big] + 2\sqrt{\left(\beta_b + \frac{1}{\gamma}\right)^2 + \frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}}. \tag{2.25}$$

When $\beta_b = \beta_f = \beta$, we recover the condition

$$\lambda\gamma > \beta + \frac{1}{\gamma}.$$

Note that $c\gamma = 1 + \delta$ and $\delta > 0$ implies $c > 0$. According to (2.23), $\mathcal{S}(t) = q(X(t), Y(t))$, and q positive definite, we deduce that there exist positive numbers c and μ such that

$$\dot{\mathcal{E}}_p(t) + c\mu\|\dot{x}(t)\|^2 + c\mu\|B(x(t)) - B(p)\|^2 \leq 0. \tag{2.26}$$

Estimates. Let us start from (2.26) that we integrate on $[0,t]$, $t \geq 0$. We obtain

$$\mathcal{E}_p(t) + c\mu \int_0^t \|\dot{x}(s)\|^2 ds + c\mu \int_0^t \|B(x(s)) - B(p)\|^2 ds \leq \mathcal{E}_p(0). \tag{2.27}$$

From (2.27) and the definition of $\mathcal{E}_p$, we immediately deduce

$$\sup_{t\geq 0} \|x(t) - p\| < +\infty, \tag{2.28}$$

$$\sup_{t\geq 0} \|x(t) - p + c(\dot{x}(t) + A_\beta(x(t)) - A_\beta(p))\| < +\infty, \tag{2.29}$$

$$\int_0^{+\infty} \|\dot{x}(t)\|^2 dt < +\infty, \tag{2.30}$$

$$\int_0^{+\infty} \|B(x(t)) - B(p)\|^2 dt < +\infty. \tag{2.31}$$

Let us return to (2.22). We recall that

$$\dot{\mathcal{E}}_p(t) + c\delta\|\dot{x}(t)\|^2 + [c^2\beta_b + c\lambda]\|B(x(t)) - B(p)\|^2 + c^2\beta_f\|\nabla f(x(t)) - \nabla f(p)\|^2 \tag{2.32}$$
$$+[c\delta\beta_b + c^2]\langle\dot{x}(t), B(x(t)) - B(p)\rangle + c^2(\beta_b + \beta_f)\left\langle B(x(t)) - B(p), \nabla f(x(t)) - \nabla f(p)\right\rangle \leq 0.$$

After integrating on $[0,t]$ and using the integral estimates $\int_0^{+\infty} \|\dot{x}(t)\|^2 dt < +\infty$ and $\int_0^{+\infty} \|B(x(t)) - B(p)\|^2 dt < +\infty$ obtained in (2.30) and (2.31), we claim the existence of a constant $C > 0$ such that

$$c^2\beta_f \int_0^t \|\nabla f(x(s)) - \nabla f(p)\|^2 ds \leq C + c^2(\beta_b+\beta_f) \int_0^t \|B(x(s)) - B(p)\|\|\nabla f(x(s)) - \nabla f(p)\| ds.$$

Therefore, for any $\epsilon > 0$, we have

$$c^2\beta_f \int_0^t \|\nabla f(x(s)) - \nabla f(p)\|^2 ds$$
$$\leq C + c^2(\beta_b + \beta_f) \int_0^t \left(\frac{1}{4\epsilon}\|B(x(s)) - B(p)\|^2 + \epsilon\|\nabla f(x(s)) - \nabla f(p)\|^2\right) ds.$$

By taking $\epsilon > 0$ such that $\beta_f > \epsilon(\beta_b+\beta_f)$, which is always possible since $\beta_f > 0$, we conclude

$$\int_0^{+\infty} \|\nabla f(x(t)) - \nabla f(p)\|^2 dt < +\infty.$$

Combining this with $\int_0^{+\infty} \|B(x(t)) - B(p)\|^2 dt < +\infty$, it follows immediately

$$\int_0^{+\infty} \|A(x(t)) - A(p)\|^2 dt < +\infty. \tag{2.33}$$

Moreover, we also have

$$\begin{aligned} \int_0^{+\infty} \|A_\beta(x(t)) - A_\beta(p)\|^2 dt &= \int_0^{+\infty} \|\beta_f(\nabla f(x(t)) - \nabla f(p)) + \beta_b(B(x(t)) - B(p))\|^2 dt \\ &\leq (\beta_f^2 + \beta_b^2) \int_0^{+\infty} \|\nabla f(x(t)) - \nabla f(p)\|^2 + \|B(x(t)) - B(p)\|^2 dt < +\infty. \end{aligned} \tag{2.34}$$

According to (2.28) the trajectory $x(\cdot)$ is bounded. Set $R := \sup_{t\geq 0} \|x(t)\|$. By assumption, ∇f is Lipschitz continuous on the bounded sets. Let $L_R < +\infty$ be the Lipschitz constant of ∇f on $B(0,R)$. Since B is λ-cocoercive, it is $\frac{1}{\lambda}$-Lipschitz continuous. Therefore A is L-Lipschitz continuous on the trajectory with $L := L_R + \frac{1}{\lambda}$. Therefore

$$\frac{d}{dt}\|A(x(t))\| \leq \left\| \frac{d}{dt} A(x(t)) \right\| \leq L\|\dot{x}(t)\| \text{ for all } t \geq 0. \tag{2.35}$$

Using (2.33) and (2.35), we deduce that $u(t) := \|A(x(t))\|$ satisfies the condition of Lemma 1.3.3 (with $p = 2$ and $r = 2$). Therefore,

$$\lim_{t\to+\infty} \|A(x(t))\| = 0. \tag{2.36}$$

Likewise, according to (2.34), we have

$$\lim_{t\to+\infty} \|A_\beta(x(t)) - A_\beta(p)\| = 0. \tag{2.37}$$

By using the same argument as in (2.35), we obtain that $\frac{d}{dt}A_\beta(x(t))$ is bounded. From (2.35) we also get that

$$\int_0^{+\infty} \left\| \frac{d}{dt} A(x(t)) \right\|^2 dt < +\infty.$$

Similarly, we also have

$$\int_0^{+\infty} \left\| \frac{d}{dt} B(x(t)) \right\|^2 dt < +\infty.$$

By using (DINAM), we have

$$\begin{aligned}\ddot{x}(t) &= -\gamma\dot{x}(t) - A(x(t)) - \frac{d}{dt}A_\beta(x(t)) \\ &= -\gamma\dot{x}(t) - A(x(t)) - \beta_f\frac{d}{dt}A(x(t)) - (\beta_b - \beta_f)\frac{d}{dt}B(x(t)).\end{aligned}$$

Since the second member of the above equality belongs to $L^2(0,+\infty;\mathcal{H})$, we finally get

$$\int_0^{+\infty} \|\ddot{x}(t)\|^2\, dt < +\infty.$$

Combining this property with (2.30) and using Lemma 1.3.3, we deduce that

$$\lim_{t\to+\infty} \|\dot{x}(t)\| = 0. \tag{2.38}$$

The limit. In oder to indicate the existence of the weak limit of $x(t)$ as $t \to +\infty$, we use Opial's lemma (see [68] for more details). Given $p \in S$, let us define the anchor function given by, for every $t \in [0,+\infty[$,

$$q_p(t) := \frac{1}{2}\|x(t) - p\|^2.$$

From $\dot{q}_p(t) = \langle \dot{x}(t), x(t) - p\rangle$ and $\ddot{q}_p(t) = \|\dot{x}(t)\|^2 + \langle \ddot{x}(t), x(t) - p\rangle$, we obtain

$$\begin{aligned}\ddot{q}_p(t) + \gamma\dot{q}_p(t) &= \|\dot{x}(t)\|^2 + \langle \ddot{x}(t) + \gamma\dot{x}(t), x(t) - p\rangle \\ &= \|\dot{x}(t)\|^2 - \langle A(x(t)) + \frac{d}{dt}A_\beta(x(t)), x(t) - p\rangle \\ &\le \|\dot{x}(t)\|^2 - \langle \frac{d}{dt}A_\beta(x(t)), x(t) - p\rangle.\end{aligned}$$

Equivalently,

$$\ddot{q}_p(t) + \gamma\dot{q}_p(t) + \langle \frac{d}{dt}A_\beta(x(t)), x(t) - p\rangle \le \|\dot{x}(t)\|^2. \tag{2.39}$$

According to the differentiation formula for a product, we can rewrite (2.39) as follows :

$$\ddot{q}_p(t) + \gamma\dot{q}_p(t) + \frac{d}{dt}\left\langle A_\beta(x(t)) - A_\beta(p), x(t) - p\right\rangle \le \|\dot{x}(t)\|^2 + \left\langle A_\beta(x(t)) - A_\beta(p), \dot{x}(t)\right\rangle.$$

By the Cauchy-Lipschitz inequality, we get

$$\ddot{q}_p(t) + \gamma \dot{q}_p(t) + \frac{d}{dt} \langle A_\beta(x(t)) - A_\beta(p), x(t) - p \rangle \leq \|\dot{x}(t)\|^2 + \|A_\beta(x(t)) - A_\beta(p)\| \|\dot{x}(t)\|. \tag{2.40}$$

Then note that the second member of (2.40)

$$g(t) := \|\dot{x}(t)\|^2 + \|A_\beta(x(t)) - A_\beta(p)\| \|\dot{x}(t)\|$$

is nonnegative and belongs to $L^1(0, +\infty)$. Indeed, we have

$$\int_0^{+\infty} \|A_\beta(x(t)) - A_\beta(p)\| \|\dot{x}(t)\| dt \leq \frac{1}{2} \int_0^{+\infty} \|A_\beta(x(t)) - A_\beta(p)\|^2 dt + \frac{1}{2} \int_0^{+\infty} \|\dot{x}(t)\|^2 dt.$$

Using (2.30) and (2.34), we deduce that

$$\int_0^{+\infty} g(t) dt < +\infty.$$

Note that the left member of (2.40) can be rewritten as a derivative of a function, precisely

$$\ddot{q}_p(t) + \gamma \dot{q}_p(t) + \frac{d}{dt} \langle A_\beta(x(t)) - A_\beta(p), x(t) - p \rangle = \dot{h}(t)$$

with

$$h(t) = \dot{q}_p(t) + \gamma q_p(t) + \langle A_\beta(x(t)) - A_\beta(p), x(t) - p \rangle . \tag{2.41}$$

So we have

$$\dot{h}(t) \leq g(t) \text{ for every } \ t \geq 0.$$

Let us prove that the function h given in (2.41) is bounded from below by some constant. Indeed, since the terms $q_p(t)$ and $\langle A_\beta(x(t)) - A_\beta(p), x(t) - p \rangle$ are nonnegative, we have

$$h(t) \geq \dot{q}_p(t) \geq -\|\dot{x}(t)\| \, \|x(t) - p\|.$$

According to the boundedness of $x(\cdot)$ and $\dot{x}(\cdot)$ (see (2.28) and (2.38)), we deduce that there exists $m \in \mathbb{R}$ such that

$$h(t) \geq m \ \text{ for every } t \geq 0.$$

Let us introduce the real-valued function $\varphi : \mathbb{R}_+ \to \mathbb{R},\ t \mapsto \varphi(t)$ defined by

$$\varphi(t) = h(t) - \int_0^t g(s)ds.$$

We have $\varphi'(t) = \dot{h}(t) - g(t) \leq 0$. Hence, the function φ is nonincreasing on $[0, +\infty[$. This classically implies that the limit of φ exists as $t \to +\infty$. Since $g \in L^1(0, +\infty)$, we deduce that $\lim_{t\to+\infty} h(t)$ exists.
Using the fact that $\langle A_\beta(x(t)) - A_\beta(p), x(t) - p\rangle$ tends to zero as $t \to +\infty$ (a consequence of (2.37) and $x(\cdot)$ bounded), we obtain

$$\dot{q}_p(t) + \gamma q_p(t) = \theta(t)$$

with limit of $\theta(t)$ exists as $t \to +\infty$. The existence of the limit of q_p can be concluded thanks to a classical general result regarding the convergence of evolution equations governed by strongly monotone operators (here $\gamma\,\mathrm{Id}$, see Theorem 3.9 p.88 in [43]). Hence, for all $p \in S$,

$$\lim_{t\to+\infty} \|x(t) - p\| \quad \text{exists.}$$

To complete the proof via Opial's lemma, we need to verify that any weak sequential cluster point of $x(t)$ belongs to S. Let $t_n \to +\infty$ such that $x(t_n) \rightharpoonup x^*, n \to +\infty$. We have

$$A(x(t_n)) \to 0 \text{ strongly in } \mathcal{H} \ \text{ and } \ x(t_n) \rightharpoonup x^* \text{ weakly in } \mathcal{H}.$$

Due to the closedness property of the graph of the operator A in $w - \mathcal{H} \times s - \mathcal{H}$, we deduce that $A(x^*) = 0$, that is, $x^* \in S$.
As a result, $x(t)$ converges weakly as t goes to $+\infty$ and its limit belongs to S. The proof of Theorem 2.3.1 is thereby completed. ■

Remark 2.3.1 In the statement of Theorem 2.3.1, if we fix the rest of parameters, then the set of λs that fulfill the inequality can easily be found. Likewise, the feasible set of γs if the other parameters are fixed can be determined explicitly.
In fact, let us rewrite condition (2.13) as follows :

$$4\lambda > \frac{\beta_b^2 + \beta_f^2}{\gamma\beta_f} + \frac{2}{\gamma^2} + \frac{2}{\gamma}\sqrt{\beta_b^2 + \frac{1}{\gamma^2} + \frac{\beta_b^2 + \beta_f^2}{\gamma\beta_f}}.$$

Equivalently,

$$4\lambda + \beta_b^2 > \beta_b^2 + \frac{1}{\gamma^2} + \frac{\beta_b^2 + \beta_f^2}{\gamma\beta_f} + \frac{2}{\gamma}\sqrt{\beta_b^2 + \frac{1}{\gamma^2} + \frac{\beta_b^2 + \beta_f^2}{\gamma\beta_f}} + \frac{1}{\gamma^2}. \tag{2.42}$$

Thanks to

$$\beta_b^2 + \frac{1}{\gamma^2} + \frac{\beta_b^2 + \beta_f^2}{\gamma\beta_f} + \frac{2}{\gamma}\sqrt{\beta_b^2 + \frac{1}{\gamma^2} + \frac{\beta_b^2 + \beta_f^2}{\gamma\beta_f}} + \frac{1}{\gamma^2} = \left(\sqrt{\beta_b^2 + \frac{1}{\gamma^2} + \frac{\beta_b^2 + \beta_f^2}{\gamma\beta_f}} + \frac{1}{\gamma}\right)^2,$$

we immediately deduce that

$$4\lambda + \beta_b^2 > \left(\sqrt{\beta_b^2 + \frac{1}{\gamma^2} + \frac{\beta_b^2 + \beta_f^2}{\gamma\beta_f}} + \frac{1}{\gamma}\right)^2.$$

Therefore (2.42) is equivalent to

$$\sqrt{\beta_b^2 + \frac{1}{\gamma^2} + \frac{\beta_b^2 + \beta_f^2}{\gamma\beta_f}} + \frac{1}{\gamma} < \sqrt{4\lambda + \beta_b^2}.$$

This in turn is equivalent to

$$\begin{cases} \dfrac{1}{\gamma} & < \sqrt{4\lambda + \beta_b^2} \\ \left(\sqrt{\beta_b^2 + \dfrac{1}{\gamma^2} + \dfrac{\beta_b^2 + \beta_f^2}{\gamma\beta_f}}\right)^2 & < \left(\sqrt{4\lambda + \beta_b^2} - \dfrac{1}{\gamma}\right)^2. \end{cases} \tag{2.43}$$

From the first inequation of (2.43), we deduce that

$$\gamma > \frac{1}{\sqrt{4\lambda + \beta_b^2}}. \tag{2.44}$$

From the second inequation of (2.43), we deduce that

$$\beta_b^2 + \frac{1}{\gamma^2} + \frac{\beta_b^2 + \beta_f^2}{\gamma\beta_f} < 4\lambda + \beta_b^2 + \frac{1}{\gamma^2} - \frac{2}{\gamma}\sqrt{4\lambda + \beta_b^2}.$$

Therefore,

$$\gamma > \frac{1}{4\lambda}\left(\frac{\beta_b^2 + \beta_f^2}{\beta_f} + 2\sqrt{4\lambda + \beta_b^2}\right). \tag{2.45}$$

Since (2.45) implies (2.44), we obtain that the feasible set of γs is defined by

$$\gamma > \frac{1}{4\lambda}\left(\frac{\beta_b^2+\beta_f^2}{\beta_f} + 2\sqrt{4\lambda+\beta_b^2}\right).$$

2.3.2 Case $\beta_b = \beta_f$

Let us specialize the preceding results in the case $\beta_b = \beta_f$. We set $\beta_b = \beta_f := \beta > 0$ and $A := \nabla f + B$. Thus, we consider the evolution system

$$\text{(DINAM)} \quad \ddot{x}(t) + \gamma\dot{x}(t) + A(x(t)) + \beta\frac{d}{dt}\left(A(x(t))\right) = 0, \quad t \geq 0.$$

The existence of strong global solutions of the system is guaranteed by Theorem 2.2.1. The asymptotic behavior of the solution trajectories of this system is a consequence of Theorem 2.3.1 and is stated as below.

Corollary 2.3.1 *Let $B : \mathcal{H} \to \mathcal{H}$ be a λ-cocoercive operator and $f : \mathcal{H} \to \mathbb{R}$ be a $\mathcal{C}^1$ convex function with a Lipschitz continuous gradient on the bounded sets. Suppose that the solution set $S = (\nabla f + B)^{-1}(0) \neq \emptyset$. Consider the evolution equation* (DINAM)*, where $A = \nabla f + B$, $\beta_b = \beta_f := \beta > 0$ and where the related parameters fulfill the following conditions :*

$$\gamma > 0,\ \beta > 0, \quad \text{and } \lambda\gamma > \beta + \frac{1}{\gamma}. \tag{2.46}$$

Then, for any solution trajectory $x : [0, +\infty[\to \mathcal{H}$ of (DINAM)*, the following properties are satisfied :*

(i) *(convergence) The trajectory $x(\cdot)$ is bounded and $x(t)$ converges weakly, as $t \to +\infty$, to an element $x^* \in S$.*

(ii) *(integral estimate)*

$$\int_0^{+\infty} \|\dot{x}(t)\|^2 dt < +\infty, \int_0^{+\infty} \|\ddot{x}(t)\|^2 dt < +\infty,$$
$$\int_0^{+\infty} \|A(x(t))\|^2 dt < +\infty, \text{ and } \int_0^{+\infty} \left\|\frac{d}{dt}A(x(t))\right\|^2 dt < +\infty.$$

(iii) *(pointwise estimate)*

$$\lim_{t\to+\infty} \|\dot{x}(t)\| = 0, \quad \text{and } \lim_{t\to+\infty} \|A(x(t))\| = 0.$$

Remark 2.3.2 It is worth presenting the result of Corollary 2.3.1 apart because this is an important case. This also enables us to emphasize this result compared to the existing work for second-order dissipative evolution systems regarding cocoercive operators. Indeed, letting β go to zero in (2.46) gives the condition

$$\lambda\gamma^2 > 1 \tag{2.47}$$

introduced by Attouch and Maingé in [24] to study the second order dynamic (2.4) without geometric damping. With respect to [24], the introduction of the geometric damping, *i.e.*, taking $\beta > 0$, provides some useful additional estimates.

2.4 Numerical illustrations : a first sight

In this section, we give some numerical illustrations of (DINAM). The detailed algorthmic analysis for these dynamical systems will be presented later. Our goal here is to make some numerical experiments to solve some certain problems by using the temporal discretization of (DINAM). At this moment, we postpose to discuss about the convergence of the algorithms.

2.4.1 From continuous dynamic to algorithms

Let us first give some indications concerning the algorithms derived from temporal discretization of the continuous dynamic (DINAM). At the moment, we postpone to next chapter the convergence analysis. Let us recall the condensed formulation of (DINAM)

$$\ddot{x}(t) + \gamma\dot{x}(t) + A(x(t)) + \frac{d}{dt}\left(A_\beta(x(t))\right) = 0, \tag{DINAM}$$

where $A := \nabla f + B$ and $A_\beta := \beta_b B + \beta_f \nabla f$. Take a fixed time step $h > 0$, and consider the following finite-difference scheme for (DINAM) :

$$\frac{1}{h^2}(x_{k+1} - 2x_k + x_{k-1}) + \frac{\gamma}{h}(x_{k+1} - x_k) + \frac{\beta_b}{h}(B(x_{k+1}) - B(x_k))$$
$$+\frac{\beta_f}{h}(\nabla f(x_k) - \nabla f(x_{k-1})) + B(x_{k+1}) + \nabla f(x_k) = 0. \tag{2.48}$$

This scheme is implicit with respect to the nonpotential B and explicit with respect to the potential operator ∇f. The temporal discretization of the Hessian driven damping $\beta_f \nabla^2 f(x(t))\dot{x}(t)$ is taken equal to $\frac{\beta_f}{h}(\nabla f(x_k) - \nabla f(x_{k-1}))$. After expanding (2.48), we obtain

$$x_{k+1} + \frac{h^2}{1+\gamma h} B(x_{k+1}) + \frac{h\beta_b}{1+\gamma h} B(x_{k+1}) = x_k + \frac{1}{1+\gamma h}(x_k - x_{k-1}) + \frac{h\beta_b}{1+\gamma h} B(x_k) - \frac{h\beta_f}{1+h\gamma}(\nabla f(x_k) - \nabla f(x_{k-1})) - \frac{h^2}{1+h\gamma}\nabla f(x_k). \tag{2.49}$$

Set $s := \frac{h}{1+\gamma h}$ and $\alpha := \frac{1}{1+\gamma h}$. So we have

$$x_{k+1} + s\mathcal{B}_h(x_{k+1}) = y_k, \tag{2.50}$$

where $\mathcal{B}_h = (h + \beta_b)B$, and

$$y_k = x_k + \alpha(x_k - x_{k-1}) + s\beta_b B(x_k) - s(h + \beta_f)\nabla f(x_k) + s\beta_f \nabla f(x_{k-1}). \tag{2.51}$$

From (2.51) we get

$$x_{k+1} = (\mathrm{Id} + s\mathcal{B}_h)^{-1}(y_k). \tag{2.52}$$

By combining (2.51) and (2.52), we obtain the following algorithm, called (DINAAM). It is a splitting algorithm which involves the operators ∇f and B separately.

(DINAAM) :

Initialize : $x_0 \in \mathcal{H}, x_1 \in \mathcal{H}$

$h > 0,$

$\alpha = \frac{1}{1+\gamma h},$

$s = \frac{h}{1+\gamma h},$

$y_k = x_k + \alpha(x_k - x_{k-1}) + s\beta_b B(x_k) - s(h + \beta_f)\nabla f(x_k) + s\beta_f \nabla f(x_{k-1}),$

$x_{k+1} = (I + s\mathcal{B}_h)^{-1}(y_k).$

(2.53)

2.4.2 Numerical experiments for the continuous dynamics (DINAM)

A general and wise method to generate monotone cocoercive operators which are not gradients of convex functions is to take Yosida approximation A_λ of a linear skew-symmetric operator A. As a model situation, take $\mathcal{H} = \mathbb{R}^2$ and start from A is the rotation of angle $\frac{\pi}{2}$. We have $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. As a result of an elementary computation, for any $\lambda > 0$, $A_\lambda = \frac{1}{1+\lambda^2}\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}$ is λ-cocoercive. Typically, for $\lambda = 1$, we obtain that the matrix $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is $\frac{1}{2}$-cocoercive. In addition, many other cocoercive operators that are not potential operators may be readily constructed by using these basic blocks. For that, use Lemma 1.3.1 which gives that the set of cocoercive operators is a convex cone.

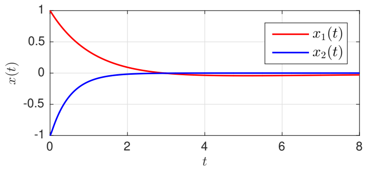
Example 2.4.1 Let us start this section with a simple illustrative example in $\mathbb{R}^2$. We take $\mathcal{H} = \mathbb{R}^2$ endowed with the normal Euclidean structure and B as a linear operator defined by $B = A_\lambda$ for $\lambda = 5$. According to the above remark, we can check that B is λ-cocoercive with $\lambda = 5$ and that B is a nonpotential operator. To observe the oscillations, in the model of heavy ball with friction, we take $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x_1, x_2) = 50x_2^2.$$

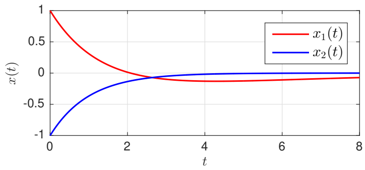
We set $\gamma = 0.9$. It is obvious that f is convex but not strongly convex. We study three cases : (1) $\beta_b = 1, \beta_f = 0.5$, (2) $\beta_b = 0.5, \beta_f = 1$, and (3) $\beta_b = \beta_f = 0.5$. As a straightforward application of Theorem 2.3.1, we obtain that the trajectory $x(t)$ generated by (DINAM) converges to x_∞, where $x_\infty \in S = (B + \nabla f)^{-1}(0) = \{0\}$. The trajectory obtained by using Matlab is depicted in Figure 2.1, where we represent the components $x_1(t)$ and $x_2(t)$ in red and blue respectively.

Now we examine the trajectory behaviour by studying more different values of β_b and β_f. We study four cases in Figures 3.2. The traces of the second solution variable have been depicted in Figure 3.2(a), while in Figure 3.2(b) the number of iterations k versus $\|B(x_k) + \nabla f(x_k)\|$ is plotted. Through Figures 2.1 and 3.2, we can conclude that by introducing the Hessian damping ($\beta_f > 0$), the oscillations of the trajectories in Figure 3.2 are attenuated. The oscillations of the solutions appear whenever β_f goes to 0.

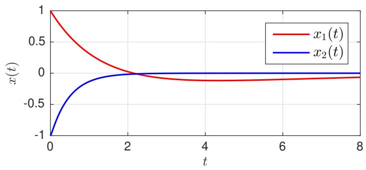
Example 2.4.2 Now we are looking at another higher dimensional example. Let us consider $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = \frac{1}{2}\|Mx - b\|^2$, where $M \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. We



(a) Case $\beta_b = 1, \beta_f = 0.5$.

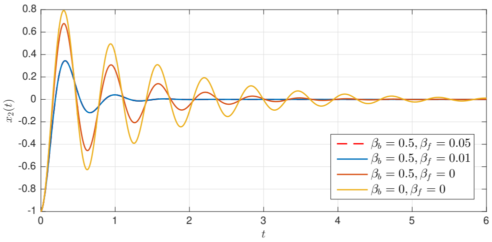


(b) Case $\beta_b = 0.5, \beta_f = 1$.

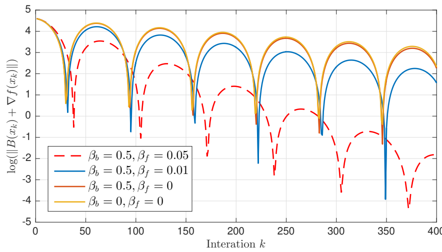


(c) Case $\beta_b = 0.5, \beta_f = 0.5$.

FIGURE 2.1 – Trajectories of (DINAM) for different values of the parameters β_b, β_f.



(a)



(b)

FIGURE 2.2 – Oscillation of the trajectories of (DINAM) for different values of β_b, β_f.

have

$$\nabla f(x) = M^\top (Mx - b), \; \nabla^2 f(x) = M^\top M.$$

Since $M^\top M$ is positive semidefinite for any matrix M, the quadratic function f is convex. Furthermore, if M has full column rank, *i.e.*, rank(M) $= n$, then $M^\top M$ is positive definite. Therefore f is strongly convex. Take $B = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & \vdots \\ 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & 1 \end{pmatrix} \in \mathbb{R}^{n \times n}$. Then B is cocoercive. Indeed, for any $x, y \in \mathbb{R}^n$,

$$\begin{aligned} \langle Bx - By, x - y \rangle &= \|x_1 - y_1\|^2 + \|x_2 - y_2\|^2 + \cdots + \|x_n - y_n\|^2 \\ &\geq \frac{1}{2} \left[2(\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2) + \|x_3 - y_3\|^2 + \cdots + \|x_n - y_n\|^2 \right] \\ &= \frac{1}{2} \|Bx - By\|^2. \end{aligned}$$

If the matrix M is not full column rank with $M^\top M + B$ nonsingular, we then have

$$B(x) + \nabla f(x) = 0 \text{ if and only if } x = (M^\top M + B)^{-1} M^\top b.$$

In our experiment, we take M a random 10×100 matrix which is not full column rank. Set $\gamma = 3$, $\beta_b = 1$, $\beta_f = 1$ and the operator B as above. Thanks to Corollary 2.3.1, we conclude that the trajectory $x(t)$ generated by the system (DINAM) converges to $x_\infty = (M^\top M + B)^{-1} M^\top b$. Implementing the algorithm (DINAAM) in Matlab, we obtain the plot of k versus the norm of $B(x_k) + \nabla f(x_k)$. Similarly, we study several cases by changing the parameters β_b, β_f. This is depicted in Figure 2.3.

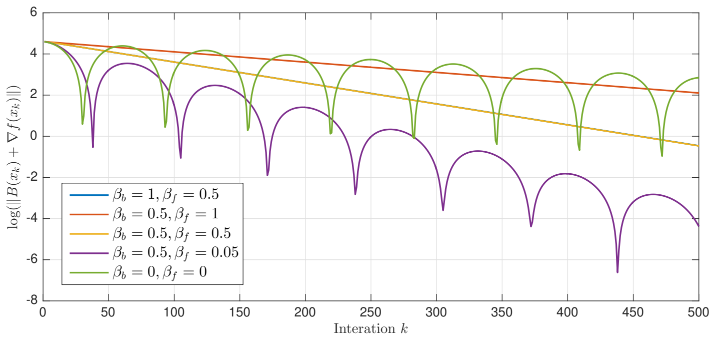


FIGURE 2.3 – The behaviour of (DINAAM) for a high dimension problem.

Before closing this part, we study an application of our model to dynamical games. The following example is taken from Attouch and Maingé [24] and adapted to our context.

Example 2.4.3 We make the following standing assumptions :

(i) $\mathcal{H} = \mathcal{X}_1 \times \mathcal{X}_2$ is the Cartesian product of two Hilbert spaces endowed with norms $\|\cdot\|_{\mathcal{X}_1}$ and $\|\cdot\|_{\mathcal{X}_2}$ respectively. In which, $x = (x_1, x_2)$, with $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$, stands for an element in $\mathcal{H}$;

(ii) $f : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathbb{R}$ is a convex function whose gradient is Lipschitz continuous on bounded sets ;

(iii) $B = (\nabla_{x_1}\mathcal{L}, -\nabla_{x_2}\mathcal{L})$ is the maximal monotone operator associated to a smooth convex-concave function $\mathcal{L} : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathbb{R}$. The operator B is assumed to be λ-cocoercive with $\lambda > 0$.

In our setting, with $x(t) = (x_1(t), x_2(t))$ the system (DINAM) is written

$$\begin{cases} \ddot{x}_1(t) & + \gamma \dot{x}_1(t) + \nabla_{x_1} f(x_1(t), x_2(t)) + \nabla_{x_1} \mathcal{L}(x_1(t), x_2(t)) \\ & + \beta_f \frac{d}{dt} (\nabla_{x_1} f(x_1(t), x_2(t))) + \beta_b \frac{d}{dt} (\nabla_{x_1} \mathcal{L}(x_1(t), x_2(t))) = 0 \\ \ddot{x}_2(t) & + \gamma \dot{x}_2(t) + \nabla_{x_2} f(x_1(t), x_2(t)) - \nabla_{x_2} \mathcal{L}(x_1(t), x_2(t)) \\ & + \beta_f \frac{d}{dt} (\nabla_{x_2} f(x_1(t), x_2(t))) - \beta_b \frac{d}{dt} (\nabla_{x_2} \mathcal{L}(x_1(t), x_2(t))) = 0. \end{cases} \tag{2.54}$$

According to Theorem 2.3.1, $x(t) \rightharpoonup x_\infty = (x_{1,\infty}, x_{2,\infty})$ weakly in $\mathcal{H}$, where $(x_{1,\infty}, x_{2,\infty})$ is solution of

$$\begin{cases} \nabla_{x_1} f(x_1(t), x_2(t)) + \nabla_{x_1} \mathcal{L}(x_1(t), x_2(t)) = 0 \\ \nabla_{x_2} f(x_1(t), x_2(t)) - \nabla_{x_2} \mathcal{L}(x_1(t), x_2(t)) = 0. \end{cases} \tag{2.55}$$

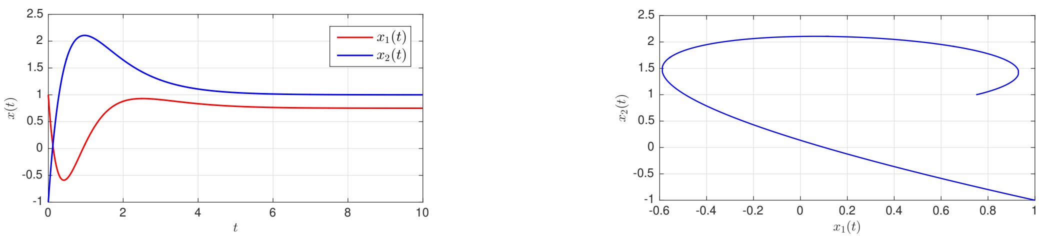
Structured systems such as (2.55) include potential and nonpotential terms which often present in decision sciences and physics. In game theory, (2.55) describes Nash equilibria of the normal form game with two players 1, 2 whose static loss functions are respectively given by

$$\begin{cases} F_1 : (x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2 \to F_1(x_1, x_2) = f(x_1, x_2) + \mathcal{L}(x_1, x_2) \\ F_2 : (x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2 \to F_2(x_1, x_2) = f(x_1, x_2) - \mathcal{L}(x_1, x_2). \end{cases} \tag{2.56}$$

$f(\cdot, \cdot)$ is their joint convex payoff, and $\mathcal{L}$ is a convex-concave payoff with zero-sum rule. For more details, we refer the reader to [24]. As an example, take $\mathcal{X}_1 = \mathcal{X}_2 = \mathbb{R}$ and $\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}$ given by $\mathcal{L}(x) = \frac{1}{2}(x_1^2 - 2x_1x_2 - x_2^2)$. Then $B = (\nabla_{x_1} \mathcal{L}, -\nabla_{x_2} \mathcal{L}) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Pick $f(x) = \frac{1}{2}(3x_1^2 - 2x_1x_2 + x_2^2) - x_1 - 2x_2$. The Nash equilibria described in (2.55) can be solved by using (DINAM). Take $\gamma = 3, \beta_b = 0.5, \beta_f = 0.5$ and $x_0 = (1, -1), \dot{x}_0 = (-10, 10)$ as initial conditions, then the numerical solution for (DINAM) converges to $x_\infty = (\frac{3}{4}, 1)$ which is the solution of (2.55) as well. The numerical trajectories and phase portrait of our model applied to dynamical games are depicted in Figure 2.4.

2.5 The nonsmooth case

The equivalence obtained in Proposition 2.2.1 between (DINAM) and a first-order evolution system in time and space enables a logical extension of our results in theoretical and numerical aspects to the case of a convex, lower semicontinuous and proper function $f : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$. It is sufficient to substitute the gradient of f with the convex



(a) **(b)**

Figure 2.4 – An application of (DINAM) to dynamical games : trajectories (a) and phase portrait (b).

subdifferential ∂f. We recall that the subdifferential of f at $x \in \mathcal{H}$ is given by

$$\partial f(x) = \{z \in \mathcal{H} : \langle z, \xi - x\rangle \leq f(\xi) - f(x) \text{ for every } \xi \in \mathcal{H}\},$$

and the domain of f is $\operatorname{dom} f = \{x \in \mathcal{H} : f(x) < +\infty\}$. This leads to consider the system

$$\text{(g-DINAM)} \begin{cases} \dot{x}(t) + \beta_f \partial f(x(t)) + \beta_b B(x(t)) + \left(\gamma - \dfrac{1}{\beta_f}\right) x(t) + y(t) \ni 0; \\ \dot{y}(t) - \left(1 - \dfrac{\beta_b}{\beta_f}\right) B(x(t)) + \dfrac{1}{\beta_f}\left(\gamma - \dfrac{1}{\beta_f}\right) x(t) + \dfrac{1}{\beta_f} y(t) = 0. \end{cases}$$

The prefix g preceding (DINAM) indicates generalized. It should be noticed that the first equation of (g-DINAM) is now a differential inclusion, because of the possibility for $\partial f(x(t))$ to be multivalued. Take $f = f_0 + \delta_C$, in which δ_C is the indicator function of a constraint set C, the system (g-DINAM) enables to model damped inelastic shocks in decision sciences and mechanics, see [25]. The original aspect comes from the fact that (g-DINAM) now involves both potential driven forces (attached to f_0) and nonpotential driven forces (attached to B). As we will discover, taking into account shocks caused by nonpotential driving forces is a source of difficulties.

Let us first state the well-posedness of the solution trajectory of the Cauchy problem.

Theorem 2.5.1 *Let $f : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semicontinuous, and proper function. Assume that $\beta_f > 0$ and $\beta_b \geq 0$. Then, for any $(x_0, y_0) \in \operatorname{dom} f \times \mathcal{H}$, there exists a unique strong global solution $(x, y) : [0, +\infty[\to \mathcal{H} \times \mathcal{H}$ of (g-DINAM) which satisfies the Cauchy data $x(0) = x_0$, $y(0) = y_0$.*

Proof. The proof is parallel to that of Theorem 2.2.1. The system (g-DINAM) can be equivalently written as

$$\dot{Z}(t) + \partial\Phi(Z(t)) + G(Z(t)) \ni 0, \quad Z(0) = (x_0, y_0), \tag{2.57}$$

where $Z := (x, y)$, and the function $\Phi(Z) = \Phi(x, y) := \beta_f f(x)$ is now convex lower semicontinuous and proper on $\mathcal{H} \times \mathcal{H}$. The operator G is unchanged and is globally Lipschitz continuous. The above equation falls under the setting of the Lipschitz perturbation of an evolution system governed by the subdifferential of a convex lower semicontinuous and proper function. The existence and uniqueness of the strong solution to (2.57) follows from Brézis [43, Proposition 3.12] and the fact that $(x_0, y_0) \in \operatorname{dom} \Phi$. Recall that strong solution means that $x(\cdot)$ and $y(\cdot)$ are locally absolutely continuous functions whose distributional derivatives $\dot{x}$ and $\dot{y}$ belong to $L^2(0, T, \mathcal{H})$ for any $T > 0$. ∎

Remark 2.5.1 As a consequence of the general theory developed above, the system (g-DINAM) satisfies a regularization effect on the initial condition. Precisely given $(x_0, y_0) \in \overline{\operatorname{dom} f} \times \mathcal{H}$, there still exists a unique strong solution to the corresponding Cauchy problem, but now with $\sqrt{t}\dot{x}(t) \in L^2(0, T, \mathcal{H})$ and $\sqrt{t}\dot{y}(t) \in L^2(0, T, \mathcal{H})$ for any $T > 0$.

The solution set S is now defined by

$$S := \{p \in \mathcal{H} : \ \partial f(p) + B(p) \ni 0\}.$$

Before stating our main result, notice that $B(p)$ is uniquely defined for $p \in S$.

Lemma 2.5.1 *$B(p)$ is uniquely defined for $p \in S$, i.e.,*

$$p_1 \in S, \ p_2 \in S \Longrightarrow B(p_1) = B(p_2).$$

Proof. The proof is similar to that of Lemma 2.3.1 and is based on the cocoercivity of the operator B and the monotonicity of the subdifferential of f. ∎

For the sake of simplicity, we give a detailed proof of the convergence analysis in the case $\beta_f = \beta_b = \beta > 0$. The system (g-DINAM) takes the simplified form :

$$\text{(g-DINAM)} \begin{cases} \dot{x}(t) + \beta \partial f(x(t)) + \beta B(x(t)) + \left(\gamma - \dfrac{1}{\beta}\right) x(t) + y(t) \ni 0; \\ \dot{y}(t) + \dfrac{1}{\beta}\left(\gamma - \dfrac{1}{\beta}\right) x(t) + \dfrac{1}{\beta} y(t) = 0. \end{cases}$$

To demonstrate the convergence results and associated estimations, we construct the first equation of (g-DINAM) as follows :

$$\dot{x}(t) + \beta \xi(t) + \beta B(x(t)) + \left(\gamma - \frac{1}{\beta}\right) x(t) + y(t) = 0, \tag{2.58}$$

where $\xi(t) \in \partial f(x(t))$, and we set $A(x(t)) = \xi(t) + B(x(t))$.

Theorem 2.5.2 *Let $B : \mathcal{H} \to \mathcal{H}$ be a λ-cocoercive operator. Let $f : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$*

be a convex, lower semicontinuous, proper function. Assume that $S = \{p \in \mathcal{H} : 0 \in \partial f(p) + B(p)\} \neq \emptyset$. *Consider the evolution equation* (g-DINAM) *where the parameters fulfill the following conditions :* $\beta_f = \beta_b = \beta > 0$ *and*

$$\gamma > 0, \beta > 0 \ \text{ and } \ \lambda\gamma > \beta + \frac{1}{\gamma}. \tag{2.59}$$

Then, for any solution trajectory $x : [0, +\infty[\to \mathcal{H}$ *of* (g-DINAM), *the following properties are satisfied :*

(i) *(integral estimates) Set* $A(x(t)) := \xi(t) + B(x(t))$ *with* $\xi(t) \in \partial f(x(t))$ *as defined in (2.58) and* $p \in S$. *Then*

$$\int_0^{+\infty} \|\dot{x}(t)\|^2 dt < +\infty, \int_0^{+\infty} \|B(x(t)) - B(p)\|^2 dt < +\infty,$$
$$\int_0^{+\infty} \|A(x(t))\|^2 dt < +\infty, \int_0^{\infty} \langle A(x(t)), x(t) - p\rangle \, dt < +\infty.$$

(ii) *(convergence) For any* $p \in S$,

1. $\lim_{t\to+\infty} \|x(t) - p\|$ *exists.*
2. $\lim_{t\to+\infty} \|B(x(t)) - B(p)\| = 0$, *where* $B(p)$ *is uniquely defined for* $p \in S$.

Proof. Let us extend the Lyapunov analysis presented in the preceding sections to the case where f is nonsmooth. However, the following points have to be paid attention to. First, we must invoke the (generalized) chain rule for derivatives over curves (see [43, Lemma 3.3]), that is, for a.e $t \geq 0$,

$$\frac{d}{dt} f(x(t)) = \langle \xi(t), \dot{x}(t)\rangle \, .$$

The second ingredient is valid as a consequence of the subdifferential inequality for convex functions.

Let us consider the Lyapunov function $t \in [0, +\infty[\mapsto \mathcal{E}_p(t) \in \mathbb{R}_+$ defined by

$$\mathcal{E}_p(t) := \frac{1}{2}\|x(t) - p + c\Big(\dot{x}(t) + \beta A(x(t))\Big)\|^2 + \frac{\delta}{2}\|x(t) - p\|^2 + [c\delta\beta + c^2]\Gamma(t), \tag{2.60}$$

where we recall that $A(x(t)) := \xi(t) + B(x(t))$ with $\xi(t) \in \partial f(x(t))$ as defined in (2.58) and $p \in S$. To differentiate $\mathcal{E}_p(t)$, we use the formulation (g-DINAM)

$$\dot{x}(t) + \beta A(x(t)) = -\left(\gamma - \frac{1}{\beta}\right) x(t) - y(t).$$

Since both x and y are locally absolutely continuous functions, this makes it possible to differentiate $\dot{x}(t)+\beta A(x(t))$ and obtain analogous formulas as in the smooth case. Then a close examination of the Lyapunov analysis indicates that we can obtain the additional estimate

$$\int_0^\infty \langle A(x(t)), x(t)-p\rangle\, dt < +\infty. \tag{2.61}$$

Set $0 \in \partial f(p)+B(p)$. To obtain (2.61), we return to (2.18) and study the following minorization, in which we divide into a sum with coefficients ϵ' and $1-\epsilon'$ (where $\epsilon' > 0$ will be taken small enough). According to the monotonicity of ∂f and the definition of $A(x(t))$, we have

$$\begin{aligned} c\langle A(x(t)), x(t)-p\rangle &= c\epsilon'\langle A(x(t)), x(t)-p\rangle + c(1-\epsilon')\langle A(x(t)) - Ap, x(t)-p\rangle \\ &\geq c\epsilon'\langle A(x(t)), x(t)-p\rangle + c(1-\epsilon')\langle B(x(t)) - B(p), x(t)-p\rangle \\ &\geq c\epsilon'\langle A(x(t)), x(t)-p\rangle + c(1-\epsilon')\lambda\|B(x(t)) - B(p)\|^2. \end{aligned} \tag{2.62}$$

In our assumptions, the inequality $\lambda\gamma > \beta + \dfrac{1}{\gamma}$ is strict and still satisfied by $(1-\epsilon')\lambda$ when ϵ' is taken small enough. Therefore, the proof continues with λ replaced by $(1-\epsilon')\lambda$ without changing the conditions on the parameters. Hence, after integrating the resulting strict Lyapunov inequality, we obtain the supplementary property (2.61). Until (2.34), the proof is substantially identical to that of a smooth function f. We obtain these estimates

$$\int_0^{+\infty} \|\dot{x}(t)\|^2 dt < +\infty, \int_0^{+\infty} \|B(x(t)) - B(p)\|^2 dt < +\infty, \int_0^{+\infty} \|A(x(t))\|^2 dt < +\infty.$$

However, we can no longer apply the Lipschitz continuity to the bounded sets of ∇f. To avoid this obstacle, we modify the rest of the proof as follows. Recall that given $p \in S$, the anchor function is defined by, for every $t \in [0, +\infty[$,

$$q_p(t) := \frac{1}{2}\|x(t)-p\|^2,$$

and that we must to show the existence of the anchor function limit as $t \to +\infty$. The goal is to exploit the fact that we have a large collection of Lyapunov functions that are parametrized by the coefficient c. Note that we have claimed that the limit of $\mathcal{E}_p(t)$ exists as $t \to +\infty$, and this is fulfilled for the whole interval of values of c. So, for such c, the limit of $W_c(t) := \dfrac{1}{c\delta\beta + c^2}\mathcal{E}_p(t)$ as $t \to +\infty$ exists, where

$$W_c(t) = \frac{1}{2(c\delta\beta + c^2)}\|x(t)-p+c\Big(\dot{x}(t)+\beta A(x(t))\Big)\|^2 + \frac{\delta}{2(c\delta\beta+c^2)}\|x(t)-p\|^2 + \Gamma(t).$$

We have

$$\begin{aligned}W_c(t) &= \frac{1+\delta}{2(c\delta\beta+c^2)}\|x(t)-p\|^2 + \frac{c^2}{2(c\delta\beta+c^2)}\|\dot{x}(t)+\beta A(x(t))\|^2 \\ &+ \frac{c}{c\delta\beta+c^2}\left\langle \dot{x}(t)+\beta A(x(t)), x(t)-p\right\rangle \\ &= \frac{\gamma}{2((c\gamma-1)\beta+c)}\|x(t)-p\|^2 + \frac{c}{2((c\gamma-1)\beta+c)}\|\dot{x}(t)+\beta A(x(t))\|^2 \\ &+ \frac{1}{(c\gamma-1)\beta+c}\left\langle \dot{x}(t)+\beta A(x(t)), x(t)-p\right\rangle.\end{aligned}$$

Thus, take two values of c, let c_1 and c_2, we immediately deduce that

$$\begin{aligned}W_{c_1}(t)-W_{c_2}(t) = \frac{\gamma}{2}&\left[\frac{1}{(c_1\gamma-1)\beta+c_1}-\frac{1}{(c_2\gamma-1)\beta+c_2}\right]\|x(t)-p\|^2 \\ +\frac{1}{2}&\left[\frac{c_1}{(c_1\gamma-1)\beta+c_1}-\frac{c_2}{(c_2\gamma-1)\beta+c_2}\right]\|\dot{x}(t)+\beta A(x(t))\|^2 \\ +&\left[\frac{1}{(c_1\gamma-1)\beta+c_1}-\frac{1}{(c_2\gamma-1)\beta+c_2}\right]\left\langle \dot{x}(t)+\beta A(x(t)), x(t)-p\right\rangle.\end{aligned}$$

Notice that

$$\begin{aligned}&\left[\frac{c_1}{(c_1\gamma-1)\beta+c_1}-\frac{c_2}{(c_2\gamma-1)\beta+c_2}\right] : \left[\frac{1}{(c_1\gamma-1)\beta+c_1}-\frac{1}{(c_2\gamma-1)\beta+c_2}\right] \\ &= \frac{c_1((c_2\gamma-1)\beta+c_2)-c_2((c_1\gamma-1)\beta+c_1)}{(c_2\gamma-1)\beta+c_2-(c_1\gamma-1)\beta-c_1} \\ &= \frac{\beta(c_2-c_1)}{(\gamma\beta+1)(c_2-c_1)} \\ &= \frac{\beta}{\gamma\beta+1}.\end{aligned}$$

Therefore,

$$W_{c_1}(t)-W_{c_2}(t) = \left[\frac{1}{(c_1\gamma-1)\beta+c_1}-\frac{1}{(c_2\gamma-1)\beta+c_2}\right]W(t)$$

in which

$$W(t) := \frac{\gamma}{2}\|x(t)-p\|^2 + \frac{\beta}{2(\gamma\beta+1)}\|\dot{x}(t)+\beta A(x(t))\|^2 + \left\langle \dot{x}(t)+\beta A(x(t)), x(t)-p\right\rangle.$$

So, we obtain the existence of the limit as $t\to+\infty$ of $W(t)$. Then note that $W(t) =$

$\gamma q_p(t) + \dfrac{d}{dt}w(t)$ where

$$w(t) := q_p(t) + \beta \int_0^t \langle A(x(s), x(s) - p\rangle \, ds + \frac{\beta}{2(\gamma\beta+1)} \int_0^t \|\dot{x}(s) + \beta A(x(s))\|^2 ds.$$

Reformulate $W(t)$ in terms of $w(t)$ as follows :

$$W(t) = \gamma w(t) + \frac{d}{dt}w(t) - \Big(\gamma\beta \int_0^t \langle A(x(s)), x(s) - p\rangle \, ds + \frac{\gamma\beta}{2(\gamma\beta+1)} \int_0^t \|\dot{x}(s) + A(x(s))\|^2 ds\Big).$$

As a consequence of (2.61) and of the former estimates, it yields the limit of the two previous integrals exists as $t \to +\infty$. According to the convergence of $W(t)$, we obtain that

$$\lim_{t\to+\infty} \Big(\gamma w(t) + \frac{d}{dt}w(t)\Big) \text{ exists.}$$

The existence of the limit of w follows from a classical general result concerning the convergence of evolution equations governed by strongly monotone operators (here $\gamma \,\mathrm{Id}$, see Theorem 3.9 p.88 in [43]). In turn, using the same argument as above, we obtain that, for all $p \in S$,

$$\lim_{t\to+\infty} \|x(t) - p\| \quad \text{exists.}$$

As in the smooth case, the strong convergence of $B(x(t))$ to $B(p)$ is a direct consequence of the integral estimates $\int_0^{+\infty} \|B(x(t)) - B(p)\|^2 dt < +\infty$, $\int_0^{+\infty} \|\dot{x}(t)\|^2 dt < +\infty$ and of the fact that B is Lipschitz continuous. The proof of Theorem 2.5.2 is thereby completed. ■

Remark 2.5.2 (i) A natural question is to know if the weak limit of the trajectory exists. Indeed we are not far from this result since $\int_0^{+\infty} \|A(x(t))\|^2 dt < +\infty$, which implies that $A(x(t))$ converges strongly to zero in an "essential" way. Opial's lemma allows to complete the convergence proof likely in the smooth case. This seems to be a challenging question to examine ahead.

(ii) A particular situation is the case $\gamma = \dfrac{1}{\beta}$, in which case the system (g-DINAM) can be written in an equivalent way

$$\dot{u}(t) + \gamma u(t) = 0,$$

where

$$\dot{x}(t) + \beta A(x(t)) \ni u(t).$$

The convergence of the trajectory $t \mapsto x(t)$ is therefore a result of the characteristic of the semigroup generated by the sum of the subdifferential of a convex, lower semicontinuous, and proper function with a cocoercive operator, see Abbas and Attouch [1]. Note that in this instance, the requirement for the convergence of the trajectories generated by (g-DINAM) is no longer dependent on the coercivity parameter λ.

2.6 Conclusion, perspectives

Throughout this chapter, in a general setting of Hilbert's real space, we have studied a dynamic inertial Newton method for solving additively structured monotone problems. In which, the corresponding dynamics are driven by the sum of two monotone operators with distinct aspects : the potential part is the gradient of a continuously convex differentiable function f, and the nonpotential one is a monotone and cocoercive operator B. The presences of the Hessian of the potential f and a Newton-type correction term attached to B have controlled the geometric damping. In addition, we have shown not only the well-posedness of the Cauchy problem but also the asymptotic convergence properties of the trajectories generated by the continuous dynamic.
Furthermore, the convergence analysis was also carried out through the parameters β_f and β_b attached to the geometric dampings along with the parameters γ and λ (the viscous damping and the coefficient of cocoercivity respectively). The oscillations, known for viscous damping of inertial systems, are controlled and attenuated by introducing geometric damping. That gives rise to faster numerical methods. It would be fascinating to extend the analysis for both the continuous dynamic and its discretization to the case of an asymptotic vanishing damping $\gamma(t) = \frac{\alpha}{t}$, with $\alpha > 0$ as in [71].This is a significant step toward developing faster methods to solve structured monotone inclusions, which correlate with Nesterov's accelerated gradient method. The work on the corresponding splitting methods is also a crucial topic which needs deeper investigation. In fact, by replacing ∇f with a general maximally monotone operator A, the resolvent of which can be easily computed, it might be worth studying a forward-backward inertial algorithm with Hessian-driven damping for solving structured monotone inclusions of the form : $Ax + Bx \ni 0$. These topics are open and challenging for future research.

3

Newton-type inertial algorithms for solving monotone equations governed by sums of potential and nonpotential operators

In the prior chapter, we deal with solving additively structured monotone problems of the type

$$\text{Find } x \in \mathcal{H} \;:\; \underbrace{\nabla f(x)}_{potential} + \underbrace{B(x)}_{nonpotential} = 0,$$

which mainly come from fields of sciences and engineering. This chapter will be the continuation of our framework presented in Chapter 2. Roughly speaking, this one is dedicated to the study of a class of first-order algorithms which aims to solve structured monotone equations involving the sum of potential and nonpotential operators. In detail, we purpose to find the zeros of an operator $A = \nabla f + B$, in which ∇f is the gradient of a differentiable convex function f, and B is a nonpotential monotone and cocoercive operator. This study can be considered as a sequel and enhanced part of the inertial autonomous dynamic previously studied by the authors, which involves dampings controlled respectively by the Hessian of f, and by a Newton-type correction term attached to B. The appearance of these geometric dampings attenuates the classical oscillations which often occur with the inertial methods and viscous damping while temporal discretization of this dynamic provides fully splitted proximal-gradient algorithms. Their convergence properties are shown to be guaranteed according to Lyapunov analysis under certain conditions on parameters. Consequently, these results give us first-order accelerated algorithms that are useful for numerical optimization taking into account the specific properties of both potential and nonpotential terms.

This chapter constitutes the subject of the published paper [6] in collaboration with S. Adly and H. Attouch.

3.1 Introduction and preliminary results

We recall that $\mathcal{H}$ is a real Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. Our study is based on the continuous inertial dynamic

$$\ddot{x}(t)+\gamma\dot{x}(t)+\nabla f(x(t))+B(x(t))+\beta_f\nabla^2 f(x(t))\dot{x}(t)+\beta_b B'(x(t))\dot{x}(t) = 0, \; t \geq 0 \quad \text{(DINAM)}$$

previously studied in Chapter 2. (DINAM) stands shortly for Dynamic Inertial Newton method for Additively structured Monotone problems. It is an autonomous dynamic which involves geometric dampings which are respectively controlled by the Hessian of the potential function f, and by a Newton-type correction term attached to B. In Chapter 2, one has been shown the well-posedness of the solution of the Cauchy problem and the weak convergence of the generated trajectories towards the zeros of $\nabla f + B$. As a remarkable property, the introduction of geometric damping attenuates notably the oscillations which naturally occur with the inertial methods. Our goal is to observe the convergence properties of the algorithms acquired by temporal discretization of (DINAM), and thus numerically solve the structured monotone equation (2.1). A particular attention will be paid to the minimal assumptions which guarantee convergence of the sequences generated by the algorithms, and which emphasize the asymmetric role of the two operators involved in the dynamic. Throughout the chapter, we also make the following standard assumptions :

$$\begin{cases} \text{(A1)} \quad f : \mathcal{H} \to \mathbb{R} \ \text{ is convex, of class } \mathcal{C}^1,\ \nabla f \ \text{ is Lipschitz continuous on the bounded sets;} \\ \text{(A2)} \quad B : \mathcal{H} \to \mathcal{H} \ \text{ is a } \lambda\text{-cocoercive operator for some } \lambda > 0; \\ \text{(A3)} \quad \gamma > 0,\ \beta_f > 0,\ \beta_b \geq 0 \ \text{ are given real damping parameters;} \\ \text{(A4)} \quad \text{the solution set } \ S := (\nabla f + B)^{-1}(0) = \{p \in \mathcal{H} :\ \nabla f(p) + B(p) = 0\} \text{ is nonempty.} \end{cases}$$

Unless specified, we do not assume the gradient of f to be globally Lipschitz continuous. The cocoercivity of the operator B is the pivot in our analysis. Recall that the operator $B : \mathcal{H} \to \mathcal{H}$ is λ-cocoercive for some $\lambda > 0$ provided that

$$\langle By - Bx, y - x \rangle \geq \lambda \|By - Bx\|^2,\ \forall x, y \in \mathcal{H}.$$

The following (DINAAM-split) algorithm is a model example of the splitting algorithms obtained by temporal discretization of the continuous dynamic (DINAM). The positive parameter h is the step size of the discretization.

(DINAAM-split) :

Initialize : $x_0 \in \mathcal{H}, x_1 \in \mathcal{H}$

$$\alpha = \frac{1}{1+\gamma h}, \quad s = \frac{h}{1+\gamma h},$$

$$y_k = x_k + \alpha(x_k - x_{k-1}) + s\beta_b B(x_k) - s(h+\beta_f)\nabla f(x_k) + s\beta_f \nabla f(x_{k-1}),$$

$$x_{k+1} = \Big(\operatorname{Id} + s(h+\beta_b)B\Big)^{-1}(y_k).$$

Its convergence properties are analyzed in Theorem 3.4.1 (section 3.4). Compared to the classical accelerated proximal gradient algorithms, it contains corrective terms where the potential and nonpotential operators appear asymmetrically, and which make it possible to attenuate the oscillations.

The outline of the chapter is the following. Following the introductory Section 3.1, we revisit some of the conclusions reported in Chapter 2 (see also [4]) concerning the continuous dynamics (DINAM) in Section 3.2. In Section 3.3, we analyze the convergence of the sequences generated by an inertial proximal algorithm acquired by implicit discretization of the continuous dynamics (DINAM). We emphasize the interplay between the damping parameters β_f, β_b, γ and the cocoercivity parameter λ, which plays a significant role in our Lyapunov analysis. In Section 3.4, we analyze an inertial proximal-gradient splitting algorithm which makes use of the gradient of f and the resolvent of B. We also analyze the effect of errors, perturbations in the algorithm. In Section 3.5, we examine a variant of this proximal-gradient algorithm, where the operators' role is reversed. In Section 3.6, we perform numerical experiments which show that the oscillations are considerably reduced with the introduction of geometric damping. Applications to structured monotone equations involving a nonpotential operator are studied.

3.2 The continuous dynamic (DINAM)

In this section, we recall the principal results reported in Chapter 2 concerning the second-order differential equation (DINAM)

$$\ddot{x}(t) + \gamma\dot{x}(t) + \nabla f(x(t)) + B(x(t)) + \beta_f \nabla^2 f(x(t))\dot{x}(t) + \beta_b B'(x(t))\dot{x}(t) = 0, \ t \geq 0. \tag{DINAM}$$

The following existence and uniqueness result for the Cauchy problem was proved in Chapter 2.

Theorem 3.2.1 *Suppose that $\beta_f > 0$ and $\beta_b \geq 0$. Then, for any $(x_0, x_1) \in \mathcal{H} \times \mathcal{H}$, there exists a unique global classical solution $x : [0, +\infty[\to \mathcal{H}$ of the continuous dynamic (DINAM) which satisfies the Cauchy data $x(0) = x_0$, $\dot{x}(0) = x_1$.*

Let us point out that $B(p)$ and $\nabla f(p)$ are uniquely defined for $p \in S := (\nabla f + B)^{-1}(0)$.

Lemma 3.2.1 *$B(p)$ is uniquely defined for $p \in S$, i.e., $p_1 \in S,\ p_2 \in S \Longrightarrow B(p_1) = B(p_2)$.*

For a proof of Lemma 3.2.1, we refer to Lemma 2.3.1 in Chapter 2. The following theorem establishes the asymptotic convergence properties of (DINAM), see Chapter 2 for the proof.

Theorem 3.2.2 *Let $B : \mathcal{H} \to \mathcal{H}$ be a λ-cocoercive operator and $f : \mathcal{H} \to \mathbb{R}$ be a $\mathcal{C}^1$ convex function whose gradient is Lipschitz continuous on the bounded sets. Suppose that the* *parameters involved in* (DINAM) *satisfy* $\beta_f > 0$ *and*

$$\lambda\gamma > \frac{(\beta_b-\beta_f)^2}{4\beta_f} + \frac{1}{2}\left(\beta_b+\frac{1}{\gamma}\right) + \frac{1}{2}\sqrt{\left(\beta_b+\frac{1}{\gamma}\right)^2 + \frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}}.$$

Then, for any solution trajectory $x : [0, +\infty[\to \mathcal{H}$ *of* (DINAM) *the following properties are satisfied :*

(i) $x(t)$ *converges weakly, as* $t \to +\infty$*, to an element of* S*.*

(ii) *Set* $A := \nabla f + B$ *and* $p \in S$*. Then,*

$$\int_0^{+\infty} \|\dot{x}(t)\|^2 dt < +\infty, \quad \int_0^{+\infty} \|\ddot{x}(t)\|^2 dt < +\infty,$$
$$\int_0^{+\infty} \|B(x(t)) - B(p)\|^2 dt < +\infty, \quad \int_0^{+\infty} \left\|\frac{d}{dt}B(x(t))\right\|^2 dt < +\infty,$$
$$\int_0^{+\infty} \|A(x(t))\|^2 dt < +\infty, \text{ and } \int_0^{+\infty} \left\|\frac{d}{dt}A(x(t))\right\|^2 dt < +\infty.$$

(iii) $\lim_{t\to+\infty} \|\dot{x}(t)\| = 0$, $\lim_{t\to+\infty} \|B(x(t)) - B(p)\| = 0$, $\lim_{t\to+\infty} \|A(x(t))\| = 0$, *where* $B(p)$ *is uniquely defined for* $p \in S$*.*

3.3 Inertial proximal algorithms associated with (DINAM)

Set $A := \nabla f + B$ and $A_\beta := \beta_f \nabla f + \beta_b B$. Consider the following implicit finite-difference scheme for (DINAM) :

$$\frac{1}{h^2}(x_{k+1} - 2x_k + x_{k-1}) + \frac{\gamma}{h}(x_{k+1} - x_k) + \frac{1}{h}(A_\beta(x_{k+1}) - A_\beta(x_k)) + A(x_{k+1}) = 0, \quad (3.1)$$

where $h > 0$ is a fixed time step. After expanding (3.1), we obtain

$$x_{k+1} + \frac{h^2}{1+\gamma h}A(x_{k+1}) + \frac{h}{1+\gamma h}A_\beta(x_{k+1}) = x_k + \frac{1}{1+\gamma h}(x_k - x_{k-1}) + \frac{h}{1+\gamma h}A_\beta(x_k). \quad (3.2)$$

Set $s := \frac{h}{1+\gamma h}$ and $\alpha := \frac{1}{1+\gamma h}$. So we have

$$x_{k+1} + s\mathcal{A}_h(x_{k+1}) = y_k, \quad (3.3)$$

where

$$\mathcal{A}_h = (h+\beta_f)\nabla f + (h+\beta_b)B, \tag{3.4}$$
$$y_k = x_k + \alpha(x_k - x_{k-1}) + sA_\beta(x_k). \tag{3.5}$$

According to (3.3) and $\mathcal{A}_h$ maximally monotone, we obtain $x_{k+1} = (\mathrm{Id} + s\mathcal{A}_h)^{-1}(y_k)$. We therefore obtain the following algorithm, where (DINAAM) stands for Dynamic Inertial Newton Algorithm for Additively structured Monotone problems.

(DINAAM) :

Initialize : $x_0 \in \mathcal{H}, x_1 \in \mathcal{H}$

$\alpha = \dfrac{1}{1+\gamma h}, \quad s = \dfrac{h}{1+\gamma h},$

$y_k = x_k + \alpha(x_k - x_{k-1}) + sA_\beta(x_k),$

$x_{k+1} = (\mathrm{Id} + s\mathcal{A}_h)^{-1}(y_k).$

The computation of the resolvent of the weighted sum $\mathcal{A}_h = (h+\beta_f)\nabla f + (h+\beta_b)B$ is required, and therefore (DINAAM) is not a splitting algorithm. Corresponding splitting algorithms will be examined in Sections 3.4 and 3.5.

3.3.1 Lyapunov analysis

Let us state the convergence characteristics of (DINAAM) as below.

Theorem 3.3.1 *Let $B : \mathcal{H} \to \mathcal{H}$ be a λ-cocoercive operator and $f : \mathcal{H} \to \mathbb{R}$ be a convex differentiable function whose gradient is Lipschitz continuous on the bounded sets. Suppose that the positive parameters $\lambda, \gamma, \beta_b, \beta_f$ fulfill*

$$\beta_f > 0, \text{ and } \lambda\gamma > \frac{(\beta_b - \beta_f)^2}{4\beta_f} + \frac{1}{2}\left(\beta_b + \frac{1}{\gamma}\right) + \frac{1}{2}\sqrt{\left(\beta_b + \frac{1}{\gamma}\right)^2 + \frac{(\beta_b - \beta_f)^2}{\gamma\beta_f}}. \tag{3.6}$$

Then, there exists h^* such that for all $0 < h < h^*$, the sequence (x_k) generated by the algorithm (DINAAM) has the following properties (where $p \in S$) :

(i) (x_k) *converges weakly to an element of S ;*

(ii) $\displaystyle\sum_{k=1}^{\infty} \|x_k - x_{k-1}\|^2 < +\infty, \quad \sum_{k=1}^{\infty} \|A(x_k)\|^2 < +\infty,$

$$\sum_{k=1}^{\infty}\|\nabla f(x_k)-\nabla f(x_{k-1})\|^2<+\infty,\ \textit{and}\ \sum_{k=1}^{\infty}\|B(x_k)-B(x_{k-1})\|^2<+\infty;$$

(iii) $\lim_{k\to\infty}\|x_{k+1}-x_k\|=0,\ \lim_{k\to\infty}\|B(x_k)-B(p)\|=0,\ \lim_{k\to\infty}\|\nabla f(x_k)-\nabla f(p)\|=0.$

Proof. **The discrete energy.** Recall that $A:=\nabla f+B$ and $A_\beta:=\beta_f\nabla f+\beta_b B$. Take $p\in S$. Consider the sequence (V_k) defined for all $k\geq 1$ by the formula

$$V_k:=\frac{1}{2}\|(x_k-p)+c\left(\frac{1}{h}(x_k-x_{k-1})+A_\beta(x_k)-A_\beta(p)\right)\|^2+\frac{\delta}{2}\|x_k-p\|^2,$$

where c and δ are positive coefficients to adjust. For each $k\geq 1$, we briefly write V_k as follows :

$$V_k=\frac{1}{2}\|v_k\|^2+\frac{\delta}{2}\|x_k-p\|^2,$$

with

$$v_k:=x_k-p+c\left(\frac{1}{h}(x_k-x_{k-1})+A_\beta(x_k)-A_\beta(p)\right).$$

By definition of v_k, we have $v_{k+1}=x_{k+1}-p+c\left(\frac{1}{h}(x_{k+1}-x_k)+A_\beta(x_{k+1})-A_\beta(p)\right)$.
Moreover, by using the formulation (3.1) of the algorithm (DINAAM), we have

$$\begin{aligned}v_k &= x_{k+1}-p+c\left(\frac{1}{h}(x_{k+1}-x_k)+\gamma(x_{k+1}-x_k)+A_\beta(x_{k+1})-A_\beta(p)+hA(x_{k+1})\right)\\ &\quad -(x_{k+1}-x_k)\\ &= v_{k+1}+(c\gamma-1)(x_{k+1}-x_k)+chA(x_{k+1}).\end{aligned}$$

Therefore, for $k\geq 1$, we have

$$\begin{aligned}&\frac{1}{2}\|v_{k+1}\|^2-\frac{1}{2}\|v_k\|^2=\frac{1}{2}\|v_{k+1}\|^2-\frac{1}{2}\|v_{k+1}+(c\gamma-1)(x_{k+1}-x_k)+chA(x_{k+1})\|^2\\ &\quad=-\frac{1}{2}(c\gamma-1)^2\|x_{k+1}-x_k\|^2-\frac{1}{2}c^2h^2\|A(x_{k+1})\|^2-hc(c\gamma-1)\langle x_{k+1}-x_k,A(x_{k+1})\rangle\\ &-\left\langle (x_{k+1}-p)+c(\frac{1}{h}(x_{k+1}-x_k)+A_\beta(x_{k+1})-A_\beta(p)),(c\gamma-1)(x_{k+1}-x_k)+chA(x_{k+1})\right\rangle\\ &\quad=-\frac{1}{2}(c\gamma-1)^2\|x_{k+1}-x_k\|^2-\frac{1}{2}c^2h^2\|A(x_{k+1})\|^2-hc(c\gamma-1)\langle x_{k+1}-x_k,A(x_{k+1})\rangle\\ &\quad -(c\gamma-1)\langle x_{k+1}-p,x_{k+1}-x_k\rangle-ch\langle x_{k+1}-p,A(x_{k+1})\rangle-\frac{c(c\gamma-1)}{h}\|x_{k+1}-x_k\|^2\\ &\quad -c^2\langle x_{k+1}-x_k,A(x_{k+1})\rangle-c(c\gamma-1)\langle A_\beta(x_{k+1})-A_\beta(p),x_{k+1}-x_k\rangle\\ &\quad -c^2h\langle A_\beta(x_{k+1})-A_\beta(p),A(x_{k+1})\rangle. \end{aligned}\tag{3.7}$$

To write the above relation in a recursive form, we use the elementary identity

$$\frac{1}{2}\|x_{k+1}-p\|^2-\frac{1}{2}\|x_k-p\|^2=-\frac{1}{2}\|x_{k+1}-x_k\|^2+\langle x_{k+1}-x_k,x_{k+1}-p\rangle.\tag{3.8}$$

Let us set $X_k := x_{k+1} - x_k,\ Y_k := B(x_{k+1}) - B(p),\ Z_k := \nabla f(x_{k+1}) - \nabla f(p)$ for $k \geq 0$.
Since $p \in S$, *i.e.*, $\nabla f(p) + B(p) = 0$, we have $A(x_{k+1}) = Y_k + Z_k$ for $k \geq 0$.
In the definition of V_k, take $\delta = c\gamma - 1$, which we assume to be nonnegative, *i.e.*, $c\gamma \geq 1$.
According to (3.7), (3.8) and the definition of V_k, we obtain after simplification

$$\begin{aligned} V_{k+1} - V_k = &- \frac{1}{2}(c\gamma - 1)^2\|X_k\|^2 - \frac{1}{2}c^2h^2\|Y_k + Z_k\|^2 - hc(c\gamma - 1)\langle X_k, Y_k + Z_k\rangle \\ &- \frac{1}{2}(c\gamma - 1)\|X_k\|^2 - ch\langle x_{k+1} - p, A(x_{k+1})\rangle - \frac{c(c\gamma - 1)}{h}\|X_k\|^2 \\ &- c^2\langle X_k, Y_k + Z_k\rangle - c(c\gamma - 1)\langle \beta_b Y_k + \beta_f Z_k, X_k\rangle - c^2h\langle \beta_b Y_k + \beta_f Z_k, Y_k + Z_k\rangle. \end{aligned}$$

Using the fact that $p \in S$, ∇f is monotone, and B is λ-cocoercive, we have

$$\begin{aligned} -ch\langle x_{k+1} - p, A(x_{k+1})\rangle &= -ch\langle x_{k+1} - p, B(x_{k+1}) - B(p)\rangle - ch\langle x_{k+1} - p, \nabla f(x_{k+1}) - \nabla f(p)\rangle \\ &\leq -ch\lambda\|B(x_{k+1}) - B(p)\|^2. \end{aligned}$$

By combining the two relations above, we obtain

$$\begin{aligned} V_{k+1} - V_k &+ \left[\frac{1}{2}(c\gamma - 1)^2 + \frac{1}{2}(c\gamma - 1) + \frac{c(c\gamma - 1)}{h}\right]\|X_k\|^2 \\ &+ \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_f\right]\langle X_k, Z_k\rangle \\ &+ \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_b\right]\langle X_k, Y_k\rangle + \left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2\right]\|Y_k\|^2 \\ &+ \left[c^2h\beta_f + \frac{1}{2}c^2h^2\right]\|Z_k\|^2 + \left[c^2h(\beta_b + \beta_f) + c^2h^2\right]\langle Z_k, Y_k\rangle \leq 0. \end{aligned} \tag{3.9}$$

Let (Γ_k) be the sequence of real numbers defined by

$$\Gamma_k := f(x_k) - f(p) - \langle \nabla f(p), x_k - p\rangle, \text{ for } k \geq 0.$$

Since f is convex, we have $\Gamma_k \geq 0$, for all $k \geq 0$. Moreover,

$$\begin{aligned} \langle X_k, Z_k\rangle &= \langle x_{k+1} - x_k, \nabla f(x_{k+1})\rangle - \langle x_{k+1} - x_k, \nabla f(p)\rangle \\ &\geq f(x_{k+1}) - f(x_k) - \langle x_{k+1} - x_k, \nabla f(p)\rangle = \Gamma_{k+1} - \Gamma_k. \end{aligned} \tag{3.10}$$

For each $k \geq 1$, let us define

$$E_k := V_k + \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_f\right]\Gamma_k.$$

(E_k) will serve us as a discrete energy function. Indeed, it is clear that (E_k) is a sequence of nonnegative numbers. From (3.9), (3.10) and the definition of (E_k), we obtain

$$\begin{aligned} E_{k+1} - E_k &+ \left[\frac{1}{2}(c\gamma-1)^2 + \frac{1}{2}(c\gamma-1) + \frac{c(c\gamma-1)}{h}\right] \|X_k\|^2 \\ &+ \left[c(c\gamma-1)h + c^2 + c(c\gamma-1)\beta_b\right] \langle X_k, Y_k\rangle + \left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2\right] \|Y_k\|^2 \\ &+ \left[c^2h\beta_f + \frac{1}{2}c^2h^2\right] \|Z_k\|^2 + \left[c^2h(\beta_b+\beta_f) + c^2h^2\right] \langle Z_k, Y_k\rangle \le 0. \end{aligned} \tag{3.11}$$

Let us eliminate Z_k from this relation by using the elementary algebraic inequality

$$\left[c^2h\beta_f + \frac{1}{2}c^2h^2\right] \|Z_k\|^2 + \left[c^2h(\beta_b+\beta_f) + c^2h^2\right] \langle Z_k, Y_k\rangle \ge -\frac{c^2h(\beta_b+\beta_f+h)^2}{4\beta_f+2h}\|Y_k\|^2.$$

Then, from (3.11), we deduce that

$$E_{k+1} - E_k + q(X_k, Y_k) \le 0, \tag{3.12}$$

where

$$\begin{aligned} q(X_k, Y_k) = &\left[\frac{1}{2}(c\gamma-1)^2 + \frac{1}{2}(c\gamma-1) + \frac{c(c\gamma-1)}{h}\right] \|X_k\|^2 \\ &+ \left[c(c\gamma-1)h + c^2 + c(c\gamma-1)\beta_b\right] \langle X_k, Y_k\rangle \\ &+ \left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2 - \frac{c^2h(\beta_b+\beta_f+h)^2}{4\beta_f+2h}\right] \|Y_k\|^2. \end{aligned}$$

Let us observe that $q : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is a quadratic form

$$q(X_k, Y_k) := a\|X_k\|^2 + b\langle X_k, Y_k\rangle + g\|Y_k\|^2,$$

with

$$\begin{aligned} a &= \frac{1}{2}(c\gamma-1)^2 + \frac{1}{2}(c\gamma-1) + \frac{c(c\gamma-1)}{h}, \\ b &= c(c\gamma-1)h + c^2 + c(c\gamma-1)\beta_b, \\ g &= ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2 - \frac{c^2h(\beta_b+\beta_f+h)^2}{4\beta_f+2h}. \end{aligned}$$

According to Lemma 1.3.5, since $a > 0$, q is positive definite if and only if $4ag - b^2 > 0$.

Equivalently,

$$4\left[\frac{1}{2}(c\gamma-1)^2+\frac{1}{2}(c\gamma-1)+\frac{c(c\gamma-1)}{h}\right]\left[ch\lambda+c^2h\beta_b+\frac{1}{2}c^2h^2-\frac{c^2h(\beta_b+\beta_f+h)^2}{4\beta_f+2h}\right] - \left[c(c\gamma-1)h+c^2+c(c\gamma-1)\beta_b\right]^2>0. \tag{3.13}$$

Our aim is to find c such that $c\gamma-1>0$ and (3.13) is satisfied. After development and simplification we obtain the following equivalent formulation of (3.13)

$$4\left[\frac{1}{2}(c\gamma-1)^2h+\frac{1}{2}(c\gamma-1)h+c(c\gamma-1)\right]\left[c\lambda+c^2\beta_b+\frac{1}{2}c^2h-\frac{c^2(\beta_b+\beta_f+h)^2}{4\beta_f+2h}\right] - \left[c(c\gamma-1)h+c^2+c(c\gamma-1)\beta_b\right]^2>0. \tag{3.14}$$

Let us denote by $\mathcal{L}(h)$ the left handside of (3.14). We have

$$\lim_{h\to0^+}\mathcal{L}(h)=4c(c\gamma-1)\left[c\lambda+c^2\beta_b-\frac{c^2(\beta_b+\beta_f)^2}{4\beta_f}\right]-\left[c^2+c(c\gamma-1)\beta_b\right]^2.$$

To guarantee the existence of $h>0$ such that the quadratic form q is positive definite, it is sufficient to find c satisfying $c\gamma-1>0$ and

$$4c(c\gamma-1)\left[c\lambda+c^2\beta_b-\frac{c^2(\beta_b+\beta_f)^2}{4\beta_f}\right]-\left[c^2+c(c\gamma-1)\beta_b\right]^2>0.$$

The preceding inequality can be rewritten equivalently as

$$4\lambda>\frac{\left[c^2+c(c\gamma-1)\beta_b\right]^2}{c^2(c\gamma-1)}-4c\beta_b+\frac{(\beta_b+\beta_f)^2}{\beta_f}c=\frac{\left[c+(c\gamma-1)\beta_b\right]^2}{c\gamma-1}+\frac{(\beta_b-\beta_f)^2}{\beta_f}c.$$

Let us reformulate this inequation by introducing $\delta=c\gamma-1>0$. Our aim is to find $\delta>0$ such that

$$4\lambda>\frac{\left[\frac{\delta+1}{\gamma}+\delta\beta_b\right]^2}{\delta}+\frac{\delta+1}{\gamma}\frac{(\beta_b-\beta_f)^2}{\beta_f}.$$

An elementary algebraic calculation gives us the equivalent formulation

$$4\lambda>\frac{2}{\gamma}\left(\beta_b+\frac{1}{\gamma}\right)+\frac{1}{\gamma}\frac{(\beta_b-\beta_f)^2}{\beta_f}+\frac{1}{\gamma^2\delta}+\left[\left(\beta_b+\frac{1}{\gamma}\right)^2+\frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}\right]\delta.$$

Therefore, in order to ensure the existence of such δ, it is sufficient to assume that

$$4\lambda > \frac{2}{\gamma}\left(\beta_b + \frac{1}{\gamma}\right) + \frac{1}{\gamma}\frac{(\beta_b-\beta_f)^2}{\beta_f} + \inf_{\delta>0}\left(\frac{1}{\gamma^2\delta} + \left[\left(\beta_b+\frac{1}{\gamma}\right)^2 + \frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}\right]\delta\right). \tag{3.15}$$

It is easy to check that

$$\inf_{\delta>0}\left(\frac{C}{\delta} + D\delta\right) = 2\sqrt{CD}, \tag{3.16}$$

for any $C, D \in \mathbb{R}_+$. Combining (3.15) and (3.16), we get the final condition

$$4\lambda > \frac{2}{\gamma}\left(\beta_b + \frac{1}{\gamma}\right) + \frac{1}{\gamma}\frac{(\beta_b-\beta_f)^2}{\beta_f} + \frac{2}{\gamma}\sqrt{\left(\beta_b+\frac{1}{\gamma}\right)^2 + \frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}}.$$

When $\beta_b = \beta_f := \beta$, we recover the condition $\lambda\gamma > \beta + \frac{1}{\gamma}$. Therefore, under the above condition, and by taking h sufficiently small, there exists a positive real number μ such that for any $k \geq 1$,

$$E_{k+1} - E_k + \mu\|X_k\|^2 + \mu\|Y_k\|^2 \leq 0. \tag{3.17}$$

Estimates. According to (3.17), the sequence of nonnegative numbers (E_k) is nonincreasing, and therefore converges. In particular, it is bounded. From this, we immediately deduce that

$$\sup_k \|(x_k - p) + c\left(\frac{1}{h}(x_k - x_{k-1}) + A_\beta(x_k) - A_\beta(p)\right)\|^2 < +\infty \tag{3.18}$$

$$\sup_k \|x_k - p\|^2 < +\infty. \tag{3.19}$$

Moreover, by summing the inequalities (3.17), we deduce that

$$\sum_{k=0}^{\infty}\|X_k\|^2 < +\infty, \quad \sum_{k=0}^{\infty}\|Y_k\|^2 < +\infty. \tag{3.20}$$

Let us return to (3.11). By using (3.20), we obtain the existence of a constant $C > 0$ such that

$$\left[c^2h\beta_f + \frac{1}{2}c^2h^2\right]\sum_{k=0}^{n}\|Z_k\|^2 \leq C + \left[c^2h(\beta_b+\beta_f) + c^2h^2\right]\sum_{k=0}^{n}\|Z_k\|\|Y_k\|.$$

Therefore, for any $\epsilon > 0$, we have

$$\left[c^2h\beta_f + \frac{1}{2}c^2h^2\right]\sum_{k=0}^{n}\|Z_k\|^2 \leq C + \left[c^2h(\beta_b+\beta_f)+c^2h^2\right]\left(\epsilon\sum_{k=0}^{n}\|Z_k\|^2 + \frac{1}{4\epsilon}\sum_{k=0}^{n}\|Y_k\|^2\right).$$

Taking $\epsilon > 0$ such that $c^2h\beta_f + \frac{1}{2}c^2h^2 > \epsilon\left[c^2h(\beta_b+\beta_f)+c^2h^2\right]$, which is always possible since $c^2h\beta_f + \frac{1}{2}c^2h^2 > 0$, we conclude that

$$\sum_{k=0}^{\infty}\|Z_k\|^2 < +\infty.$$

Since $A(x_{k+1}) = Y_k + Z_k$, we immediately deduce

$$\sum_{k=1}^{\infty}\|A(x_k)\|^2 < +\infty.$$

Furthermore, according to (3.19) the trajectory (x_k) is bounded. Set $R := \sup_{k\geq 0}\|x_k\|$. By assumption, ∇f is Lipschitz continuous on the bounded sets. Let $L_R < +\infty$ be the Lipschitz constant of ∇f on the ball $\mathbb{B}(0,R)$. Since B is λ-cocoercive, it is $\frac{1}{\lambda}$-Lipschitz continuous. Therefore, A is L-Lipschitz continuous on the trajectory with $L := L_R + \frac{1}{\lambda}$, which implies,

$$\|A(x_{k+1}) - A(x_k)\| \leq L\|x_{k+1}-x_k\| \text{ for all } k \geq 0.$$

Therefore, $\sum_{k=1}^{\infty}\|A(x_{k+1}) - A(x_k)\|^2 \leq \sum_{k=1}^{\infty}L^2\|x_{k+1}-x_k\|^2 < +\infty$. By the same argument, we get

$$\sum_{k=1}^{\infty}\|B(x_{k+1}) - B(x_k)\|^2 < +\infty, \text{ and } \sum_{k=1}^{\infty}\|\nabla f(x_{k+1}) - \nabla f(x_k)\|^2 < +\infty.$$

Since the general term of a convergent series goes to zero, we deduce that

$$\begin{aligned}&\lim_{k\to\infty}\|x_{k+1}-x_k\| = 0, \lim_{k\to\infty}\|A(x_k)\| = 0, \text{ and } \lim_{k\to\infty}\|A(x_{k+1}) - A(x_k)\| = 0,\\ &\lim_{k\to\infty}\|B(x_{k+1}) - B(x_k)\| = 0, \lim_{k\to\infty}\|\nabla(x_{k+1}) - \nabla(x_k)\| = 0.\end{aligned} \tag{3.21}$$

Likewise, we also have

$$\lim_{k\to\infty}\|B(x_k) - B(p)\| = 0, \text{ and } \lim_{k\to\infty}\|\nabla f(x_k) - \nabla f(p)\| = 0. \tag{3.22}$$

Convergence of (x_k)**.** Let us first show that every weak cluster point x^* of the sequence (x_k) belongs to S. Consider a subsequence (x_{k_n}) of (x_k), such that $x_{k_n} \rightharpoonup x^*$, as $n \to +\infty$. We have

$$A(x_{k_n}) \to 0 \text{ strongly in } \mathcal{H} \text{ and } x_{k_n} \rightharpoonup x^* \text{ weakly in } \mathcal{H}.$$

From the closedness property of the graph of the maximally monotone operator A in $w - \mathcal{H} \times s - \mathcal{H}$, we deduce that $A(x^*) = 0$, that is $x^* \in S$. Since the limit of E_k exists, and according to the above strong convergence results, we deduce that there exists a positive number r such that, for any $p \in S$

$$\lim_{k\to\infty} \left[\|x_k - p\|^2 + r\left(f(x_k) - \langle \nabla f(p), x_k - p\rangle\right) \right] \quad \text{exists.}$$

Suppose that the bounded sequence (x_k) has two weak limit points, let p and p'. By the above argument p and p' belong to S. Therefore, $\lim_{k\to\infty} \left[\|x_k - p\|^2 + r\left(f(x_k) - \langle \nabla f(p), x_k - p\rangle\right) \right]$ and $\lim_{k\to\infty} \left[\|x_k - p'\|^2 + r\left(f(x_k) - \langle \nabla f(p'), x_k - p'\rangle\right) \right]$ exist. By taking the difference, and using that $\nabla f(p) = \nabla f(p')$, we deduce that $\lim_{k\to\infty} \|x_k - p\|^2 - \|x_k - p'\|^2$ exists. Equivalently $\lim_{k\to\infty} \langle x_k, p - p'\rangle$ exists. By specializing this result to the subsequences defining p and p' we get

$$\langle p, p - p'\rangle = \langle p', p - p'\rangle,$$

that is $\|p - p'\|^2 = 0$, which gives $p = p'$. The sequence (x_k) has a unique weak cluster point, and hence converges weakly.

3.3.2 Estimation of the upper bound on the time step h

The former results are valid whenever h is taken small enough. For numerical purposes, it is important to specify this result, and find $h^* > 0$ such that the convergence properties hold true for all $h \in]0, h^*[$. So let us come back to (3.13), which is the key estimate for our Lyapunov analysis. As a result of elementary calculation, it can be written as follows

$$2(c\gamma - 1)(2 + \gamma h)\left[\lambda + c\beta_b + \frac{1}{2}ch - \frac{c(\beta_b + \beta_f + h)^2}{4\beta_f + 2h}\right] - \left[(c\gamma - 1)h + c + (c\gamma - 1)\beta_b\right]^2 > 0. \tag{3.23}$$

After dividing by $c\gamma - 1 > 0$ and a few elementary calculation steps, we get

$$(2+\gamma h)\big((\lambda+c\beta_b+\frac{1}{2}ch)(4\beta_f+2h)-c(\beta_b+\beta_f+h)^2\big)-(c\gamma-1)(2\beta_f+h)\left[h+\frac{c}{c\gamma-1}+\beta_b\right]^2 > 0.$$

Let us develop the preceding expression. Then, we acquire a third-order polynomial with respect to h, namely $P_c(h) = a_0 + a_1 h + a_2 h^2 + a_3 h^3$ with

$$\begin{aligned}
a_0 &= 2\Big(4\lambda\beta_f - c(\beta_b-\beta_f)^2 - \beta_f\frac{(c+\beta_b(c\gamma-1))^2}{c\gamma-1}\Big),\\
a_1 &= 4\lambda(1+\gamma\beta_f) - c\gamma(\beta_b-\beta_f)^2 - 4\beta_f\left(c+\beta_b(c\gamma-1)\right) - \frac{1}{c\gamma-1}\left(c+\beta_b(c\gamma-1)\right)^2,\\
a_2 &= 2\lambda\gamma - 2c - 2(c\gamma-1)(\beta_b+\beta_f),\\
a_3 &= -(c\gamma-1).
\end{aligned}$$

We discovered that choosing appropriately $c > 0$ with $c\gamma - 1 > 0$ yields $a_0 > 0$ under the growth condition (3.6). We can precisely pick $c = c^*$ where $c^*\gamma - 1 = \frac{1}{\gamma}\left(\left(\beta_b+\frac{1}{\gamma}\right)^2 + \frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}\right)^{-\frac{1}{2}}$. Hence $P_{c^*}(0) > 0$. In fact, let us consider

$$\frac{1}{2}\gamma a_0(c) = 4\lambda\gamma\beta_f - c\gamma(\beta_b-\beta_f)^2 - \beta_f\frac{(c\gamma+\beta_b\gamma(c\gamma-1))^2}{\gamma(c\gamma-1)}.$$

Since

$$\lambda\gamma > \frac{(\beta_b-\beta_f)^2}{4\beta_f} + \frac{1}{2}\left(\beta_b+\frac{1}{\gamma}\right) + \frac{1}{2}\sqrt{\left(\beta_b+\frac{1}{\gamma}\right)^2 + \frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}},$$

we have

$$\begin{aligned}
\frac{1}{2}\gamma a_0(c) > (\beta_b-\beta_f)^2 + 2\beta_f\left(\beta_b+\frac{1}{\gamma}\right) + 2\beta_f\sqrt{\left(\beta_b+\frac{1}{\gamma}\right)^2 + \frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}}\\
-c\gamma(\beta_b-\beta_f)^2 - \beta_f\frac{(c\gamma+\beta_b\gamma(c\gamma-1))^2}{\gamma(c\gamma-1)}.
\end{aligned}$$

Take $c = c^*$ where $c^*\gamma - 1 = \frac{1}{\gamma}\left(\left(\beta_b+\frac{1}{\gamma}\right)^2 + \frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}\right)^{-\frac{1}{2}}$. Shortly, we set

$$y = \sqrt{\left(\beta_b+\frac{1}{\gamma}\right)^2 + \frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}}.$$

Then, we immediately obtain

$$\begin{aligned}\frac{1}{2}\gamma a_0(c^*) &> -\frac{1}{\gamma y}(\beta_b-\beta_f)^2+2\beta_f\left(\beta_b+\frac{1}{\gamma}\right)+2\beta_f y-\beta_f y\left(1+\frac{1}{\gamma y}+\frac{\beta_b}{y}\right)^2\\ &= -\frac{1}{\gamma y}(\beta_b-\beta_f)^2+2\beta_f y-\beta_f y-\frac{\beta_f}{y}\left(\frac{1}{\gamma}+\beta_b\right)^2\\ &= \frac{\beta_f}{y}\left[-\frac{1}{\gamma\beta_f}(\beta_b-\beta_f)^2+2y^2-y^2-\left(\frac{1}{\gamma}+\beta_b\right)^2\right]\\ &= 0.\end{aligned}$$

Note that for large h, $P_{c^*}(h) \sim -(c^*\gamma-1)h^3$, and so $P_{c^*}(h)$ is negative. Therefore, $h^* > 0$ is the smallest positive zero (which exists) of P_{c^*}. Its explicit determination is quite technical in our general setting. In practical situations, its calculation results from elementary numerical analysis. Let us stress the fact that h^* depends only on the parameters that enter (DINAAM) (not on f).

3.3.3 Case $\beta_b = \beta_f$

In this case, the formulas are simplified, and we have the following result. Set $\beta_b = \beta_f := \beta > 0$, and $A := \nabla f + B$. The algorithm (DINAAM) is written as follows :

(DINAAM) : $\beta_b = \beta_f = \beta$

Initialize : $x_0 \in \mathcal{H}, x_1 \in \mathcal{H}$

$\alpha = \dfrac{1}{1+\gamma h}, \quad s = \dfrac{h(h+\beta)}{1+\gamma h},$

$y_k = x_k + \alpha(x_k - x_{k-1}) + h\alpha\beta A(x_k),$

$x_{k+1} = (\mathrm{Id} + sA)^{-1}(y_k).$

The following result is a particular case of Theorem 3.3.1.

Corollary 3.3.1 *Let $B : \mathcal{H} \to \mathcal{H}$ be a λ-cocoercive operator and $f : \mathcal{H} \to \mathbb{R}$ be a $\mathcal{C}^1$ convex function whose gradient is Lipschitz continuous on the bounded sets. Suppose that $\beta_b = \beta_f := \beta > 0$, and that the parameters γ, λ, β satisfy the following conditions*

$$\gamma > 0,\ \beta > 0 \text{ and } \lambda\gamma > \beta + \frac{1}{\gamma}.$$

Then, there exists h^* such that for all $0 < h < h^*$, the sequence (x_k) generated by the *algorithm* (DINAAM) *has the following properties :*

(i) (x_k) *converges weakly to an element in* S *;*

(ii) $\sum_{k=1}^{\infty} \|x_k - x_{k-1}\|^2 < +\infty, \quad \sum_{k=1}^{\infty} \|A(x_k)\|^2 < +\infty$ *;*

(iii) $\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0, \quad \lim_{k\to\infty} \|A(x_k)\| = 0.$

3.4 An inertial proximal-gradient algorithm

In this section, we assume that f is a $\mathcal{C}^1$ convex function whose gradient is L-Lipschitz continuous. Set $A := \nabla f + B$ and $A_\beta := \beta_f \nabla f + \beta_b B$. We take a fixed time step $h > 0$, and consider the following finite-difference scheme for (DINAM) :

$$\frac{1}{h^2}(x_{k+1} - 2x_k + x_{k-1}) + \frac{\gamma}{h}(x_{k+1} - x_k) + \frac{\beta_b}{h}(B(x_{k+1}) - B(x_k)) + \frac{\beta_f}{h}(\nabla f(x_k) - \nabla f(x_{k-1})) + B(x_{k+1}) + \nabla f(x_k) = 0. \tag{3.24}$$

This scheme is implicit with respect to the nonpotential B and explicit with respect to the potential operator ∇f. After expanding (3.24), we obtain

$$x_{k+1} + \frac{h^2}{1+\gamma h}B(x_{k+1}) + \frac{h\beta_b}{1+\gamma h}B(x_{k+1}) = x_k + \frac{1}{1+\gamma h}(x_k - x_{k-1}) + \frac{h\beta_b}{1+\gamma h}B(x_k) - \frac{h\beta_f}{1+h\gamma}(\nabla f(x_k) - \nabla f(x_{k-1})) - \frac{h^2}{1+h\gamma}\nabla f(x_k). \tag{3.25}$$

Set $s := \dfrac{h}{1+\gamma h}$ and $\alpha := \dfrac{1}{1+\gamma h}$. So we have

$$x_{k+1} + s\mathcal{B}_h(x_{k+1}) = y_k, \tag{3.26}$$

where $\mathcal{B}_h = (h + \beta_b)B$, and

$$y_k = x_k + \alpha(x_k - x_{k-1}) + s\beta_b B(x_k) - s(h+\beta_f)\nabla f(x_k) + s\beta_f \nabla f(x_{k-1}). \tag{3.27}$$

From (3.26) we get $x_{k+1} = (\mathrm{Id} + s\mathcal{B}_h)^{-1}(y_k)$, which gives the following algorithm.

(DINAAM-split) :

Initialize : $x_0 \in \mathcal{H}, x_1 \in \mathcal{H}$

$\alpha = \dfrac{1}{1+\gamma h}, \quad s = \dfrac{h}{1+\gamma h},$

$y_k = x_k + \alpha(x_k - x_{k-1}) + s\beta_b B(x_k) - s(h+\beta_f)\nabla f(x_k) + s\beta_f \nabla f(x_{k-1}),$

$x_{k+1} = (\mathrm{Id} + s\mathcal{B}_h)^{-1}(y_k).$

3.4.1 Lyapunov analysis

Theorem 3.4.1 *Let $B : \mathcal{H} \to \mathcal{H}$ be a λ-cocoercive operator and $f : \mathcal{H} \to \mathbb{R}$ a $\mathcal{C}^1$ convex function whose gradient is $L-$Lipschitz continuous. Suppose that the positive parameters $\lambda, \gamma, \beta_b, \beta_f$ satisfy*

$$\beta_f > 0, \;\; \textit{and } \lambda\gamma > \frac{(\beta_b - \beta_f)^2}{4\beta_f} + \frac{1}{2}\left(\beta_b + \frac{1}{\gamma}\right) + \frac{1}{2}\sqrt{\left(\beta_b + \frac{1}{\gamma}\right)^2 + \frac{(\beta_b - \beta_f)^2}{\gamma\beta_f}}.$$

Then, there exists h^* (depending on L) such that for all $0 < h < h^*$, the sequence (x_k) generated by the algorithm (DINAAM-split) has the following properties :

(i) (x_k) *converges weakly to an element in S ;*

(ii) $\displaystyle\sum_{k=1}^{\infty} \|x_k - x_{k-1}\|^2 < +\infty, \sum_{k=1}^{\infty} \|A(x_k)\|^2 < +\infty,$

$$\sum_{k=1}^{\infty} \|\nabla f(x_k) - \nabla f(x_{k-1})\|^2 < +\infty, \;\; \textit{and } \sum_{k=1}^{\infty} \|B(x_k) - B(x_{k-1})\|^2 < +\infty;$$

(iii) $\displaystyle\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0, \lim_{k\to\infty} \|B(x_k) - B(p)\| = 0, \;\; \textit{and } \lim_{k\to\infty} \|\nabla f(x_k) - \nabla f(p)\| = 0.$

Proof. The structure of the proof is similar to that of Theorem 3.3.1. The main difference in Lyapunov's analysis is the use of the global Lipschitz continuity of ∇f to deal with the corresponding gradient method. Take $p \in S$. Let us consider the sequence (V_k) defined by, for each $k \geq 1$

$$V_k = \frac{1}{2}\|(x_k - p) + c\Big(\frac{1}{h}(x_k - x_{k-1}) + \beta_b B(x_k) + \beta_f \nabla f(x_{k-1}) - A_\beta(p)\Big)\|^2 + \frac{\delta}{2}\|x_k - p\|^2,$$

where c, δ are positive coefficients to adjust. For $k \geq 1$, let us briefly write V_k as follows

$$V_k = \frac{1}{2}\|v_k\|^2 + \frac{\delta}{2}\|x_k - p\|^2,$$

with $v_k := x_k - p + c\left(\frac{1}{h}(x_k - x_{k-1}) + \beta_b B(x_k) + \beta_f \nabla f(x_{k-1}) - A_\beta(p)\right)$.
According to the formulation of the algorithm (DINAAM-split), we have

$$\begin{aligned}
v_k &= c\left(\frac{1}{h}(x_{k+1} - x_k) + \gamma(x_{k+1} - x_k) + \beta_b B(x_{k+1}) + \beta_f \nabla f(x_k) - A_\beta(p) + hB(x_{k+1}) + h\nabla f(x_k)\right) \\
&\quad + (x_{k+1} - p) - (x_{k+1} - x_k) \\
&= v_{k+1} + (c\gamma - 1)(x_{k+1} - x_k) + chB(x_{k+1}) + ch\nabla f(x_k).
\end{aligned}$$

Set $X_k = x_{k+1} - x_k, Y_k = B(x_{k+1}) - B(p), Z_k = \nabla f(x_k) - \nabla f(p)$. Taking $\delta := c\gamma - 1$, we obtain

$$\begin{aligned}
V_{k+1} - V_k = &- \frac{1}{2}(c\gamma - 1)^2\|X_k\|^2 - \frac{1}{2}c^2h^2\|Y_k + Z_k\|^2 - c(c\gamma - 1)h\langle X_k, Y_k + Z_k\rangle \\
&- \frac{1}{2}(c\gamma - 1)\|X_k\|^2 - ch\langle x_{k+1} - p, B(x_{k+1}) + \nabla f(x_k)\rangle - \frac{c(c\gamma - 1)}{h}\|X_k\|^2 \\
&- c^2\langle X_k, Y_k + Z_k\rangle - c(c\gamma - 1)\langle \beta_b Y_k + \beta_f Z_k, X_k\rangle - c^2h\langle \beta_b Y_k + \beta_f Z_k, Y_k + Z_k\rangle.
\end{aligned}$$

Using the fact that $p \in S$, ∇f is monotone, and B is λ-cocoercive, we have

$$\begin{aligned}
&-ch\langle x_{k+1} - p, B(x_{k+1}) + \nabla f(x_k)\rangle \\
&= -ch\langle x_{k+1} - p, B(x_{k+1}) - B(p)\rangle - ch\langle x_{k+1} - p, \nabla f(x_k) - \nabla f(p)\rangle \\
&\leq -ch\lambda\|B(x_{k+1}) - B(p)\|^2 - ch\langle x_{k+1} - x_k, \nabla f(x_k) - \nabla f(p)\rangle.
\end{aligned}$$

By combining the two relations above, we obtain

$$\begin{aligned}
V_{k+1} - V_k &+ \left[\frac{1}{2}(c\gamma - 1)^2 + \frac{1}{2}(c\gamma - 1) + \frac{c(c\gamma - 1)}{h}\right]\|X_k\|^2 \\
&+ \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_f + ch\right]\langle X_k, Z_k\rangle \\
&+ \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_b\right]\langle X_k, Y_k\rangle + \left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2\right]\|Y_k\|^2 \\
&+ \left[c^2h\beta_f + \frac{1}{2}c^2h^2\right]\|Z_k\|^2 + \left[c^2h(\beta_b + \beta_f) + c^2h^2\right]\langle Z_k, Y_k\rangle \leq 0. \qquad (3.28)
\end{aligned}$$

Let (Γ_k) be the sequence defined by

$$\Gamma_k = f(x_k) - f(p) - \langle \nabla f(p), x_k - p\rangle, \text{ for } k \geq 0.$$

Since f is convex, we have $\Gamma_k \geq 0$, for all $k \geq 0$. By the gradient descent lemma, since

∇f is L-Lipschitz, we have

$$\begin{aligned}\langle X_k, Z_k\rangle &= \langle x_{k+1} - x_k, \nabla f(x_k)\rangle - \langle x_{k+1} - x_k, \nabla f(p)\rangle \\ &\geq f(x_{k+1}) - f(x_k) - \frac{L}{2}\|x_{k+1} - x_k\|^2 + \Gamma_{k+1} - \Gamma_k + f(x_k) - f(x_{k+1}) \\ &= \Gamma_{k+1} - \Gamma_k - \frac{L}{2}\|X_k\|^2. \end{aligned} \tag{3.29}$$

Let us define

$$E_k = V_k + \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_f + ch\right]\Gamma_k,$$

for $k \geq 1$. (E_k) will serve us as a discrete energy function. Indeed, it is clear that (E_k) is a sequence of nonnegative numbers. From (3.28), (3.29) and the definition of (E_k), we obtain

$$\begin{aligned} E_{k+1} - E_k &+ \left[\frac{1}{2}(c\gamma - 1)^2 + \frac{1}{2}(c\gamma - 1) + \frac{c(c\gamma-1)}{h} - \frac{L}{2}\left(c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_f + ch\right)\right]\|X_k\|^2 \\ &+ \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_b\right]\langle X_k, Y_k\rangle + \left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2\right]\|Y_k\|^2 \\ &+ \left[c^2h\beta_f + \frac{1}{2}c^2h^2\right]\|Z_k\|^2 + \left[c^2h(\beta_b + \beta_f) + c^2h^2\right]\langle Z_k, Y_k\rangle \leq 0. \end{aligned} \tag{3.30}$$

Let us eliminate Z_k from this relation by using the elementary algebraic inequality

$$\left[c^2h\beta_f + \frac{1}{2}c^2h^2\right]\|Z_k\|^2 + \left[c^2h(\beta_b + \beta_f) + c^2h^2\right]\langle Z_k, Y_k\rangle \geq -\frac{c^2h^2(\beta_b + \beta_f + h)^2}{4h\beta_f + 2h^2}\|Y_k\|^2.$$

Then, from (3.30) we deduce that

$$E_{k+1} - E_k + \mathcal{S}_k \leq 0,$$

where

$$\begin{aligned} \mathcal{S}_k = &\left[\frac{1}{2}(c\gamma - 1)^2 + \frac{1}{2}(c\gamma - 1) + \frac{c(c\gamma-1)}{h} - \frac{L}{2}\left(c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_f + ch\right)\right]\|X_k\|^2 \\ &+ \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_b\right]\langle X_k, Y_k\rangle \\ &+ \left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2 - \frac{c^2h^2(\beta_b + \beta_f + h)^2}{4h\beta_f + 2h^2}\right]\|Y_k\|^2. \end{aligned}$$

We have $\mathcal{S}_k = q(X_k, Y_k)$ where $q : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is the quadratic form

$$q(X_k, Y_k) := a\|X_k\|^2 + b\langle X_k, Y_k\rangle + g\|Y_k\|^2,$$

with

$$a = \frac{1}{2}(c\gamma-1)^2 + \frac{1}{2}(c\gamma-1) + \frac{c(c\gamma-1)}{h} - \frac{L}{2}\left(c(c\gamma-1)h + c^2 + c(c\gamma-1)\beta_f + ch\right),$$
$$b = c(c\gamma-1)h + c^2 + c(c\gamma-1)\beta_b,$$
$$g = ch\lambda + c^2 h\beta_b + \frac{1}{2}c^2h^2 - \frac{c^2h^2(\beta_b+\beta_f+h)^2}{4h\beta_f+2h^2}.$$

The above coefficients differ from those involved in the Lyapunov analysis of Theorem 3.3.1 only by a, where L enters. Since, for h small, $a \sim \dfrac{c(c\gamma-1)}{h}$ it is immediate to verify that $a>0$ for h sufficiently small (depending now on L). Moreover the term with coefficient L induces a negligable perturbation. So, by using the same argument as the proof of Theorem 3.3.1, under the condition

$$4\lambda > \frac{2}{\gamma}\left(\beta_b + \frac{1}{\gamma}\right) + \frac{1}{\gamma}\frac{(\beta_b-\beta_f)^2}{\beta_f} + \frac{2}{\gamma}\sqrt{\left(\beta_b+\frac{1}{\gamma}\right)^2 + \frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}}.$$

there exists c such that $c\gamma-1>0$ and $4ag-b^2>0$ is satisfied for h sufficiently small. Therefore, there exists a positive real number μ such that for any $k\geq 1$,

$$E_{k+1} - E_k + \mu\|X_k\|^2 + \mu\|Y_k\|^2 \leq 0. \tag{3.31}$$

The rest of the proof is similar to that of Theorem 3.3.1, so we omit it.

For numerical purposes, it is interseting to give an estimate value of the upper bound h^* in the theorem. We can obtain this value by proceeding in the same way as in the proof of the Theorem 3.3.1. Precisely, according to Lemma 1.3.5, since $a>0$ (for h small enough), q is positive definite if and only if $4ag-b^2>0$. After a few steps of elementary calculation as in the proof of Theorem 3.3.1, we obtain

$$\left(2+\gamma h - Lh(h+\beta_f) - \tfrac{Lh(c+h)}{c\gamma-1}\right)\left((\lambda + c\beta_b + \frac{1}{2}ch)(4\beta_f+2h) - c(\beta_b+\beta_f+h)^2\right)$$
$$-(c\gamma-1)(2\beta_f+h)\left[h + \frac{c}{c\gamma-1} + \beta_b\right]^2 > 0.$$

Let us develop the above expression. We obtain a third-order polynomial with respect to h, namely $P_c(h) = a_0 + a_1 h + a_2 h^2 + a_3 h^3$ with

$$\begin{aligned}
a_0 &= 2\Big(4\lambda\beta_f - c(\beta_b-\beta_f)^2 - \beta_f\frac{(c+\beta_b(c\gamma-1))^2}{c\gamma-1}\Big),\\
a_1 &= 4\lambda(1+\gamma\beta_f) - c\gamma(\beta_b-\beta_f)^2 - 4\beta_f\left(c+\beta_b(c\gamma-1)\right) - \frac{1}{c\gamma-1}\left(c+\beta_b(c\gamma-1)\right)^2\\
&\quad -L(\beta_f+\frac{c}{c\gamma-1})\left(4\lambda\beta_f - c(\beta_b-\beta_f)^2\right),\\
a_2 &= 2\lambda\gamma - 2c - 2(c\gamma-1)(\beta_b+\beta_f) - L(1+\frac{1}{c\gamma-1})\left(4\lambda\beta_f - c(\beta_b-\beta_f)^2\right)\\
&\quad -2\lambda L(\beta_f+\frac{c}{c\gamma-1}),\\
a_3 &= -(c\gamma-1) - 2\lambda L - \frac{2\lambda L}{c\gamma-1}.
\end{aligned}$$

Choosing adequately $c > 0$ with $c\gamma - 1 > 0$ gives that $a_0 > 0$ under the growth condition in Theorem 3.4.1. Precisely, we can take $c = c^*$ where $c^*\gamma - 1 = \frac{1}{\gamma}\left(\left(\beta_b + \frac{1}{\gamma}\right)^2 + \frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}\right)^{-\frac{1}{2}}$. Hence $P_{c^*}(0) > 0$. Note that for large h, $P_{c^*}(h) \sim -(c^*\gamma - 1 + 2\lambda L + \frac{2\lambda L}{c^*\gamma-1})h^3$, and so $P_{c^*}(h)$ is negative. Set $h_{\min}$ is the smallest positive zero (which exists) of P_{c^*}. Therefore, $P_{c^*}(h) > 0$ for all $h \in]0, h_{\min}[$. Note that the choice of h must ensure that $a > 0$ as well. For $c = c^*$, we see that

$$\begin{aligned}
a := a(c^*) &= \frac{1}{2}(c^*\gamma-1)^2 + \frac{1}{2}(c^*\gamma-1) + \frac{c^*(c^*\gamma-1)}{h}\\
&\quad - \frac{L}{2}\left(c^*(c^*\gamma-1)h + c^{*2} + c^*(c^*\gamma-1)\beta_f + c^*h\right) > 0.
\end{aligned}$$

Therefore, $a := a(c^*) > 0$ if and only if

$$L(1+\frac{1}{c^*\gamma-1})h^2 + \left[L(\frac{c^*}{c^*\gamma-1}+\beta_f) - \gamma\right]h - 2 < 0.$$

Let h_+ be the positive root (which exists) of the second-order polynomial (with respect to h), that is, $L(1+\frac{1}{c^*\gamma-1})h^2 + \left[L(\frac{c^*}{c^*\gamma-1}+\beta_f) - \gamma\right]h - 2$. Then $a > 0$ for all $h \in]0, h_+[$. Therefore, the upper bound h^* for the (DINAAM-split) scheme can be defined as

$$h^* = \min\{h_{min}, h_+\}.$$

■

3.4.2 Errors, perturbations

Now, we examine the impact of introducing perturbations, or errors in the algorithm (DINAAM-split). Let us begin with the perturbed version of (DINAM)

$$\ddot{x}(t) + \gamma\dot{x}(t) + \nabla f(x(t)) + B(x(t)) + \beta_f \nabla^2 f(x(t))\dot{x}(t) + \beta_b B'(x(t))\dot{x}(t) = e(t), \tag{DINAM-pert}$$

where the right-handside $e(\cdot)$ accounts for perturbations, errors. An analogous discretization to the one used previously yields

$$\begin{aligned}\frac{1}{h^2}(x_{k+1} - 2x_k + x_{k-1}) + \frac{\gamma}{h}(x_{k+1} - x_k) + \frac{\beta_b}{h}(B(x_{k+1}) - B(x_k))\\ + \frac{\beta_f}{h}(\nabla f(x_k) - \nabla f(x_{k-1})) + B(x_{k+1}) + \nabla f(x_k) = e_k.\end{aligned} \tag{3.32}$$

After expanding and rearranging all terms of (3.32), we obtain

$$\begin{aligned} x_{k+1} + \frac{h(\beta_b + h)}{1+\gamma h}B(x_{k+1}) &= x_k + \frac{1}{1+\gamma h}(x_k - x_{k-1}) + \frac{h\beta_b}{1+\gamma h}B(x_k) \qquad (3.33)\\ &- \frac{h\beta_f}{1+\gamma h}(\nabla f(x_k) - \nabla f(x_{k-1})) - \frac{h^2}{1+\gamma h}\nabla f(x_k) + \frac{h^2}{1+\gamma h}e_k.\end{aligned}$$

Set $s := \dfrac{h}{1+\gamma h}$ and $\alpha := \dfrac{1}{1+\gamma h}$. So we have

$$x_{k+1} + s\mathcal{B}_h(x_{k+1}) = y_k, \tag{3.34}$$

where $\mathcal{B}_h = (h + \beta_b)B$, and

$$y_k = x_k + \alpha(x_k - x_{k-1}) + s\beta_b B(x_k) - s(h + \beta_f)\nabla f(x_k) + s\beta_f \nabla f(x_{k-1}) + she_k. \tag{3.35}$$

From (3.34), we get $x_{k+1} = (\mathrm{Id} + s\mathcal{B}_h)^{-1}(y_k)$. Combining the aforementioned facts, we obtain the following algorithm :

(DINAAM-split-pert) :

Initialize : $x_0 \in \mathcal{H}, x_1 \in \mathcal{H}$

$$\alpha = \frac{1}{1+\gamma h}, \quad s = \frac{h}{1+\gamma h},$$

$$y_k = x_k + \alpha(x_k - x_{k-1}) + s\beta_b B(x_k) - s(h+\beta_f)\nabla f(x_k) + s\beta_f \nabla f(x_{k-1}) + she_k,$$

$$x_{k+1} = (\mathrm{Id} + s\mathcal{B}_h)^{-1}(y_k).$$

Theorem 3.4.2 *Let us make the assumptions of Theorem 3.4.1, and suppose that the sequence (e_k) of perturbations, errors satisfies :*

$$\sum_{k=1}^{\infty} \|e_k\| < +\infty.$$

Then there exists h^* such that the sequence (x_k) generated by the algorithm (DINAAM-split-pert) *has the following properties for all $0 < h < h^*$:*

(i) (x_k) *converges weakly to an element in S ;*

(ii) $$\sum_{k=1}^{\infty} \|x_k - x_{k-1}\|^2 < +\infty, \quad \sum_{k=1}^{\infty} \|A(x_k)\|^2 < +\infty,$$

$$\sum_{k=1}^{\infty} \|\nabla f(x_k) - \nabla f(x_{k-1})\|^2 < +\infty, \text{ and } \sum_{k=1}^{\infty} \|B(x_k) - B(x_{k-1})\|^2 < +\infty;$$

(iii) $\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0, \lim_{k\to\infty} \|B(x_k) - B(p)\| = 0,$ *and* $\lim_{k\to\infty} \|\nabla f(x_k) - \nabla f(p)\| = 0.$

Passing from the unperturbed Lyapunov analysis to the perturbed case is a typical procedure ; see [19] for example. It is based on a comparable Lyapunov analysis and the application of the following discrete variant of the Gronwall Lemma ; see Lemma 1.3.7 or [19, Lemma A.9.].

Proof. The proof is similar to that of Theorem 3.4.1. It uses the following sequence (E_k) as a discrete energy function

$$E_k = V_k + \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_f\right] \Gamma_k,$$

where $c > \frac{1}{\gamma}$ is coefficient to adjust, and

$$
\begin{aligned}
V_k &= \frac{1}{2}\|(x_k - p) + c(\frac{1}{h}(x_k - x_{k-1}) + A_\beta(x_k) - A_\beta(p))\|^2 + \frac{c\gamma - 1}{2}\|x_k - p\|^2, \\
\Gamma_k &= f(x_k) - f(p) - \langle \nabla f(p), x_k - p\rangle.
\end{aligned}
$$

By setting $X_k = x_{k+1} - x_k, Y_k = B(x_{k+1}) - B(p), Z_k = \nabla f(x_k) - \nabla f(p)$ for $k \geq 0$ and following the same argument as in the proof of Theorem 3.4.1, we have

$$
\begin{aligned}
E_{k+1} - E_k &+ \left[\frac{1}{2}(c\gamma - 1)^2 + \frac{1}{2}(c\gamma - 1) + \frac{c(c\gamma-1)}{h}\right]\|X_k\|^2 \\
&+ \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_b\right]\langle X_k, Y_k\rangle + \left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2\right]\|Y_k\|^2 \\
&+ \left[c^2h\beta_f + \frac{1}{2}c^2h^2\right]\|Z_k\|^2 + \left[c^2h(\beta_b + \beta_f) + c^2h^2\right]\langle Z_k, Y_k\rangle \leq \epsilon_k. \qquad (3.36)
\end{aligned}
$$

Here,

$$
\begin{aligned}
\epsilon_k =& -\frac{1}{2}c^2h^2\|e_k\|^2 + c^2h^2\langle Y_k + Z_k, e_k\rangle + \left[c(c\gamma - 1)h + c^2\right]\langle X_k, e_k\rangle \\
&+ ch\langle x_{k+1} - p, e_k\rangle + c^2h\langle \beta_b Y_k + \beta_f Z_k, e_k\rangle. \qquad (3.37)
\end{aligned}
$$

According to an elementary inequality, we have that

$$
\langle X_k, e_k\rangle \leq \frac{1}{2\eta}\|X_k\|^2 + \frac{\eta}{2}\|e_k\|^2, \qquad (3.38)
$$

holds for any $\eta > 0$. Moreover, by using Cauchy-Schwarz inequality, and the fact that $B, \nabla f$ are Lipschitz, we have

$$
\begin{aligned}
\langle Y_k, e_k\rangle &\leq \|Y_k\|.\|e_k\| \leq \frac{1}{\lambda}\|x_{k+1} - p\|.\|e_k\|, \qquad (3.39) \\
\langle Z_k, e_k\rangle &\leq \|Z_k\|.\|e_k\| \leq L\|x_{k+1} - p\|.\|e_k\|. \qquad (3.40)
\end{aligned}
$$

Combining (3.37)-(3.40), we obtain

$$
\begin{aligned}
\epsilon_k \leq& -\frac{1}{2}c^2h^2\|e_k\|^2 + \frac{c(c\gamma - 1)h + c^2}{2\eta}\|X_k\|^2 + \frac{\eta}{2}[c(c\gamma - 1)h + c^2]\|e_k\|^2 \\
&+ \left[ch + \frac{c^2h^2 + c^2h\beta_b}{\lambda} + (c^2h^2 + c^2h\beta_f)L\right]\|x_{k+1} - p\|\|e_k\|. \qquad (3.41)
\end{aligned}
$$

From (3.36) and (3.41), we deduce that

$$\begin{aligned}E_{k+1}-E_k &+ \left[\frac{1}{2}(c\gamma-1)^2+\frac{1}{2}(c\gamma-1)+\frac{c(c\gamma-1)}{h}-\frac{c(c\gamma-1)h+c^2}{2\eta}\right]\|X_k\|^2\\ &+\left[c(c\gamma-1)h+c^2+c(c\gamma-1)\beta_b\right]\langle X_k,Y_k\rangle\\ &+\left[ch\lambda+c^2h\beta_b+\frac{1}{2}c^2h^2\right]\|Y_k\|^2\\ &+\left[c^2h\beta_f+\frac{1}{2}c^2h^2\right]\|Z_k\|^2\\ &+\left[c^2h(\beta_b+\beta_f)+c^2h^2\right]\langle Z_k,Y_k\rangle\le\epsilon'_k,\end{aligned}\tag{3.42}$$

with

$$\epsilon'_k=\frac{\eta}{2}[c(c\gamma-1)h+c^2]\|e_k\|^2+\left[ch+\frac{c^2h^2+c^2h\beta_b}{\lambda}+(c^2h^2+c^2h\beta_f)L\right]\|x_{k+1}-p\|\|e_k\|.$$

Let us eliminate Z_k from this relation by using the elementary algebraic inequality

$$\begin{aligned}\left[c^2h\beta_f+\frac{1}{2}c^2h^2\right]\|Z_k\|^2+\left[c^2h(\beta_b+\beta_f)+c^2h^2\right]\langle Z_k,Y_k\rangle\\ \ge-\frac{c^2h^2\left(\beta_b+\beta_f+h\right)^2}{4\left(h\beta_f+\frac{1}{2}h^2\right)}\|Y_k\|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}E_{k+1}-E_k &+ \left[\frac{1}{2}(c\gamma-1)^2+\frac{1}{2}(c\gamma-1)+\frac{c(c\gamma-1)}{h}-\frac{c(c\gamma-1)h+c^2}{2\eta}\right]\|X_k\|^2\\ &+\left[c(c\gamma-1)h+c^2+c(c\gamma-1)\beta_b\right]\langle X_k,Y_k\rangle\\ &+\left[ch\lambda+c^2h\beta_b+\frac{1}{2}c^2h^2-\frac{c^2h^2\left(\beta_b+\beta_f+h\right)^2}{4\left(h\beta_f+\frac{1}{2}h^2\right)}\right]\|Y_k\|^2\le\epsilon'_k.\end{aligned}\tag{3.43}$$

Equivalently

$$E_{k+1}-E_k+\mathcal{S}_k\le\epsilon'_k,\tag{3.44}$$

where

$$\mathcal{S}_k = \left[\frac{1}{2}(c\gamma-1)^2 + \frac{1}{2}(c\gamma-1) + \frac{c(c\gamma-1)}{h} - \frac{c(c\gamma-1)h+c^2}{2\eta}\right]\|X_k\|^2$$
$$+ \left[c(c\gamma-1)h + c^2 + c(c\gamma-1)\beta_b\right]\langle X_k, Y_k\rangle$$
$$+ \left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2 - \frac{c^2h^2\left(\beta_b+\beta_f+h\right)^2}{4\left(h\beta_f+\frac{1}{2}h^2\right)}\right]\|Y_k\|^2.$$

Similarly, we have $\mathcal{S}_k = q(X_k, Y_k)$ where $q : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ is the quadratic form

$$q(X_k, Y_k) := a\|X_k\|^2 + b\langle X_k, Y_k\rangle + g\|Y_k\|^2,$$

with

$$a = \frac{1}{2}(c\gamma-1)^2 + \frac{1}{2}(c\gamma-1) + \frac{c(c\gamma-1)}{h} - \frac{c(c\gamma-1)h+c^2}{2\eta},$$
$$b = c(c\gamma-1)h + c^2 + c(c\gamma-1)\beta_b,$$
$$g = ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2 - \frac{c^2h^2\left(\beta_b+\beta_f+h\right)^2}{4\left(h\beta_f+\frac{1}{2}h^2\right)}.$$

We choose $\eta > 0$ such that $a > 0$. That means

$$\eta > \frac{c(c\gamma-1)h^2 + c^2h}{(c\gamma-1)^2h + (c\gamma-1)h + c(c\gamma-1)}.$$

Since the time step h will be taken small, there exists $\eta_0 > 0$ such that $\eta < \eta_0$.
Again, thanks to Lemma 1.3.5, we have that q is positive definite if and only if $4ag - b^2 > 0$. By using the same argument as in the proof of Theorem 3.3.1, we have the existence of c such that $\mathcal{S}_k > 0$. To ensure the existence of such c, we need

$$4\lambda > \frac{2}{\gamma}\left(\beta_b + \frac{1}{\gamma}\right) + \frac{1}{\gamma}\frac{(\beta_b-\beta_f)^2}{\beta_f} + \frac{2}{\gamma}\sqrt{\left(\beta_b+\frac{1}{\gamma}\right)^2 + \frac{(\beta_b-\beta_f)^2}{\gamma\beta_f}}.$$

Therefore, there exists positive real number μ such that for any $k \geq 1$,

$$E_{k+1} - E_k + \mu\|X_k\|^2 + \mu\|Y_k\|^2 \leq \epsilon'_k. \tag{3.45}$$

From (3.45) we deduce that

$$E_{k+1} \leq E_1 + \sum_{1\leq i<k+1} \epsilon'_i.$$

Taking into account the form of the energy sequence (E_k), we obtain

$$\frac{c\gamma - 1}{2}\|x_{k+1} - p\|^2 \leq E_1 + \sum_{1\leq i<k+1} \epsilon'_i. \tag{3.46}$$

According to the assumption $\sum_{k=1}^{\infty} \|e_k\| < +\infty$, this implies that $\sum_{k=1}^{\infty} \|e_k\|^2 < +\infty$. Therefore, there exists $C > 0$ such that

$$\begin{aligned}\sum_{1\leq i<k+1} \epsilon'_i &= \left[ch + \frac{c^2h^2 + c^2h\beta_b}{\lambda} + (c^2h^2 + c^2h\beta_f)L\right] \sum_{1\leq i<k+1} \|x_{i+1} - p\|\|e_i\| \\ &\quad + \frac{\eta}{2}[c(c\gamma - 1)h + c^2] \sum_{1\leq i<k+1} \|e_k\|^2 \\ &\leq \left[ch + \frac{c^2h^2 + c^2h\beta_b}{\lambda} + (c^2h^2 + c^2h\beta_f)L\right] \sum_{1\leq i<k+1} \|x_{i+1} - p\|\|e_i\| + C. \end{aligned} \tag{3.47}$$

From (3.46) and (3.47), we deduce that

$$\frac{c\gamma - 1}{2}\|x_{k+1}-p\|^2 \leq E_1+C+\left[ch + \frac{c^2h^2 + c^2h\beta_b}{\lambda} + (c^2h^2 + c^2h\beta_f)L\right] \sum_{1\leq i<k+1} \|e_i\|\|x_{i+1}-p\|.$$

More precisely, we have

$$\frac{1}{2}\|x_{k+1} - p\|^2 \leq \frac{1}{2}C_0^2 + c_0 \sum_{1\leq i<k+1} \|e_i\|\|x_{i+1} - p\|, \tag{3.48}$$

where

$$C_0 = \sqrt{\frac{2(E_1 + C)}{c\gamma - 1}}, \quad c_0 = ch + \frac{c^2h^2 + c^2h\beta_b}{\lambda} + (c^2h^2 + c^2h\beta_f)L.$$

Now, applying Lemma 1.3.7 to (3.48), we obtain

$$\|x_{k+1} - p\| \leq C_0 + c_0 \sum_{1\leq i<k+1} \|e_i\| < +\infty. \tag{3.49}$$

Therefore, $(\|x_{k+1} - p\|)$ and consequently $(\|x_k\|)$ is a bounded sequence.
Returning to (3.47), according to the boundedness of $(\|x_{k+1} - p\|)$ and the assumption of (e_k), we obtain

$$\sum_{k=1}^{\infty} \epsilon'_k < +\infty.$$

The rest of the proof is similar to that of Theorem 3.4.1, so we omit here. The above inequality allows us to estimate $\sum_{k=1}^{\infty} \|X_k\|^2$ and $\sum_{k=1}^{\infty} \|Y_k\|^2$. ■

3.5 A variant of the proximal-gradient algorithm

In this section, we study a variant of the preceding proximal-gradient algorithm in which the operators' role is reversed. This allows us to weaken the hypothesis on f, assuming that f is a $\mathcal{C}^1$ convex function with a Lipschitz gradient on the bounded sets (instead of globally Lipschitz). We examine the semi-implicit finite-difference scheme for (DINAM) shown below :

$$\frac{1}{h^2}(x_{k+1} - 2x_k + x_{k-1}) + \frac{\gamma}{h}(x_{k+1} - x_k) + \frac{\beta_b}{h}(B(x_k) - B(x_{k-1})) \\ + \frac{\beta_f}{h}(\nabla f(x_{k+1}) - \nabla f(x_k)) + B(x_k) + \nabla f(x_{k+1}) = 0. \tag{3.50}$$

The temporal discretization of the Hessian-driven damping term $\beta_b B(x(t))\dot{x}(t)$ is taken equal to $\frac{\beta_b}{h}(B(x_k) - B(x_{k-1}))$ instead of $\frac{\beta_b}{h}(B(x_{k+1}) - B(x_k))$. After expanding (3.50), we obtain

$$x_{k+1} + \frac{h^2}{1+\gamma h}\nabla f(x_{k+1}) + \frac{h\beta_f}{1+\gamma h}\nabla f(x_{k+1}) = x_k + \frac{1}{1+\gamma h}(x_k - x_{k-1}) + \frac{h\beta_f}{1+\gamma h}\nabla f(x_k) \\ - \frac{h\beta_b}{1+h\gamma}(B(x_k) - B(x_{k-1})) - \frac{h^2}{1+h\gamma}B(x_k). \tag{3.51}$$

Set $s := \frac{h}{1+\gamma h}$ and $\alpha := \frac{1}{1+\gamma h}$. So we have

$$x_{k+1} + s\mathcal{F}_h(x_{k+1}) = y_k, \tag{3.52}$$

where

$$\mathcal{F}_h = (h + \beta_f)\nabla f, \tag{3.53}$$

$$y_k = x_k + \alpha(x_k - x_{k-1}) + s\beta_f \nabla f(x_k) - s(h + \beta_b)B(x_k) + s\beta_b B(x_{k-1}). \tag{3.54}$$

From (3.52) we get $x_{k+1} = (\mathrm{Id} + s\mathcal{F}_h)^{-1}(y_k)$, which gives the following algorithm :

(DINAAM-split-var) :

Initialize : $x_0 \in \mathcal{H}, x_1 \in \mathcal{H}$
$\alpha = \dfrac{1}{1+\gamma h}, \quad s = \dfrac{h}{1+\gamma h},$
$y_k = x_k + \alpha(x_k - x_{k-1}) + s\beta_f \nabla f(x_k) - s(h+\beta_b)B(x_k) + s\beta_b B(x_{k-1}),$
$x_{k+1} = (\mathrm{Id} + s\mathcal{F}_h)^{-1}(y_k) = \mathrm{prox}_{s(h+\beta_f)f}(y_k).$

Theorem 3.5.1 *Let $B : \mathcal{H} \to \mathcal{H}$ be a λ-cocoercive operator and $f : \mathcal{H} \to \mathbb{R}$ be a $\mathcal{C}^1$ convex function whose gradient is Lipschitz continuous on the bounded sets. Suppose the positive parameters $\lambda, \gamma, \beta_b, \beta_f$ satisfy*

$$\beta_f > 0, \ \text{and} \ \lambda\gamma > \frac{(\beta_b - \beta_f)^2}{4\beta_f} + \frac{1}{2}\left(\beta_b + \frac{1}{\gamma}\right) + \frac{1}{2}\sqrt{\left(\beta_b + \frac{1}{\gamma}\right)^2 + \frac{(\beta_b - \beta_f)^2}{\gamma\beta_f}}.$$

Then, there exists h^* such that for all $0 < h < h^*$, the sequence (x_k) generated by the algorithm (DINAAM-split-var) has the following properties :

(i) *(x_k) converges weakly to an element in S ;*

(ii) $\displaystyle\sum_{k=1}^{\infty} \|x_k - x_{k-1}\|^2 < +\infty, \ \sum_{k=1}^{\infty} \|A(x_k)\|^2 < +\infty,$

$$\sum_{k=1}^{\infty} \|\nabla f(x_k) - \nabla f(x_{k-1})\|^2 < +\infty, \ \text{and} \ \sum_{k=1}^{\infty} \|B(x_k) - B(x_{k-1})\|^2 < +\infty;$$

(iii) *(pointwise estimates)*

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0, \lim_{k\to\infty} \|B(x_k) - B(p)\| = 0, \ \text{and} \ \lim_{k\to\infty} \|\nabla f(x_k) - \nabla f(p)\| = 0.$$

Proof. Let us consider the sequence (V_k) given by, for each $k \geq 1$

$$V_k = \frac{1}{2}\|(x_k - p) + c(\frac{1}{h}(x_k - x_{k-1}) + \beta_f \nabla f(x_k) + \beta_b B(x_{k-1}) - A_\beta(p))\|^2 + \frac{\delta}{2}\|x_k - p\|^2,$$

where c, δ are positive coefficients to adjust. For $k \geq 1$, let us briefly write V_k as follows

$$V_k = \frac{1}{2}\|v_k\|^2 + \frac{\delta}{2}\|x_k - p\|^2,$$

with $v_k = (x_k - p) + c\left(\frac{1}{h}(x_k - x_{k-1}) + \beta_f \nabla f(x_k) + \beta_b B(x_{k-1}) - A_\beta(p)\right)$.
By using the formulation (3.50) of the algorithm, we have

$$\begin{aligned}
v_k &= c\big(\frac{1}{h}(x_{k+1} - x_k) + \gamma(x_{k+1} - x_k) + \beta_f \nabla f(x_{k+1}) + \beta_b B(x_k) - A_\beta(p) + h\nabla f(x_{k+1}) + hB(x_k)\big) \\
&\quad + (x_{k+1} - p) - (x_{k+1} - x_k) \\
&= v_{k+1} + (c\gamma - 1)(x_{k+1} - x_k) + ch\nabla f(x_{k+1}) + chB(x_k).
\end{aligned}$$

Set $X_k = x_{k+1} - x_k, Y_k = B(x_k) - B(p), Z_k = \nabla f(x_{k+1}) - \nabla f(p)$. Taking $\delta := c\gamma - 1$, we get

$$\begin{aligned}
V_{k+1} - V_k = &- \frac{1}{2}(c\gamma - 1)^2\|X_k\|^2 - \frac{1}{2}c^2h^2\|Y_k + Z_k\|^2 - c(c\gamma - 1)h\langle X_k, Y_k + Z_k\rangle \\
&- \frac{1}{2}(c\gamma - 1)\|X_k\|^2 - ch\langle x_{k+1} - p, \nabla f(x_{k+1}) + B(x_k)\rangle - \frac{c(c\gamma - 1)}{h}\|X_k\|^2 \\
&- c^2\langle X_k, Y_k + Z_k\rangle - c(c\gamma - 1)\langle \beta_b Y_k + \beta_f Z_k, X_k\rangle - c^2h\langle \beta_b Y_k + \beta_f Z_k, Y_k + Z_k\rangle.
\end{aligned}$$

Using the fact that $p \in S$, ∇f is monotone, and B is λ-cocoercive, we have

$$\begin{aligned}
&-ch\langle x_{k+1} - p, \nabla f(x_{k+1}) + B(x_k)\rangle \\
&= -ch\langle x_{k+1} - p, \nabla f(x_{k+1}) - \nabla f(p)\rangle - ch\langle x_{k+1} - p, B(x_k) - B(p)\rangle \\
&\leq -ch\lambda\|B(x_k) - B(p)\|^2 - ch\langle x_{k+1} - x_k, B(x_k) - B(p)\rangle.
\end{aligned}$$

By combining the two relations above, we obtain

$$\begin{aligned}
V_{k+1} - V_k &+ \left[\frac{1}{2}(c\gamma - 1)^2 + \frac{1}{2}(c\gamma - 1) + \frac{c(c\gamma - 1)}{h}\right]\|X_k\|^2 \\
&+ \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_f\right]\langle X_k, Z_k\rangle \\
&+ \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_b + ch\right]\langle X_k, Y_k\rangle + \left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2\right]\|Y_k\|^2 \\
&+ \left[c^2h\beta_f + \frac{1}{2}c^2h^2\right]\|Z_k\|^2 + \left[c^2h(\beta_b + \beta_f) + c^2h^2\right]\langle Z_k, Y_k\rangle \leq 0.
\end{aligned} \tag{3.55}$$

Let (Γ_k) be the sequence defined by

$$\Gamma_k = f(x_k) - f(p) - \langle \nabla f(p), x_k - p\rangle, \text{ for } k \geq 0.$$

Since f is convex, we have $\Gamma_k \geq 0$, for all $k \geq 0$. Moreover,

$$\begin{aligned}\langle X_k, Z_k\rangle &= \langle x_{k+1} - x_k, \nabla f(x_{k+1})\rangle - \langle x_{k+1} - x_k, \nabla f(p)\rangle \\ &\geq f(x_{k+1}) - f(x_k) + \Gamma_{k+1} - \Gamma_k + f(x_k) - f(x_{k+1}) \\ &= \Gamma_{k+1} - \Gamma_k. \end{aligned} \tag{3.56}$$

Let us define

$$E_k := V_k + \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_f\right] \Gamma_k,$$

for $k \geq 1$. (E_k) will serve us as a discrete energy function. Indeed, it is clear that (E_k) is a sequence of nonnegative numbers. From (3.55), (3.56) and the definition of (E_k), we obtain

$$\begin{aligned}&E_{k+1} - E_k + \left[\frac{1}{2}(c\gamma - 1)^2 + \frac{1}{2}(c\gamma - 1) + \frac{c(c\gamma - 1)}{h}\right] \|X_k\|^2 \\ &\quad + \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_b + ch\right] \langle X_k, Y_k\rangle + \left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2\right] \|Y_k\|^2 \\ &\quad + \left[c^2h\beta_f + \frac{1}{2}c^2h^2\right] \|Z_k\|^2 + \left[c^2h(\beta_b + \beta_f) + c^2h^2\right] \langle Z_k, Y_k\rangle \leq 0. \end{aligned} \tag{3.57}$$

Let us eliminate Z_k from this relation by using the elementary algebraic inequality

$$\left[c^2h\beta_f + \frac{1}{2}c^2h^2\right] \|Z_k\|^2 + \left[c^2h(\beta_b + \beta_f) + c^2h^2\right] \langle Z_k, Y_k\rangle \geq -\frac{c^2h^2(\beta_b + \beta_f + h)^2}{4h\beta_f + 2h^2}\|Y_k\|^2.$$

From (3.57) we deduce that $E_{k+1} - E_k + \mathcal{S}_k \leq 0$, where

$$\begin{aligned}\mathcal{S}_k = &\left[\frac{1}{2}(c\gamma - 1)^2 + \frac{1}{2}(c\gamma - 1) + \frac{c(c\gamma - 1)}{h}\right] \|X_k\|^2 \\ &+ \left[c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_b + ch\right] \langle X_k, Y_k\rangle \\ &+ \left[ch\lambda + c^2h\beta_b + \frac{1}{2}c^2h^2 - \frac{c^2h^2(\beta_b + \beta_f + h)^2}{4h\beta_f + 2h^2}\right] \|Y_k\|^2. \end{aligned}$$

We have $\mathcal{S}_k = q(X_k, Y_k)$ where $q : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is the quadratic form

$$q(X_k, Y_k) := a\|X_k\|^2 + b\langle X_k, Y_k\rangle + g\|Y_k\|^2,$$

with

$$a = \frac{1}{2}(c\gamma - 1)^2 + \frac{1}{2}(c\gamma - 1) + \frac{c(c\gamma - 1)}{h},$$
$$b = c(c\gamma - 1)h + c^2 + c(c\gamma - 1)\beta_b + ch,$$
$$g = ch\lambda + c^2 h\beta_b + \frac{1}{2}c^2h^2 - \frac{c^2h^2(\beta_b + \beta_f + h)^2}{4h\beta_f + 2h^2}.$$

By using the same argument as the proof of Theorem 3.3.1, we obtain the existence of c such that $\mathcal{S}_k > 0$. To ensure the existence of such c, we need

$$4\lambda > \frac{2}{\gamma}\left(\beta_b + \frac{1}{\gamma}\right) + \frac{1}{\gamma}\frac{(\beta_b - \beta_f)^2}{\beta_f} + \frac{2}{\gamma}\sqrt{\left(\beta_b + \frac{1}{\gamma}\right)^2 + \frac{(\beta_b - \beta_f)^2}{\gamma\beta_f}}.$$

Therefore, there exists positive real number μ such that for any $k \geq 1$,

$$E_{k+1} - E_k + \mu\|X_k\|^2 + \mu\|Y_k\|^2 \leq 0. \tag{3.58}$$

The rest of the proof is similar to Theorem 3.3.1, so we omit it. ■

3.6 Numerical illustrations

Remark 3.6.1 As we discussed in Chapter 2, a general and effective method to generate monotone cocoercive operators which are not gradients of convex functions is taking Yosida approximation A_λ of a linear skew symmetric operator A . For example, starting from A equal to the counterclockwise rotation of angle $\pi/2$ in the plane, we obtain that, for any $\lambda > 0$, the following operator is λ-cocoercive

$$A_\lambda = \frac{1}{1+\lambda^2}\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}. \tag{3.59}$$

Example 3.6.1 Take $\mathcal{H} = \mathbb{R}^2$ equipped with the Euclidean structure. Let us consider the linear operator B whose matrix in the canonical basis of $\mathbb{R}^2$ is given by (3.59) with $\lambda = 5$, *i.e.*, $B = A_\lambda$. According to Remark 3.6.1, B is a nonpotential operator which is λ-cocoercive with $\lambda = 5$. In Chapter 2, we observed the oscillations, in the heavy ball with friction, when $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x_1, x_2) = 50x_2^2.$$

We set $\gamma = 0.9$. It is clear that f is convex but not strongly convex and its gradient ∇f is $L-$Lipschitz with $L = 100$. We study three cases : (1) $\beta_b = 1, \beta_f = 0.5$, (2) $\beta_b = 0.5, \beta_f = 1$

and (3) $\beta_b = \beta_f = 0.5$. As a straight application of Theorem 3.3.1 and 3.4.1, we obtain that the sequences (x_k) generated by (DINAAM) and (DINAAM-split) converge to x_∞, where $x_\infty \in S = (\nabla f + B)^{-1}(0) = \{0\}$. The trajectory obtained by using Matlab is depicted in Figure 1 in Chapter 2. In order to compare the two algorithms, we observe the norm of $x_k - x_\infty$. In Figure 3.1, we can see that the two algorithms give almost the same numerical results. The difference between them is the use or not of the resolvent operator of the sum of B and ∇f. In our numerical experiment, we took $h = 5.10^{-3}$ as a time-step. According to the proof of Theorem 3.3.1 and 3.4.1, we can estimate the upper bound h^* for these schemes. The numerical results are shown in Table 1.

Case	DINAAM scheme	DINAAM-split scheme
$\beta_b = 1, \beta_f = 0.5$	10.9276	0.00526
$\beta_b = 0.5, \beta_f = 1$	8.8208	0.00533
$\beta_b = 0.5, \beta_f = 0.5$	8.7365	0.00619

TABLE 3.1 – Numerical values for the upper bound h^* in (DINAAM) and (DINAAM-split) schemes.

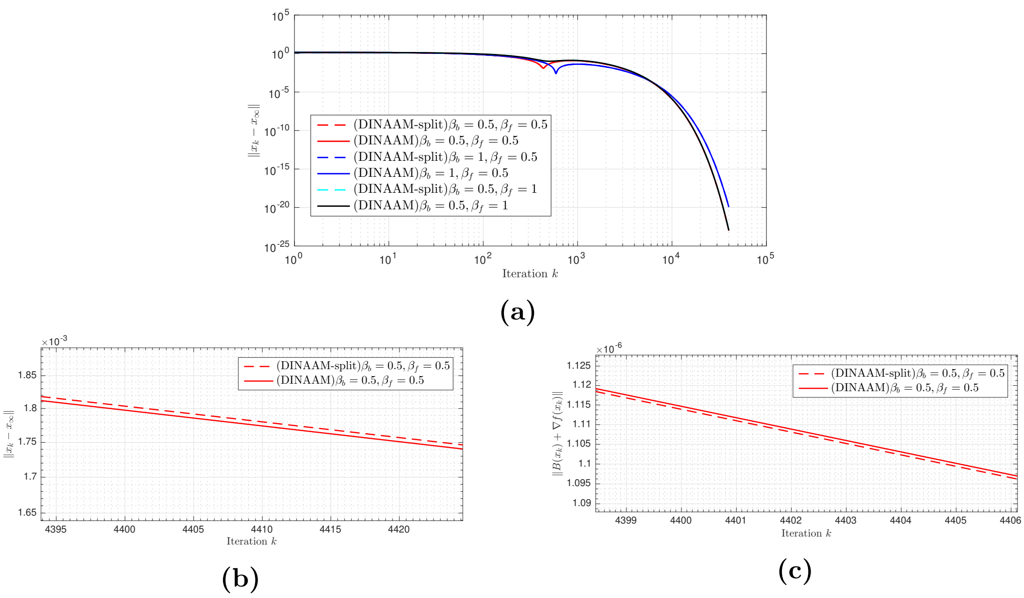
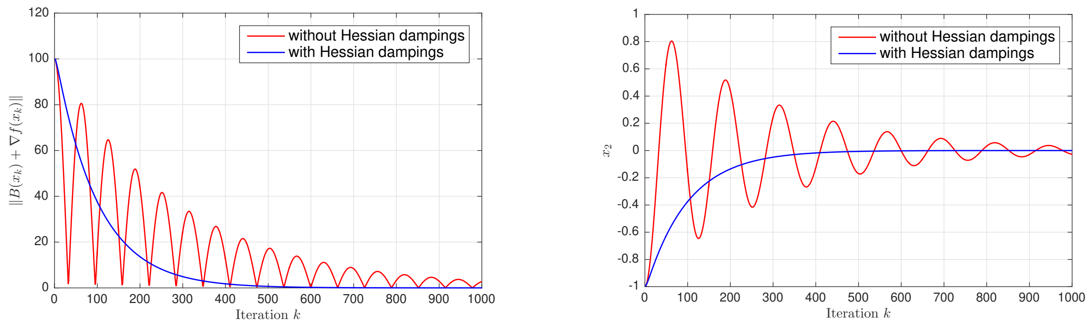


FIGURE 3.1 – A comparison between (DINAAM) and (DINAAM-split).

Before moving on to another example, let us numerically compare the performance of our algorithms with or without the correcting terms associated with the Hessien and Newton-like damping. For convenience, we keep the function f and the operator B, and the values of the parameters λ, γ as before. We compare the performance of (DINAAM) in two situations, one when $\beta_b = \beta_f = 0$, and the other when β_b and β_f are different from 0 (for example, we take $\beta = 1, \beta_f = 0.5$). Figure 3.2 illustrates the typical situation of an ill-conditioned problem, where the wild oscillations of (2.4) are neutralized by introducing the Hessian dampings. This shows that the presence of the Hessian-driven damping and the Newton-type correction term attached to B attenuate the oscillations which occur with the inertial methods with viscous damping.



(a) **(b)**

Figure 3.2 – Evolution of the objective (left) and trajectories (right) for (DINAAM) when adding the Hessian dampings.

The interested reader can find in Chapter 2 an application of our continuous model to dynamic games. Our corresponding algorithmic results provide a new class of best response dynamics with inertia and cost to change, the detailed analysis of which goes beyond the scope of the article. It is an interesting subject for futher studies. Now we study another example to see how our algorithm can be applied to find the zeros of $\nabla f + B$.

Example 3.6.2 Nonpotential version of sparse logistic regression. Let us recall the following sparse logistic regression problem for binary classification :

$$\min_{x\in\mathbb{R}^n} \frac{1}{m}\sum_{i=1}^{m} \log(1 + e^{-v_i u_i^\top x}) + \mu\|x\|_1,$$

where $(u_i, v_i)_{1\leq i\leq m}$ is the training set with $u_i \in \mathbb{R}^n$ is the feature vector of each data sample, and $v_i \in \{-1, 1\}$ is the binary label. Here $\mu > 0$ is a regularization parameter. We set

$$f(x) = \frac{1}{m}\sum_{i=1}^{m} \log(1 + e^{-v_i u_i^\top x}).$$

The gradient of f is given by $\nabla f(x) = -\frac{1}{m}A^\top(1_m - q(x))$, with $A^\top = \begin{pmatrix} v_1u_1 & v_2u_2 & \dots & v_mu_m \end{pmatrix} \in \mathbb{R}^{n\times m}$, $1_m = \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix} \in \mathbb{R}^m$ and $q(x) = 1_m./(1 + e^{-Ax})$ (./ denotes the component-wise division). Consider the following problem : Solve $B_n(x) + \nabla f(x) = 0$,

$$\text{where } \ x \in \mathbb{R}^n \text{ and } \ B_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & \dots & 0 & -1 & 2 \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

Let us show that B_n is positive definite for all $n \geq 2$. Let us denote by y_k the k-th leading principal minor of a matrix B_n which is the determinant of its upper-left $k \times k$ sub-matrix. We have

$$y_k = \det(B_k), \ \text{ for } 1 \leq k \leq n.$$

For simplify, we define $B_1 = 2$. By the definition of B_n, we have that

$$\det(B_{n+1}) = 2\det(B_n) - \det(B_{n-1}),$$

for $n \geq 2$. An elementary calculation gives $\det(B_{n+1}) = n+2$ for $n \geq 1$. Thus, $y_k = k+1 > 0$ for all $1 \leq k \leq n$. Hence, B_n is positive definite. Furthermore, B_n is cocoercive. Indeed, for any $x, y \in \mathbb{R}^n$, there exists $\lambda > 0$ such that

$$\langle B_n x - B_n y, x - y\rangle \geq \lambda \|B_n x - B_n y\|^2. \tag{3.60}$$

Since $B_n, B_n^\top B_n$ are positive (semi)definite, for any $x, y \in \mathbb{R}^n$ we have

$$\langle B_n x - B_n y, x - y\rangle \geq \lambda_{min}(B_n)\|x - y\|^2, \tag{3.61}$$

and

$$\lambda_{max}(B_n^\top B_n)\|x - y\|^2 \geq \|B_n x - B_n y\|^2. \tag{3.62}$$

Here, $\lambda_{min}(B_n), \lambda_{max}(B_n^\top B_n)$ are the smallest eigenvalue of B_n and the greatest eigenvalue of $B_n^\top B_n$ respectively. For instance, take $\lambda = \dfrac{\lambda_{min}(B_n)}{\lambda_{max}(B_n^\top B_n)}$, from (3.61) and (3.62), we deduce that (3.60) holds.

Let us check that ∇f is Lipschitz continuous. In fact, for any $x, y \in \mathbb{R}^n$, we have

$$\begin{aligned} m\|\nabla f(x) - \nabla f(y)\| &= \|A^\top q(x) - A^\top q(y)\| \\ &= \Big\| \sum_{i=1}^{m} \frac{v_i}{1+e^{-v_i u_i^\top x}} u_i - \frac{v_i}{1+e^{-v_i u_i^\top y}} u_i \Big\| \\ &\leq \sum_{i=1}^{m} \|u_i\|.|v_i|.\Big| \frac{1}{1+e^{-v_i u_i^\top x}} - \frac{1}{1+e^{-v_i u_i^\top y}} \Big| \\ &\leq \frac{1}{4} \sum_{i=1}^{m} \|u_i\|.|u_i^\top x - u_i^\top y| \leq \frac{1}{4} \sum_{i=1}^{m} \|u_i\|^2.\|x-y\|. \end{aligned}$$

Therefore, $\|\nabla f(x) - \nabla f(y)\| \leq \frac{1}{4m}\|x-y\| \sum_{i=1}^{m} \|u_i\|^2$.

It is clear that for any matrix B is λ-cocoercive if $B - \lambda B^\top B$ is positive semidefinite for some $\lambda > 0$. For example, we take $n = 3, m = 2$. Then, B_3 is $\frac{1}{4}$-cocoercive. Set $\gamma = 4, \beta_b = \beta_f = 0.5$, and $x_0 = 0_n, \dot{x}_0 = 1_n$ as initial conditions. According to Theorem 3.4.1, we can conclude that the sequence (x_k) generated by (DINAAM-split) converges to the zeros of $\nabla f + B_3$. Implementing the algorithm (DINAAM-split) in Matlab, we obtain the plot of k versus the norm of $\nabla f(x_k) + B_3(x_k)$, see Figure 3.3. Here, the training set is taken randomly for numerical test purposes and we took $h = 5.10^{-3}$.

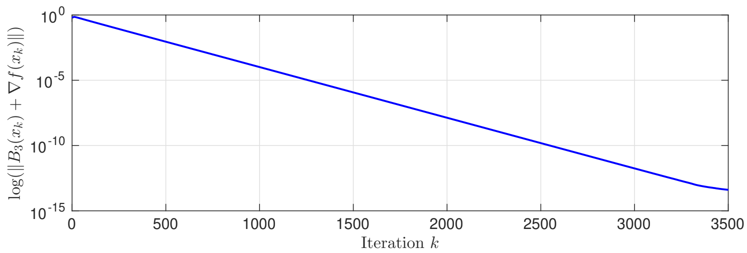


FIGURE 3.3 – The plot of k versus the norm of $\nabla f(x_k) + B_3(x_k)$ obtained by (DINAAM-split).

Remark 3.6.2 In Example 3.6.2, since the resolvent operator $(\mathrm{Id} + s\mathcal{A}_h)^{-1}$ can not be computed easily, we used the algorithm (DINAAM-split) instead of (DINAAM). Then, our algorithm requires to compute $(\mathrm{Id} + s\mathcal{B}_h)^{-1}$ and in this situation it is easier to operate.

Example 3.6.3 Let us return to Example 3.6.1 and consider the effect of the introduction of perturbations, errors. With the same numerical values of the involved parameters, we just add the errors $e_k = \frac{1}{k^2}$ and $\bar{e}_k = \frac{1}{\sqrt{k}}$. Clearly, the errors (e_k) satisfy the assumptions of Theorem 3.4.2 while $(\bar{e}_k)$ does not. Running algorithm (DINAAM-split-pert) in Matlab, the plot of $\|x_k - x_\infty\|$ versus k is depicted in Figure 3.4. We observe that if the perturbed term e_k satisfies the assumptions of Theorem 3.4.2, then algorithm (DINAAM-split-pert) behaves as well as the nonperturbed version.

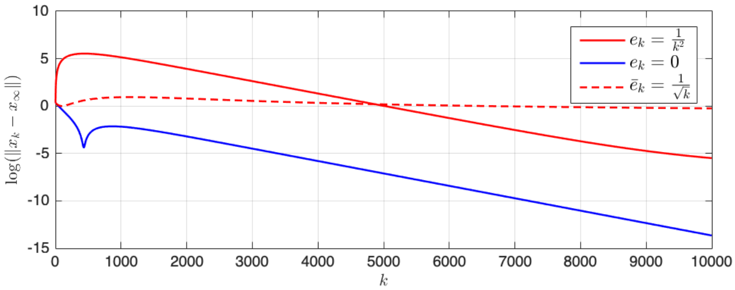


FIGURE 3.4 – The effect of perturbations, errors in the algorithm (DINAAM-split).

3.7 Conclusion, perspectives

In Chapters 2 and 3, the importance of Hessian-driven damping in the convergence characteristics of inertial algorithms in convex optimization is well demonstrated. While these algorithms preserve the convergence rates associated with the Nesterov accelerated gradient method, they give quick convergence to gradient zeros and significantly reduce oscillations. Our contribution is to combine these two aspects inside the same algorithms and to develop inertial algorithms for structured monotone inclusions with potential and nonpotential components (skew-symmetric operators as a typical instance). As a result, this is critical for numerical reasoning and modeling in engineering and decision sciences with cooperative and noncooperative aspects.

Furthermore, our Lyapunov analysis highlighted the two operators' nonsymmetrical roles. That is a major improvement over previous research in which we handled the two operators as a whole. Addressing the issue when B is a generic maximally monotone operator (for example, linear skew-symmetric) is a crucial challenge for dealing with primal-dual techniques from several angles. In this sense, the Yosida approximation of B (a cocoercive operator) allows us to return to the situation discussed in our topic. It is an intriguing subject for future research. Lastly, we finalize a similar methodology to cope with the problem of asymptotic vanishing viscous damping to cover the case of Nesterov's accelerated gradient method.

4

Convergence of inertial dynamics driven by sums of potential and nonpotential operators with implicit Newton-like damping

In this chapter, we propose and investigate the convergence properties of trajectories produced by a damped inertial dynamic driven by the sum of potential and nonpotential operators. More specifically, we desire asymptotically zero sums of the potential term (the gradient of a continuously differentiable convex function) and nonpotential monotone and cocoercive operator. In addition to viscous friction, the dynamic includes implicit Newton-type damping, which differs from the preceding chapters' investigation, which used explicit Newton-type damping as the potential term and related to Hessian-driven damping. We will study and demonstrate the weak convergence of the generated trajectories towards the zeros of the sum of the potential and nonpotential operators as the time approaches infinity. These results are based on Lyapunov analysis and the appropriate choice of damping settings. The addition of geometric dampings enables for the control and attenuation of the oscillations associated with inertial viscous damping. We might extend the convergence analysis to nonsmooth convex potentials by rewriting the second-order evolution equation as a system containing only first-order derivatives in time and space. Even though our research focuses on the autonomous case with positive fixed parameters, these observations pave the way for their extension to the nonautonomous case and the development of new first-order accelerated algorithms in optimization that take into account the special features of potential and nonpotential terms. Because of the presence of the nonpotential term, the proofs and methodologies are unique.

This chapter constitutes the subject of the published paper [5] in collaboration with S. Adly and H. Attouch.

4.1 Problem statement and related works

4.1.1 General presentation

Let $\mathcal{H}$ denote a real Hilbert space endowed with the scalar product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. Our research keeps concentrating on the dynamic approach to addressing the additively structured monotone problem

$$\text{Find } x \in \mathcal{H} \ : \ \nabla f(x) + B(x) = 0, \tag{4.1}$$

where ∇f is the gradient of a continuously differentiable convex function $f : \mathcal{H} \to \mathbb{R}$ (this is the potential part), and $B : \mathcal{H} \to \mathcal{H}$ is a monotone and cocoercive operator (this is the nonpotential part). Specifically, our study focuses the convergence properties of the trajectories generated by the second-order evolution equation

$$\ddot{x}(t) + \gamma\dot{x}(t) + \nabla f\Big(x(t) + \beta_f \dot{x}(t)\Big) + B\Big(x(t) + \beta_b \dot{x}(t)\Big) = 0, \qquad \text{(iDINAM)}$$

whose stationary points are solutions of (4.1). The nonnegative coefficients β_f and β_b in (iDINAM) can be understood as geometric damping parameters, as we will demonstrate. (iDINAM) is an abbreviation for implicit Dynamic Inertial Newton method for Additively structured Monotone problems. In addition to the modeling characteristics discussed above, this system is part of a large family of inertial systems that have recently been studied for designing fast first-order optimization methods.

4.1.2 Related works

In the potential case (*i.e.*, $B = 0$), Alesca-Lazlo-Pinta initiated this in [7]. For f being a strongly convex function f, the associated autonomous system can be found in [58]. This ODE, known as (ISIHD) or Inertial System with Implicit Hessian Damping, takes the form

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \nabla f\Big(x(t) + \beta(t)\dot{x}(t)\Big) = 0, \qquad \text{(ISIHD)}$$

where $\alpha \geq 3$ and $\beta(t) = \gamma + \dfrac{\beta}{t}$, $\gamma,\ \beta \geq 0$. That motivated us extend the results for the case $B \neq 0$. In addition, the explicit version with the introduction of nonpotential term B

$$\ddot{x}(t) + \gamma\dot{x}(t) + \nabla f\left(x(t)\right) + B\left(x(t)\right) + \beta_f \nabla^2 f(x(t))\dot{x}(t) + \beta_b B'(x(t))\dot{x}(t) = 0,\ t \geq 0 \qquad \text{(DINAM)}$$

was previously studied by the authors in [4] or in Chapter 2. (DINAM) is an autonomous dynamic including geometric dampings controlled by the Hessian of the potential function f, and by a Newton-type correction term attached to B. The following explains the connection between the two dynamics described above and the rationale of their respective explicit and implicit qualifying. When $t \to +\infty$ we have $\dot{x}(t) \to 0$, therefore, thanks to the Taylor expansion, we obtain as $t \to +\infty$

$$\begin{aligned}\nabla f\left(x(t) + \beta_f \dot{x}(t)\right) &\approx \nabla f(x(t)) + \beta_f \nabla^2 f(x(t))\dot{x}(t),\\ B\left(x(t) + \beta_b \dot{x}(t)\right) &\approx B(x(t)) + \beta_b B'(x(t))(\dot{x}(t)).\end{aligned}$$

When these terms in (iDINAM) are replaced by their equivalent expressions, the outcome is (DINAM). As a results, when $t \to +\infty$, both systems should behave comparably. The primary goal of this chapter is to investigate the new system (iDINAM) and compare it to (DINAM). In the potential case, ($i.e. B = 0$), such comparision research was carried out in [21] from the viewpoint of the dynamics' stability concerning disturbances and errors. The study is also related to the recent works by Attouch-Laszlo [22, 23] who considered the general case of monotone equations. In contrast to [22, 23], we do not apply the Yosida regularization and exhibit minimum assumptions involving just the nonpotential component based on the cocoercivity of B.

Our main motivation for studying these dynamical systems originates from the fact that geometric damping allows us to regulate and attenuate the oscillations known for the viscous damping of the inertial methods. This is critical for the development of appropriate fast optimization algorithms acquired by temporal discretization.

Throughout the chaper we set up the following standing assumptions [1] :

$$\begin{cases} \text{(A1)} \;\; f : \mathcal{H} \to \mathbb{R} \;\text{ is convex, of class } \mathcal{C}^1,\; \nabla f \;\text{ is Lipschitz continuous;} \\ \text{(A2)} \;\; B : \mathcal{H} \to \mathcal{H} \;\text{ is a } \lambda\text{-cocoercive operator for some } \lambda > 0; \\ \text{(A3)} \;\; \gamma > 0,\; \beta_f > 0,\; \beta_b > 0 \;\text{ are given real damping parameters;} \\ \text{(A4)} \;\;\text{ the solution set }\; S := (\nabla f + B)^{-1}(0) = \{p \in \mathcal{H} :\; \nabla f(p) + B(p) = 0\} \neq \emptyset. \end{cases}$$

We emphasize that the assumption of cocoercivity on the operator B is crucial our analysis. The content of this chapter is organized as follows. After the introductory Section 4.1, in Section 4.2, we show the well-posedness of the Cauchy problem for (iDINAM). In Section 4.3, we analyze the convergence properties of the solution trajectories generated by the continuous dynamics (iDINAM). We highlight the interplay between the damping parameters β_f, β_b, γ and the cocoercivity parameter λ, which plays a significant role in our Lyapunov analysis. In Section 4.4, we analyze various inertial proximal-gradient splitting algorithms which come naturally from the temporal discretization of (iDINAM). We also examine the effect of errors, perturbations in these algorithms. In Section 4.5, we perform numerical experiments which show that the oscillations are considerably reduced with the introduction of geometric damping. Applications to structured monotone equations involving a nonpotential operator are considered.

1. At several places the assumption (A1) will be relaxed, just assuming ∇f to be Lipschitz continuous on the bounded sets

4.2 Well-posedness of the Cauchy problem for (iDINAM)

In this section, we show the existence and the uniqueness of solution trajectory for the Cauchy problem asscociated with the dynamical system (iDINAM). Depending on the hypothesis on the potential function f, we will provide two distinct approaches and results. The first is a very easy example in which f is differentiable with ∇f globally continuous Lipschitz on $\mathcal{H}$. It is based on a straighforward application of the Cauchy-Lipschitz theorem to the Hamiltonian formulation of (iDINAM). The second, more complicated proof concerns the case where $f : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is a convex lower semi-continuous proper function. In both circumstances, we will employ the concept of a strong solution, as defined below.

Definition 4.2.1 *The function $x : [0, +\infty[\to \mathcal{H}$ is called a strong global solution of the dynamical system* (iDINAM) *if it satisfies the following properties :*

(i) $x, \dot{x} : [0, +\infty[\to \mathcal{H}$ *are locally absolutely continuous ;*

(ii) $\ddot{x}(t) + \gamma\dot{x}(t) + \nabla f(x(t) + \beta_f \dot{x}(t)) + B(x(t) + \beta_b \dot{x}(t)) = 0$ *for almost every* $t \geq 0$ *;*

(iii) $x(0) = x_0$ *and* $\dot{x}(0) = x_1$.

Recall that a map $x : [t_0, +\infty[\to \mathcal{H}$ is said to be locally absolutely continuous if it is absolutely continuous on any compact interval $[t_0, T]$, where $T > t_0$. Moreover, we have the following equivalent characterizations of an absolutely continuous function $x : [t_0, T] \to \mathcal{H}$, (see, for example [2, 32]) :

(a) there exists $y : [t_0, T] \to \mathcal{H}$ a Lebesgue-integrable function, such that

$$x(t) = x(0) + \int_0^t y(s)ds, \quad \forall t \in [0, T];$$

(b) x is continuous and its distributional derivative is Lebesgue integrable on the interval $[0, T]$;

(c) for every $\epsilon > 0$, there exists $\eta > 0$ such that for every finite family $I_k = (a_k, b_k)$ from $[0, T]$, the following implication is valid :

$$\left[I_k \cap I_j = \emptyset, \forall k \neq j \text{ and } \sum_k |b_k - a_k| < \eta \right] \Longrightarrow \left[\sum_k \|x(b_k) - x(a_k)\| < \epsilon \right].$$

4.2.1 Existence and uniqueness : the smooth case

Theorem 4.2.1 *Suppose that $f : \mathcal{H} \to \mathbb{R}$ is differentiable with ∇f globally continuous Lipschitz on $\mathcal{H}$. Suppose that $\beta_f > 0$ and $\beta_b > 0$. Then, for any $(x_0, x_1) \in \mathcal{H} \times \mathcal{H}$, there* *exists a unique strong global solution* $x : [0, +\infty[\to \mathcal{H}$ *of the continuous dynamic* (iDINAM) *which satisfies the Cauchy data* $x(0) = x_0$, $\dot{x}(0) = x_1$.

Proof Let us reformulate (iDINAM) as a first-order evolution equation. According to its Hamiltonian formulation, the system (iDINAM) can be rewritten as

$$\begin{cases} \dot{Z}(t) = F(Z(t)) \\ Z(0) = (x_0, x_1), \end{cases} \tag{4.2}$$

where $Z(t) = (x(t), y(t))$ and $F : \mathcal{H}^2 \to \mathcal{H}^2$ is given by

$$F(x,y) = \begin{pmatrix} y \\ -\gamma y - \nabla f(x + \beta_f y) - B(x + \beta_b y) \end{pmatrix}.$$

The Lipschitz continuity properties of ∇f and B make it obvious that F is a Lipschitz continuous map. We obtain the existence and uniqueness of the solution of (4.2), and therefore of the Cauchy problem, by applying the classical Cauchy-Lipschitz theorem to (iDINAM). Because the vector field F is only Lipschitz continuous, we find a strong solution rather than a classical $\mathcal{C}^2$ solution when no further assumptions are made.

4.2.2 Existence and uniqueness : the nonsmooth case

Let us denote by $\Gamma_0(\mathcal{H})$ the set of proper, lower semi-continuous and convex functions on $\mathcal{H}$. We now present another first order formulation of (iDINAM) which is based on the new function

$$y(t) := x(t) + \beta_f \dot{x}(t).$$

Equivalently,

$$\dot{x}(t) = \frac{1}{\beta_f}(y(t) - x(t)). \tag{4.3}$$

Elementary algebra gives

$$x(t) + \beta_b \dot{x}(t) = \frac{\beta_b}{\beta_f} y(t) + \left(1 - \frac{\beta_b}{\beta_f}\right) x(t). \tag{4.4}$$

The time derivation of $y(t)$ using the aforementioned formula and the constitutive equation (iDINAM) yields

$$\begin{aligned}\dot{y}(t) &= \dot{x}(t) + \beta_f \ddot{x}(t)\\ &= \dot{x}(t) - \beta_f\Big(\gamma \dot{x}(t) + \nabla f(y(t)) + B\Big(\frac{\beta_b}{\beta_f} y(t) + \Big(1 - \frac{\beta_b}{\beta_f}\Big) x(t)\Big)\Big)\\ &= (1-\gamma\beta_f)\dot{x}(t) - \beta_f \nabla f(y(t)) - \beta_f B\Big(\frac{\beta_b}{\beta_f} y(t) + \Big(1 - \frac{\beta_b}{\beta_f}\Big) x(t)\Big). \qquad (4.5)\end{aligned}$$

Replacing $\dot{x}(t)$ with $\frac{1}{\beta_f}(y(t) - x(t))$, as given by (4.3), gives

$$\dot{y}(t) = \frac{1-\gamma\beta_f}{\beta_f}(y(t) - x(t)) - \beta_f \nabla f(y(t)) - \beta_f B\Big(\frac{\beta_b}{\beta_f} y(t) + \Big(1 - \frac{\beta_b}{\beta_f}\Big) x(t)\Big). \quad (4.6)$$

In a similar way, the reverse transformation which consists in passing from (4.3), (4.6) to (iDINAM) is obtained. The results are stated in the following theorem.

Theorem 4.2.2 *Let $f \in \mathcal{C}^1(\mathcal{H})$. Suppose that $\beta_f > 0$. The following statements are equivalent :*

1. $x : [0, +\infty[\to \mathcal{H}$ *is a solution trajectory of* (iDINAM) *with initial conditions* $x(0) = x_0$, $\dot{x}(0) = x_1$.
2. $(x, y) : [0, +\infty[\to \mathcal{H} \times \mathcal{H}$ *is a solution trajectory of the first-order system*

$$\begin{cases} \dot{x}(t) + \frac{1}{\beta_f} x(t) - \frac{1}{\beta_f} y(t) = 0.\\ \dot{y}(t) + \beta_f \nabla f(y(t)) + \beta_f B\Big(\frac{\beta_b}{\beta_f} y(t) + \Big(1 - \frac{\beta_b}{\beta_f}\Big) x(t)\Big) + \frac{1-\gamma\beta_f}{\beta_f}(x(t) - y(t)) = 0.\end{cases}$$

with initial conditions $x(0) = x_0$, $y(0) = x_0 + \beta_f x_1$.

By substituting the gradient ∇f with the subdifferential ∂f, we can readily extend the preceding formulation to the situation when $f \in \Gamma_0(\mathcal{H})$.

Definition 4.2.2 *Let $\beta_f > 0$, $f \in \Gamma_0(\mathcal{H})$. Given $(x_0, y_0) \in \mathcal{H} \times \mathrm{dom}(f)$, the Cauchy* *problem associated with the generalized* (iDINAM) *system is defined by*

$$\begin{cases} \dot{x}(t) + \dfrac{1}{\beta_f}x(t) - \dfrac{1}{\beta_f}y(t) = 0 \\ \dot{y}(t) + \beta_f \partial f(y(t)) + \beta_f B\Big(\dfrac{\beta_b}{\beta_f}y(t) + \Big(1 - \dfrac{\beta_b}{\beta_f}\Big)x(t)\Big) + \dfrac{1-\gamma\beta_f}{\beta_f}(x(t)-y(t)) \ni 0. \\ x(0) = x_0, y(0) = y_0. \end{cases} \tag{4.7}$$

The following theorem establishes the well-posedness of a global strong solution of the Cauchy problem (4.7).

Theorem 4.2.3 *Let $f \in \Gamma_0(\mathcal{H})$. Suppose that $\beta_f > 0, \beta_b > 0$. Then, for any Cauchy data $(x_0, y_0) \in \mathcal{H} \times \mathrm{dom}(f)$, there exists a unique global strong solution $(x, y) : [0, +\infty[\to \mathcal{H} \times \mathcal{H}$ of the generalized* (iDINAM) *system* (4.7) *satisfying the initial condition $x(0) = x_0$, $y(0) = y_0$. Moreover when $f \in \mathcal{C}^1(\mathcal{H})$, $x(\cdot)$ is a classical (i.e. $\mathcal{C}^2$) global solution of the Cauchy problem associated with* (iDINAM)*.*

Proof We reformulate (4.7) in the product space $\mathcal{H} \times \mathcal{H}$ by setting $Z(t) = (x(t), y(t)) \in \mathcal{H} \times \mathcal{H}$, and thus (4.7) can be equivalently written as

$$\dot{Z}(t) + \beta_f \partial \mathcal{G}(Z(t)) + \mathcal{D}(Z(t)) \ni 0, \tag{4.8}$$

where the function $\mathcal{G} \in \Gamma_0(\mathcal{H} \times \mathcal{H})$ is defined as $\mathcal{G}(Z) = f(y)$, and operator $\mathcal{D} : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ is given by

$$\mathcal{D}(Z) = \left(\frac{1}{\beta_f}(x-y), \beta_f B\Big(\frac{\beta_b}{\beta_f}y + \Big(1 - \frac{\beta_b}{\beta_f}\Big)x\Big) + \frac{1-\gamma\beta_f}{\beta_f}(x-y)\right).$$

The sum of the convex subdifferential operator $\beta_f \partial \mathcal{G}$ and the Lipschitz continuous operator $\mathcal{D}(\cdot)$ drives the differential inclusion (4.8). A straightforward application of [43, Proposition 3.12] results in the existence and uniqueness of a global strong solution for the Cauchy problem (4.8), and hence for (4.7). In turn, (4.7) admits a unique $\mathcal{C}^1([0, +\infty[)$ global solution (x, y) if $f \in \mathcal{C}^1(\mathcal{H})$. The first equation in (4.7) implies that $\dot{x}$ is a $\mathcal{C}^1([0, +\infty[)$ function, and hence $x \in \mathcal{C}^2([0, +\infty[)$ function. Based on the equivalence in Theorem 4.2.2, the existence and uniqueness of a classical global solution to the Cauchy problem associated with (iDINAM) are then established.

4.3 Asymptotic convergence properties of (iDINAM)

In this part, we examine the asymptotic behavior of the solution trajectories of (iDINAM). We demonstrate that the weak limit, w-$\lim_{t\to+\infty} x(t) = x_\infty$ exists for each solution trajectory $t \mapsto x(t)$ of (iDINAM), and fulfills $x_\infty \in S$, where

$$S := \{p \in \mathcal{H} : \ \nabla f(p) + B(p) = 0\}.$$

We complete these results by producing integral and pointwise convergence rates.

4.3.1 Main results

Our main contributions are Theorems 4.3.1 and 4.4.1. These demonstrate that a wise adjustment of the damping parameters guarantees the weak convergence of the trajectories generated by (iDINAM) and the associated proximal-gradient algorithms achieved by temporal discretization.

Take $p \in S$. Let $x(\cdot)$ be a solution trajectory of the dynamical system (iDINAM). Applying Lyapunov analysis, we obtain the convergence properties of $x(\cdot)$. Let us introduce the function $\mathcal{E}_p : [0, +\infty[\to \mathbb{R}+$ defined by

$$\begin{aligned}\mathcal{E}_p(t) &:= a\Big(f(x(t) + \beta_f \dot{x}(t)) - f(p) - \langle \nabla f(p), x(t) + \beta_f \dot{x}(t) - p\rangle\Big) \\ &\quad + \frac{1}{2}\|x(t) - p + \beta_f \dot{x}(t)\|^2 + \frac{d}{2}\|x(t) - p\|^2, \qquad (4.9)\end{aligned}$$

that will serve us as a Lyapunov function. The convexity of f indicates that $\mathcal{E}_p(\cdot)$ is a nonnegative function. Our aim is to adjust the constants $a > 0$ and $d > 0$ such that $\dot{\mathcal{E}}_p(t) \le 0$ for every $t \ge 0$.

Theorem 4.3.1 *Let $B : \mathcal{H} \to \mathcal{H}$ be a λ-cocoercive operator and $f : \mathcal{H} \to \mathbb{R}$ a $\mathcal{C}^1$ convex function whose gradient is Lipschitz continuous on the bounded sets. Suppose that $S = (\nabla f + B)^{-1}(0) \neq \emptyset$. Consider the evolution equation* (iDINAM) *where the involved parameters fulfill the following conditions :*

$$\gamma\beta_f > 1 \quad and \quad \lambda > \frac{(\beta_b - \beta_f)^2}{4(\gamma\beta_f - 1)}. \qquad (4.10)$$

Then, for any solution trajectory $x : [0, +\infty[\to \mathcal{H}$ of (iDINAM) *the following properties are satisfied :*

(i) *(convergence)*

$$\begin{aligned}
&x(t) \textit{ converges weakly, as } t \to +\infty, \textit{ to an element of } S.\\
&\lim_{t\to+\infty} \|\dot{x}(t)\| = 0,\\
&\lim_{t\to+\infty} \|B(x(t)) - B(p)\| = 0, \lim_{t\to+\infty} \|\nabla f(x(t)) - \nabla f(p)\| = 0,
\end{aligned}$$

where $B(p)$ and $\nabla f(p)$ are uniquely defined for $p \in S$.

(ii) *(integral estimates)*

$$\begin{aligned}
&\int_0^{+\infty} \|\dot{x}(t)\|^2 dt < +\infty, \int_0^{+\infty} \|\ddot{x}(t)\|^2 dt < +\infty,\\
&\int_0^{+\infty} \|B(x(t) + \beta_b \dot{x}(t)) - B(p)\|^2 dt < +\infty,\\
&\int_0^{+\infty} \|\nabla f(x(t) + \beta_f \dot{x}(t)) - \nabla f(p)\|^2 dt < +\infty,\\
&\int_0^{+\infty} \left\| \frac{d}{dt} B(x(t) + \beta_b \dot{x}(t)) \right\|^2 dt < +\infty,\\
&\int_0^{+\infty} \left\| \frac{d}{dt} \nabla f(x(t) + \beta_f \dot{x}(t)) \right\|^2 dt < +\infty.
\end{aligned}$$

Proof **Lyapunov analysis.** Let us derivate the function $\mathcal{E}_p(\cdot)$ defined in (4.9). The derivation chain rule gives

$$\begin{aligned}
\dot{\mathcal{E}}_p(t) = {} & a\langle \nabla f(x(t) + \beta_f \dot{x}(t)) - \nabla f(p), \dot{x}(t) + \beta_f \ddot{x}(t)\rangle\\
& + \langle x(t) - p + \beta_f \dot{x}(t), \dot{x}(t) + \beta_f \ddot{x}(t)\rangle + d\langle x(t) - p, \dot{x}(t)\rangle.
\end{aligned}$$

According to the constitutive equation (iDINAM) we have

$$\ddot{x}(t) = -\gamma \dot{x}(t) - \nabla f\,(x(t) + \beta_f \dot{x}(t)) - B(x(t) + \beta_b \dot{x}(t)).$$

Therefore,

$$\begin{aligned}
\dot{\mathcal{E}}_p(t) = {} & a\langle \nabla f(x(t) + \beta_f \dot{x}(t)) - \nabla f(p), \dot{x}(t)\rangle + d\langle x(t) - p, \dot{x}(t)\rangle\\
& + a\langle \nabla f(x(t) + \beta_f \dot{x}(t)) - \nabla f(p), \beta_f(-\gamma \dot{x}(t) - \nabla f\,(x(t) + \beta_f \dot{x}(t)) - B(x(t) + \beta_b \dot{x}(t)))\rangle\\
& + \langle x(t) - p + \beta_f \dot{x}(t), \dot{x}(t) + \beta_f(-\gamma \dot{x}(t) - \nabla f\,(x(t) + \beta_f \dot{x}(t)) - B(x(t) + \beta_b \dot{x}(t)))\rangle.
\end{aligned}$$

Let us denote shortly

$$X(t) := \nabla f\Big(x(t) + \beta_f \dot{x}(t)\Big) - \nabla f(p),$$
$$Y(t) := B\Big(x(t) + \beta_b \dot{x}(t)\Big) - B(p).$$

Since $p \in S$, we have $\nabla f(p) + B(p) = 0$. So, we can arrange $\dot{\mathcal{E}}_p(t)$ as follows

$$\begin{aligned}
\dot{\mathcal{E}}_p(t) &= a\langle X(t), \dot{x}(t) + \beta_f(-\gamma\dot{x}(t) - X(t) - Y(t))\rangle \\
&\quad + \langle x(t) - p + \beta_f\dot{x}(t), \dot{x}(t) + \beta_f(-\gamma\dot{x}(t) - X(t) - Y(t))\rangle + d\langle x(t) - p, \dot{x}(t)\rangle \\
&= -a\beta_f\|X(t)\|^2 + a(1-\gamma\beta_f)\langle X(t)), \dot{x}(t)\rangle - a\beta_f\langle X(t), Y(t)\rangle + \beta_f(1-\gamma\beta_f)\|\dot{x}(t)\|^2 \\
&\quad + (d+1-\gamma\beta_f)\langle x(t) - p, \dot{x}(t)\rangle - \beta_f\langle x(t) - p + \beta_f\dot{x}(t), X(t) + Y(t)\rangle. \qquad (4.11)
\end{aligned}$$

We may deduce from the convexity of f, that ∇f is monotone. By definition of $X(t)$ this gives

$$\langle x(t) - p + \beta_f\dot{x}(t), X(t)\rangle \geq 0.$$

Furthermore, since B is λ-cocoercive, that implies

$$\begin{aligned}
\langle x(t) - p + \beta_f\dot{x}(t), Y(t)\rangle &= \langle x(t) - p + \beta_b\dot{x}(t), Y(t)\rangle + (\beta_f - \beta_b)\langle\dot{x}(t), Y(t)\rangle \\
&\geq \lambda\|Y(t)\|^2 + (\beta_f - \beta_b)\langle\dot{x}(t), Y(t)\rangle.
\end{aligned}$$

Combining the aforementioned facts, and assuming $d = \gamma\beta_f - 1 > 0$, we derive from (4.11) that

$$\begin{aligned}
\dot{\mathcal{E}}_p(t) &\leq -a\beta_f\|X(t)\|^2 + a(1-\gamma\beta_f)\langle X(t)), \dot{x}(t)\rangle - a\beta_f\langle X(t), Y(t)\rangle \\
&\quad + \beta_f(1-\gamma\beta_f)\|\dot{x}(t)\|^2 - \lambda\beta_f\|Y(t)\|^2 - \beta_f(\beta_f - \beta_b)\langle\dot{x}(t), Y(t)\rangle. \qquad (4.12)
\end{aligned}$$

Let us use the following elementary inequalities to majorize the scalar products that appear in (4.12) : for any parameters $\rho > 0$ and $r > 0$ that will be adjusted (recall that $\gamma\beta_f > 1$)

$$a(1-\gamma\beta_f)\langle X(t)), \dot{x}(t)\rangle \leq \frac{1}{2}\rho a(\gamma\beta_f - 1)\|X(t)\|^2 + \frac{1}{2\rho}a(\gamma\beta_f - 1)\|\dot{x}(t)\|^2, \qquad (4.13)$$

$$-a\beta_f\langle X(t), Y(t)\rangle \leq \frac{1}{2}ar\beta_f\|X(t)\|^2 + \frac{1}{2r}a\beta_f\|Y(t)\|^2. \qquad (4.14)$$

Combining (4.12) with (4.13) and (4.14), one has

$$\begin{aligned}\dot{\mathcal{E}}_p(t) \leq & -a\beta_f\|X(t)\|^2 + \frac{1}{2}\rho a(\gamma\beta_f - 1)\|X(t)\|^2 + \frac{1}{2\rho}a(\gamma\beta_f - 1)\|\dot{x}(t)\|^2 \\ & + \frac{1}{2}ar\beta_f\|X(t)\|^2 + \frac{1}{2r}a\beta_f\|Y(t)\|^2 \\ & + \beta_f(1 - \gamma\beta_f)\|\dot{x}(t)\|^2 - \lambda\beta_f\|Y(t)\|^2 - \beta_f(\beta_f - \beta_b)\langle\dot{x}(t), Y(t)\rangle. \end{aligned} \tag{4.15}$$

After rearranging the terms, we get

$$\begin{aligned}\dot{\mathcal{E}}_p(t) \leq & -a\left(\beta_f - \frac{1}{2}\rho(\gamma\beta_f - 1) - \frac{1}{2}r\beta_f\right)\|X(t)\|^2 - (\gamma\beta_f - 1)\left(\beta_f - \frac{a}{2\rho}\right)\|\dot{x}(t)\|^2 \\ & - \beta_f\left(\lambda - \frac{a}{2r}\right)\|Y(t)\|^2 - \beta_f(\beta_f - \beta_b)\langle\dot{x}(t), Y(t)\rangle. \end{aligned} \tag{4.16}$$

Equivalently,

$$\dot{\mathcal{E}}_p(t) + a\left(\beta_f - \frac{1}{2}\rho(\gamma\beta_f - 1) - \frac{1}{2}r\beta_f\right)\|X(t)\|^2 + \beta_f\mathcal{S}(t) \leq 0, \tag{4.17}$$

where

$$\mathcal{S}(t) := \left(\lambda - \frac{a}{2r}\right)\|Y(t)\|^2 + (\beta_f - \beta_b)\langle\dot{x}(t), Y(t)\rangle + (\gamma\beta_f - 1)\left(1 - \frac{a}{2\rho\beta_f}\right)\|\dot{x}(t)\|^2.$$

We have $\mathcal{S}(t) = q(Y(t), \dot{x}(t))$ where $q : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is the quadratic form

$$q(Y, Z) := \left(\lambda - \frac{a}{2r}\right)\|Y\|^2 + (\beta_f - \beta_b)\langle Y, Z\rangle + (\gamma\beta_f - 1)\left(1 - \frac{a}{2\rho\beta_f}\right)\|Z\|^2.$$

The system of constraints on the positive parameters a, r, ρ shown below guarantees not only is the coefficient of $\|X(t)\|^2$ in (4.17) positive but also the quadratic form q is positive definite :

$$\beta_f - \frac{1}{2}\rho(\gamma\beta_f - 1) - \frac{1}{2}r\beta_f > 0; \tag{4.18}$$

$$\lambda - \frac{a}{2r} > 0; \tag{4.19}$$

$$1 - \frac{a}{2\rho\beta_f} > 0; \tag{4.20}$$

$$4\left(\lambda - \frac{a}{2r}\right)(\gamma\beta_f - 1)\left(1 - \frac{a}{2\rho\beta_f}\right) - (\beta_f - \beta_b)^2 > 0. \tag{4.21}$$

Constraints (4.19) and (4.20) are equivalent to $r > \frac{a}{2\lambda}$ and $\rho > \frac{a}{2\beta_f}$. So they are satisfied by taking $r = \dfrac{\tau a}{2\lambda}$ and $\rho = \dfrac{\tau a}{2\beta_f}$ with $\tau > 1$. Reinjecting these values in (4.18) we has the condition as follows

$$\tau a \leq \frac{4\lambda\beta_f^2}{\lambda(\gamma\beta_f - 1) + \beta_f^2}. \tag{4.22}$$

Let us now examine the last constraint (4.21). Because of the choice of r and ρ, it is simplified as

$$\Delta(\tau) := 4\lambda\left(1 - \frac{1}{\tau}\right)^2(\gamma\beta_f - 1) - (\beta_f - \beta_b)^2 > 0.$$

We have

$$\lim_{\tau + \infty}\Delta(\tau) = 4\lambda(\gamma\beta_f - 1) - (\beta_f - \beta_b)^2$$

and it is positive thanks to our assumption (4.10) on the parameters. Hence, by taking τ large enough, and adjusting a small enough according to (4.22), we obtain that the coefficient of $\|X(t)\|^2$ in (4.17) is positive, and the quadratic form q is positive definite as well. We infer there exist positive real numbers η and μ such that

$$\dot{\mathcal{E}}_p(t) + \eta\|X(t)\|^2 + \mu\beta_f\|\dot{x}(t)\|^2 + \mu\beta_f\|Y(t)\|^2 \leq 0. \tag{4.23}$$

Estimates. We rely on the estimate (4.23) and integrate it on $[0,t], t \geq 0$. Then, one has

$$\mathcal{E}_p(t) + \eta\int_0^t \|X(s)\|^2 ds + \mu\beta_f\int_0^t \|\dot{x}(s)\|^2 ds + \mu\beta_f\int_0^t \|Y(s)\|^2 ds \leq \mathcal{E}_p(0). \tag{4.24}$$

From this we immediately obtain that $\mathcal{E}_p(t) \leq \mathcal{E}_p(0)$, *i.e.* $\mathcal{E}_p(t)$ is bounded from above. According to the definition of $\mathcal{E}_p(\cdot)$ we deduce that

$$\sup_{t\geq 0}\|x(t) - p\| < +\infty, \tag{4.25}$$

$$\sup_{t\geq 0}\|x(t) - p + \beta_f\dot{x}(t)\| < +\infty. \tag{4.26}$$

From (4.25)-(4.26) and $\beta_f > 0$, according to the triangle inequality we claim that

$$\sup_{t\geq 0}\|\dot{x}(t)\| < +\infty. \tag{4.27}$$

Moreover, we immediately deduce from (2.29) and nonnegative property of $\mathcal{E}_p(t)$ the following integral estimates

$$\int_0^{+\infty} \|X(t)\|^2 dt < +\infty, \quad \int_0^{+\infty} \|\dot{x}(t)\|^2 dt < +\infty, \quad \int_0^{+\infty} \|Y(t)\|^2 dt < +\infty. \tag{4.28}$$

Let us rewrite (iDINAM) equivalently as follows (recall that $\nabla f(p) + Bp = 0$)

$$\ddot{x}(t) = -\gamma \dot{x}(t) - X(t) - Y(t).$$

According to (4.28) the second member of the above equality belongs to $L^2(0, +\infty; \mathcal{H})$. Therefore

$$\int_0^{+\infty} \|\ddot{x}(t)\|^2 \, dt < +\infty. \tag{4.29}$$

From (4.28) and (4.29) we have $\dot{x} \in L^2([0, +\infty[; \mathcal{H})$ and $\ddot{x} \in L^2([0, +\infty[; \mathcal{H})$. By Lemma 1.3.3 applied to $u = \dot{x}$ with $p = r = 2$ we deduce that

$$\lim_{t \to +\infty} \dot{x}(t) = 0. \tag{4.30}$$

Furthermore, since B is λ-cocoercive, it is $\frac{1}{\lambda}$-Lipschitz continuous. Therefore,

$$\left\| \frac{d}{dt} B(x(t) + \beta_b \dot{x}(t)) \right\| \leq \frac{1}{\lambda} \|\dot{x}(t) + \beta_b \ddot{x}(t)\|, \text{ for all } t \geq 0. \tag{4.31}$$

Hence,

$$\begin{aligned}
\int_0^{+\infty} \left\| \frac{d}{dt} B(x(t) + \beta_b \dot{x}(t)) \right\|^2 dt &\leq \frac{1}{\lambda^2} \int_0^{+\infty} \|\dot{x}(t) + \beta_b \ddot{x}(t)\|^2 dt \\
&\leq \frac{2}{\lambda^2} \int_0^{+\infty} \|\dot{x}(t)\|^2 dt + \frac{2\beta_b^2}{\lambda^2} \int_0^{+\infty} \|\ddot{x}(t)\|^2 dt < +\infty.
\end{aligned}$$

Similarly, we have

$$\int_0^{+\infty} \left\| \frac{d}{dt} \nabla f(x(t) + \beta_f \dot{x}(t)) \right\|^2 dt < +\infty$$

where have used that $x(t) + \beta_f \dot{x}(t)$ remains bounded (according to (4.25) and (4.27)) and that ∇f is Lipschitz continuous on the bounded sets. So, according to the definition of $X(t)$ and $Y(t)$ we have

$$\int_0^{+\infty} \left\| \frac{d}{dt} X(t) \right\|^2 dt < +\infty, \quad \int_0^{+\infty} \left\| \frac{d}{dt} Y(t) \right\|^2 dt < +\infty. \tag{4.32}$$

From (4.28)-(4.32), by applying Lemma 1.3.3 we deduce that $\lim_{t\to+\infty} X(t) = \lim_{t\to+\infty} Y(t) = 0$, that is

$$\lim_{t\to+\infty} \|B((x(t)+\beta_b\dot{x}(t)) - B(p)\| = 0, \lim_{t\to+\infty} \|\nabla f(x(t)+\beta_f\dot{x}(t)) - \nabla f(p)\| = 0 \quad (4.33)$$

According to the Lipschitz continuity of B, and the Lipschitz continuity of ∇f on the bounded sets (recall that $x(t)$ and $\dot{x}(t)$ are bounded) we immediately deduce from (4.33) and $\lim_{t\to+\infty} \dot{x}(t) = 0$, that

$$\lim_{t\to+\infty} \|B(x(t)) - B(p)\| = 0, \lim_{t\to+\infty} \|\nabla f(x(t)) - \nabla f(p)\| = 0. \quad (4.34)$$

Convergence of the trajectory. In oder to show the existence of the weak limit of $x(t)$ as $t \to +\infty$, we use Opial's lemma (see [68] for more details). Given $p \in S$, let us define the anchor function given by, for every $t \in [0,+\infty[$

$$q_p(t) := \frac{1}{2}\|x(t) - p\|^2.$$

From $\dot{q}_p(t) = \langle \dot{x}(t), x(t) - p\rangle$ and $\ddot{q}_p(t) = \|\dot{x}(t)\|^2 + \langle \ddot{x}(t), x(t) - p\rangle$, we obtain

$$\begin{aligned}\ddot{q}_p(t) + \gamma\dot{q}_p(t) &= \|\dot{x}(t)\|^2 + \langle \ddot{x}(t) + \gamma\dot{x}(t), x(t) - p\rangle \\ &= \|\dot{x}(t)\|^2 - \langle \nabla f\,(x(t) + \beta_f\dot{x}(t)) + B(x(t) + \beta_b\dot{x}(t)), x(t) - p\rangle.\end{aligned}$$

According to the monotonicity of ∇f and B, we have

$$\begin{aligned}&\langle \nabla f\,(x(t) + \beta_f\dot{x}(t)) + B(x(t) + \beta_b\dot{x}(t)), x(t) - p\rangle \\ &= \langle X(t) + Y(t), x(t) - p\rangle \\ &\geq -\beta_f\langle X(t), \dot{x}(t)\rangle - \beta_b\langle Y(t), \dot{x}(t)\rangle.\end{aligned}$$

Therefore,

$$\ddot{q}_p(t) + \gamma\dot{q}_p(t) \leq \|\dot{x}(t)\|^2 + \beta_f\langle X(t), \dot{x}(t)\rangle + \beta_b\langle Y(t), \dot{x}(t)\rangle. \quad (4.35)$$

Applying the Cauchy-Schwarz inequality, we get

$$\ddot{q}_p(t) + \gamma\dot{q}_p(t) \leq \|\dot{x}(t)\|^2 + \beta_f\|X(t)\|\|\dot{x}(t)\| + \beta_b\|Y(t)\|\|\dot{x}(t)\|. \quad (4.36)$$

Then note that the second member of (4.36)

$$g(t) := \|\dot{x}(t)\|^2 + \beta_f\|X(t)\|\|\dot{x}(t)\| + \beta_b\|Y(t)\|\|\dot{x}(t)\|$$

is nonnegative and belongs to $L^1([0,+\infty[,\mathbb{R})$. Indeed, we have

$$\int_0^{+\infty}\|X(t)\|\|\dot{x}(t)\|dt \leq \frac{1}{2}\int_0^{+\infty}\|X(t)\|^2dt + \frac{1}{2}\int_0^{+\infty}\|\dot{x}(t)\|^2dt,$$
$$\int_0^{+\infty}\|Y(t)\|\|\dot{x}(t)\|dt \leq \frac{1}{2}\int_0^{+\infty}\|Y(t)\|^2dt + \frac{1}{2}\int_0^{+\infty}\|\dot{x}(t)\|^2dt.$$

Using (4.28), we deduce that

$$\int_0^{+\infty} g(t)dt < +\infty.$$

Since q_p is nonnegative, Lemma 1.3.4 shows that $\lim_{t\to+\infty} q_p(t)$ exists. To complete the proof using Opial's lemma, we need to show that every weak sequential cluster point of $x(t)$ belongs to S. Let $t_n \to +\infty$ such that $x(t_n) \rightharpoonup x^*, n \to +\infty$. According to (4.34)

$$\nabla f(x(t_n)) \to \nabla f(p);\ B(x(t_n)) \to B(p) \text{ strongly in } \mathcal{H}$$

and

$$x(t_n) \rightharpoonup x^* \text{ weakly in } \mathcal{H}.$$

We may deduce that $\nabla f(x^*) = \nabla f(p)$, and $B(x^*) = B(p)$ rrom the closedness property of the graph of the maximally monotone operators ∇f and B in $w-\mathcal{H}\times s-\mathcal{H}$. As a result, $\nabla f(x^*) + B(x^*) = \nabla f(p) + B(p) = 0$, that is $x^* \in S$. Consequently, $x(t)$ converges weakly towards an element of S as t goes to $+\infty$. The proof of Theorem 4.3.1 is thus completed. Let us specialize the preceding results in the case $\beta_b = \beta_f$. We set $\beta_b = \beta_f := \beta > 0$ and $A := \nabla f + B$. Thus, we consider the evolution system

$$\text{(iDINAM)} \quad \ddot{x}(t) + \gamma\dot{x}(t) + A(x(t) + \beta\dot{x}(t)) = 0, \quad t \geq 0.$$

The well-posedness of the strong global solution to this system is guaranteed by Theorem 4.2.1 while its convergence properties are a consequence of Theorem 4.3.1 and are given below.

Corollary 4.3.1 *Let $B : \mathcal{H} \to \mathcal{H}$ be a λ-cocoercive operator and $f : \mathcal{H} \to \mathbb{R}$ be a $\mathcal{C}^1$ convex function whose gradient is Lipschitz continuous on the bounded sets. Suppose that the solution set $S = (\nabla f + B)^{-1}(0) \neq \emptyset$. Consider the evolution equation* (iDINAM)*, where $A = \nabla f + B$, $\beta_b = \beta_f := \beta > 0$ and where the involved parameters satisfy the following condition $\gamma\beta > 1$. Then, for any solution trajectory $x : [0,+\infty[\to \mathcal{H}$ of* (iDINAM)*, the following properties are satisfied :*

(i) *(convergence)* *The trajectory* $x(t)$ *converges weakly, as* $t \to +\infty$, *to an element* $x^* \in S$. *Moreover* $\lim_{t\to+\infty} \|\dot{x}(t)\| = 0$, *and* $\lim_{t\to+\infty} \|A(x(t) + \beta\dot{x}(t))\| = 0$.

(ii) *(integral estimate)*

$$\int_0^{+\infty} \|\dot{x}(t)\|^2 dt < +\infty, \int_0^{+\infty} \|\ddot{x}(t)\|^2 dt < +\infty,$$
$$\int_0^{+\infty} \|A(x(t) + \beta\dot{x}(t))\|^2 dt < +\infty, \text{ and } \int_0^{+\infty} \left\| \frac{d}{dt} A(x(t) + \beta\dot{x}(t)) \right\|^2 dt < +\infty.$$

4.3.2 Comparison of the dynamics with explicit and implicit Newton-type damping

For the sake of simplicity, let us compare the dynamics in the case $\beta_f = \beta_b = \beta > 0$. According to the authors' prior research in [4] concerning the dynamic (DINAM) with explicit Newton-type damping, the condition on the parameters ensuring the trajectory convergence is

$$\lambda\gamma > \beta + \frac{1}{\gamma} \tag{4.37}$$

On the other hand, the corresponding condition for (iDINAM), as given by Corollary 4.3.1 is

$$\gamma\beta > 1. \tag{4.38}$$

As a result, in contrast to (DINAM), the cocoercivity parameter λ no longer enters the condition relative to (iDINAM). This implies that it would be particularly interesting to study the case of an asymptotic vanishing damping coefficient $\gamma(t) = \frac{\alpha}{t}$ in accordance with the Nesterov accelerated scheme. By modifying the coefficient $\beta(t)$, which now tends to infinity, it is feasible to achieve fast convergence results for general monotone inclusions. In fact, first results in this approach have been obtained for the ADMM algorithm, see [17].

4.4 Inertial proximal algorithms associated with (iDINAM)

We focus on the convergence properties of several splitting algorithms with inertial features acquired by temporal discretization of the second-order (in time) evolution equation :

$$\ddot{x}(t) + \gamma\dot{x}(t) + \nabla f\left(x(t) + \beta_f\dot{x}(t)\right) + B\left(x(t) + \beta_b\dot{x}(t)\right) = 0. \tag{iDINAM}$$

Under appropriate parameter and discretization step adjustments, our aim is to achieve continuous convergence results of the same kind as those obtained in the preceding section.

4.4.1 An inertial proximal-gradient algorithm

In this section, f is a $\mathcal{C}^1$ convex function with an L-Lipschitz continuous gradient. Take a fixed time step $h > 0$, and consider the following finite-difference scheme for (iDINAM) :

$$\frac{1}{h^2}(x_{k+1} - 2x_k + x_{k-1}) + \frac{\gamma}{h}(x_{k+1} - x_k) + \nabla f\left(x_k + \frac{\beta_f}{h}(x_k - x_{k-1})\right) + B\left(x_{k+1} + \frac{\beta_b}{h}(x_{k+1} - x_k)\right) = 0. \tag{4.39}$$

This scheme is implicit in terms of the nonpotential B but explicit in terms of the potential operator ∇f. When $B = 0$, we may anticipate the algorithm's gradient-like structure to yield convergence results if the step size h is small enough.
After expanding (4.39), we obtain

$$(1 + \gamma h)(x_{k+1} - x_k) + h^2 B\left(x_{k+1} + \frac{\beta_b}{h}(x_{k+1} - x_k)\right) = (x_k - x_{k-1}) - h^2 \nabla f\left(x_k + \frac{\beta_f}{h}(x_k - x_{k-1})\right). \tag{4.40}$$

Set $\alpha := 1 + \frac{\beta_b}{h}$. After arranging (4.40), we obtain equivalently

$$x_{k+1} = \frac{\alpha - 1}{\alpha} x_k + \frac{1}{\alpha}(\mathrm{Id} + \frac{\alpha h^2}{1 + \gamma h} B)^{-1}(\xi_k),$$

with

$$\xi_k = x_k + \frac{\alpha}{1 + \gamma h}\left((x_k - x_{k-1}) - h^2 \nabla f\left(x_k + \frac{\beta_f}{h}(x_k - x_{k-1})\right)\right).$$

We thus obtain the following algorithm.

(iDINAAM-split) :

Initialize : $x_0 \in \mathcal{H}, x_1 \in \mathcal{H}$

$$\alpha = 1 + \frac{\beta_b}{h},$$

$$\xi_k = x_k + \frac{\alpha}{1+\gamma h}(x_k - x_{k-1}) - \frac{\alpha h^2}{1+\gamma h}\nabla f\left(x_k + \frac{\beta_f}{h}(x_k - x_{k-1})\right),$$

$$x_{k+1} = \frac{\alpha - 1}{\alpha}x_k + \frac{1}{\alpha}\left(\mathrm{Id} + \frac{\alpha h^2}{1+\gamma h}B\right)^{-1}(\xi_k).$$

Here, (iDINAAM) stands for Implicit Dynamic Inertial Newton Algorithm for Additively structured Monotone problems.

Theorem 4.4.1 *Let $B : \mathcal{H} \to \mathcal{H}$ be a λ-cocoercive operator and $f : \mathcal{H} \to \mathbb{R}$ a differentiable convex function whose gradient is L-Lipschitz continuous. Suppose the positive parameters $\lambda, \gamma, \beta_b, \beta_f$ satisfy*

$$0 < h < \frac{2}{L\beta_f}, \ \gamma\beta_f > 1 \ \text{and} \ \lambda > \frac{(\beta_b - \beta_f)^2}{4(\gamma\beta_f - 1)}. \tag{4.41}$$

Then, the sequence (x_k) generated by the algorithm (iDINAAM-split) *has the following properties :*

(i) *(x_k) converges weakly to an element in S ;*

(ii) $\lim_{k\to\infty} \|\nabla f(x_k) - \nabla f(p)\| = 0, \quad \lim_{k\to\infty} \|B(x_k) - B(p)\| = 0$;

(iii) $\sum_{k=1}^{\infty} \|x_k - x_{k-1}\|^2 < +\infty, \ \sum_{k=1}^{\infty} \|\nabla f(x_k) - \nabla f(p)\|^2 < +\infty, \ \sum_{k=1}^{\infty} \|B(x_k) - B(p)\|^2 < +\infty$

where $\nabla f(p)$, $B(p)$ are independent of the choice of $p \in S$.

Proof **The discrete energy.** Take $p \in S$. Let us consider the sequence (E_k) defined for all $k \geq 1$ by

$$E_k := \frac{1}{2}\|(x_k - p) + \frac{\beta_f}{h}(x_k - x_{k-1})\|^2 + \frac{\delta}{2}\|x_k - p\|^2,$$

where δ is an adjustable positive coefficient.

For each $k \geq 1$, E_k can be rewritten as follows :

$$E_k = \frac{1}{2}\|v_k\|^2 + \frac{\delta}{2}\|x_k - p\|^2,$$

with

$$v_k := x_k - p + \frac{\beta_f}{h}(x_k - x_{k-1}).$$

By definition of v_k and (4.39), we have

$$\begin{aligned} & v_{k+1} - v_k = x_{k+1} - x_k + \frac{\beta_f}{h}(x_{k+1} - 2x_k + x_{k-1}) \\ & = (1-\gamma\beta_f)(x_{k+1} - x_k) - h\beta_f \nabla f\left(x_k + \frac{\beta_f}{h}(x_k - x_{k-1})\right) - h\beta_f B\left(x_{k+1} + \frac{\beta_b}{h}(x_{k+1} - x_k)\right) \\ & = (1-\gamma\beta_f)(x_{k+1} - x_k) - h\beta_f \nabla f(y_k) - h\beta_f B(z_k), \end{aligned}$$

where we write shortly

$$\begin{aligned} y_k &:= x_k + \frac{\beta_f}{h}(x_k - x_{k-1}), \\ z_k &:= x_{k+1} + \frac{\beta_b}{h}(x_{k+1} - x_k). \end{aligned}$$

Therefore, for $k \geq 1$, we have

$$\begin{aligned} & \frac{1}{2}\|v_{k+1}\|^2 - \frac{1}{2}\|v_k\|^2 = -\frac{1}{2}\|v_{k+1} - v_k\|^2 + \langle v_{k+1} - v_k, v_{k+1}\rangle \\ & = -\frac{1}{2}(\gamma\beta_f - 1)^2\|x_{k+1} - x_k\|^2 - \frac{1}{2}h^2\beta_f^2\|\nabla f(y_k) + B(z_k)\|^2 \\ & -h\beta_f(\gamma\beta_f - 1)\langle x_{k+1} - x_k, \nabla f(y_k) + B(z_k)\rangle \\ & -\langle x_{k+1} - p + \frac{\beta_f}{h}(x_{k+1} - x_k), (\gamma\beta_f - 1)(x_{k+1} - x_k) + h\beta_f\nabla f(y_k) + h\beta_f B(z_k)\rangle. \end{aligned} \tag{4.42}$$

Then using the elementary identity

$$\frac{1}{2}\|x_{k+1} - p\|^2 - \frac{1}{2}\|x_k - p\|^2 = -\frac{1}{2}\|x_{k+1} - x_k\|^2 + \langle x_{k+1} - x_k, x_{k+1} - p\rangle. \tag{4.43}$$

Take $\delta = \gamma\beta_f - 1$. As the result of the first condition on the parameters, it requires that

$$\gamma\beta_f > 1. \tag{4.44}$$

From (4.42) and (4.43), we deduce that

$$\begin{aligned} E_{k+1} - E_k = & -\left(\frac{1}{2}\delta^2 + \frac{\delta\beta_f}{h} + \frac{1}{2}\delta\right)\|x_{k+1} - x_k\|^2 - \frac{1}{2}h^2\beta_f^2\|\nabla f(y_k) + B(z_k)\|^2 \\ & - h\beta_f\delta\langle x_{k+1} - x_k, \nabla f(y_k) + B(z_k)\rangle \\ & - \langle x_{k+1} - p + \frac{\beta_f}{h}(x_{k+1} - x_k), h\beta_f\nabla f(y_k) + h\beta_f B(z_k)\rangle. \end{aligned}$$

According to $\nabla f(p) + B(p) = 0$, the previous relation can be rewritten as follows

$$\begin{aligned} E_{k+1} - E_k = &-\left(\frac{1}{2}\delta^2 + \frac{\delta\beta_f}{h} + \frac{1}{2}\delta\right)\|x_{k+1} - x_k\|^2 - \frac{1}{2}h^2\beta_f^2\|Y_k + Z_k\|^2 \\ &- h\beta_f\delta\langle x_{k+1} - x_k, Y_k + Z_k\rangle - h\beta_f\langle x_{k+1} - p + \frac{\beta_f}{h}(x_{k+1} - x_k), Y_k + Z_k\rangle, \end{aligned} \tag{4.45}$$

where $Y_k = \nabla f(y_k) - \nabla f(p)$ and $Z_k = B(z_k) - B(p)$.
B is λ-cocoercive and we thus obtain

$$\begin{aligned} \langle x_{k+1} - p + \frac{\beta_f}{h}(x_{k+1} - x_k), Z_k\rangle &= \langle z_k - p + \frac{1}{h}(\beta_f - \beta_b)(x_{k+1} - x_k), B(z_k) - B(p)\rangle \\ &\geq \lambda\|B(z_k) - B(p)\|^2 + \frac{1}{h}(\beta_f - \beta_b)\langle x_{k+1} - x_k, B(z_k) - B(p)\rangle \\ &= \lambda\|Z_k\|^2 + \frac{1}{h}(\beta_f - \beta_b)\langle x_{k+1} - x_k, Z_k\rangle. \end{aligned}$$

In the same way, since ∇f is $1/L$-cocoercive, using the constitutive equation (4.39), we have

$$\begin{aligned} &\langle x_{k+1} - p + \frac{\beta_f}{h}(x_{k+1} - x_k), Y_k\rangle \\ &= \langle y_k - p + x_{k+1} - x_k + \frac{\beta_f}{h}(x_{k+1} - 2x_k + x_{k-1}), \nabla f(y_k) - \nabla f(p)\rangle \\ &\geq \frac{1}{L}\|Y_k\|^2 + \langle x_{k+1} - x_k + \frac{\beta_f}{h}(x_{k+1} - 2x_k + x_{k-1}), \nabla f(y_k) - \nabla f(p)\rangle, \\ &= \frac{1}{L}\|Y_k\|^2 + \langle x_{k+1} - x_k - \gamma\beta_f(x_{k+1} - x_k) - h\beta_f\nabla f(y_k) - h\beta_f B(z_k), \nabla f(y_k) - \nabla f(p)\rangle \\ &= \frac{1}{L}\|Y_k\|^2 - \langle\delta(x_{k+1} - x_k) + h\beta_f Y_k + h\beta_f Z_k, Y_k\rangle. \end{aligned}$$

Combining the aforementioned relations with (4.45), then we obtain

$$\begin{aligned} E_{k+1} - E_k \leq &\left(\frac{1}{2}h^2\beta_f^2 - \frac{h\beta_f}{L}\right)\|Y_k\|^2 - \left(\frac{1}{2}\delta^2 + \frac{\delta\beta_f}{h} + \frac{1}{2}\delta\right)\|x_{k+1} - x_k\|^2 \\ &- (h\beta_f\delta + \beta_f(\beta_f - \beta_b))\langle x_{k+1} - x_k, Z_k\rangle - \left(\frac{1}{2}h^2\beta_f^2 + h\beta_f\lambda\right)\|Z_k\|^2. \end{aligned} \tag{4.46}$$

Equivalently,

$$E_{k+1} - E_k + S_k \leq \left(\frac{1}{2}h^2\beta_f^2 - \frac{h\beta_f}{L}\right)\|Y_k\|^2, \tag{4.47}$$

where

$$\begin{aligned} S_k &= \left(\frac{1}{2}\delta^2 + \frac{\delta\beta_f}{h} + \frac{1}{2}\delta\right)\|x_{k+1} - x_k\|^2 + (h\beta_f\delta + \beta_f(\beta_f - \beta_b))\langle x_{k+1} - x_k, Z_k\rangle \\ &+ \left(\frac{1}{2}h^2\beta_f^2 + h\beta_f\lambda\right)\|Z_k\|^2. \end{aligned}$$

Acorrding to second conditions on the parameters, we ask $\frac{1}{2}h^2\beta_f^2 - \frac{h\beta_f}{L} < 0$, hence that is

$$0 < h < \frac{2}{L\beta_f}. \tag{4.48}$$

Then note that $S_k = q(x_{k+1} - x_k, Z_k) > 0$ if $4ag - b^2 > 0$ where $q : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is the quadratic form

$$q(u, v) := a\|u\|^2 + b\langle u, v\rangle + g\|v\|^2,$$

where

$$\begin{aligned} a &= \frac{1}{2}\delta^2 + \frac{\delta\beta_f}{h} + \frac{1}{2}\delta, \\ b &= h\beta_f\delta + \beta_f(\beta_f - \beta_b), \\ g &= \frac{1}{2}h^2\beta_f^2 + h\beta_f\lambda. \end{aligned}$$

The third and last condition on the parameters will be fulfilled provided that the quadratic form q is positive definite. Since both a and g are positive, the positivity of q equates to $4ag - b^2 > 0$. In addition, we have

$$\begin{aligned} 4ag - b^2 &= 4\left(\frac{1}{2}\delta^2 + \frac{\delta\beta_f}{h} + \frac{1}{2}\delta\right)\left(\frac{1}{2}h^2\beta_f^2 + h\beta_f\lambda\right) - (h\beta_f\delta + \beta_f(\beta_f - \beta_b))^2 \\ &= \beta_f^2\Big(4\lambda\delta - (\beta_f - \beta_b)^2\Big) + 2h\delta\beta_f\Big(\lambda(\delta + 1) + \beta_f\beta_b\Big) + h^2\beta_f^2\delta \\ &\geq \beta_f^2\Big(4\lambda\delta - (\beta_f - \beta_b)^2\Big) > 0, \end{aligned} \tag{4.49}$$

where the last inequality is a consequence of our assumptions. Hence, q is positive definite. As a result, we claim that there exist positive real numbers μ and η such that for any $k \geq 1$,

$$E_{k+1} - E_k + \mu\|x_{k+1} - x_k\|^2 + \mu\|B(z_k) - B(p)\|^2 + \eta\|\nabla f(y_k) - \nabla f(p)\|^2 \leq 0. \tag{4.50}$$

It should be noted that μ is dependent on all of the damping coefficients in the algorithm as well as the step size h. Its accurate estimate is a fascinating topic for numerical purposes.

Estimates. According to (4.50), the sequence of nonnegative numbers (E_k) is nonincreasing, thus being convergent. In particular, it is bounded. For this reason, we immediately deduce that

$$\sup_k \|(x_k - p) + \frac{\beta_f}{h}(x_k - x_{k-1})\| < +\infty,$$
$$\sup_k \|x_k - p\| < +\infty.$$

Moreover, by summing the inequalities (4.50), we deduce that

$$\sum_{k=1}^{\infty} \|x_k - x_{k-1}\|^2 < \infty, \quad \sum_{k=1}^{\infty} \|\nabla f(y_k) - \nabla f(p)\|^2 < \infty, \quad \sum_{k=1}^{\infty} \|B(z_k) - B(p)\|^2 < \infty. \tag{4.51}$$

Elementary algebra and the Lipschitz continuity of ∇f give, for each $k \geq 1$

$$\begin{aligned}
\|\nabla f(x_k) - \nabla f(p)\|^2 &\leq (\|\nabla f(y_k) - \nabla f(p)\| + \|\nabla f(x_k) - \nabla f(y_k)\|)^2 \\
&\leq 2\|\nabla f(y_k) - \nabla f(p)\|^2 + 2\|\nabla f(x_k) - \nabla f(y_k)\|^2 \\
&\leq 2\|\nabla f(y_k) - \nabla f(p)\|^2 + 2L^2\|x_k - y_k\|^2 \\
&\leq 2\|\nabla f(y_k) - \nabla f(p)\|^2 + \frac{2L^2\beta_f^2}{h^2}\|x_k - x_{k-1}\|^2.
\end{aligned} \tag{4.52}$$

According to (4.51), one has

$$\sum_{k=1}^{\infty} \|\nabla f(x_k) - \nabla f(p)\|^2 < +\infty.$$

Likewise, since B is $1/\lambda$-Lipschitz, we consequently obtain

$$\sum_{k=1}^{\infty} \|B(x_k) - B(p)\|^2 < +\infty.$$

The general term of a convergent series goes to zero, we thus deduce (ii).

Convergence of (x_k). Firstly, we show that every weak cluster point x^* of the sequence (x_k) belongs to S. Consider a subsequence (x_{k_n}) of (x_k) satisfying $x_{k_n} \rightharpoonup x^*$, as $n \to +\infty$. According to the item (ii) already proved we have

$$\nabla f\left(x_{k_n}\right) \to \nabla f(p), \quad B\left(x_{k_n}\right) \to B(p) \text{ strongly in } \mathcal{H},$$

and

$$x_{k_n} \rightharpoonup x^* \text{ weakly in } \mathcal{H}.$$

The closedness property of the graph of the maximally monotone operators ∇f and B in $w - \mathcal{H} \times s - \mathcal{H}$ leads us to the conclusion that $\nabla f(x^*) = \nabla f(p)$, and $B(x^*) = B(p)$. Therefore, $\nabla f(x^*) + B(x^*) = \nabla f(p) + B(p) = 0$, that means $x^* \in S$.

Thanks to the estimate (iii), we have $\sum_{k=1}^{\infty} \|x_k - x_{k-1}\|^2 < +\infty$. The general term of a convergent series always goes to zero that implies $\lim_k \|x_k - x_{k-1}\| = 0$. Furthermore, according to the definition of E_k, and since $\lim_k E_k$ exists (indeed it is nonincreasing), we claim that, for any $p \in S$

$$\lim_{k\to\infty} \|x_k - p\| \quad \text{exists.}$$

As a result, the two requirements of the Opial's lemma are fulfilled, which shows the convergence of (x_k).

4.4.2 Errors, perturbations

We will now examine the impact of introducing perturbations, or errors, into the algorithm (iDINAAM-split). Let us commence with the perturbed version of (iDINAM) shown below :

$$\ddot{x}(t) + \gamma\dot{x}(t) + \nabla f\left(x(t) + \beta_f \dot{x}(t)\right) + B\left(x(t) + \beta_b \dot{x}(t)\right) = e(t), \tag{iDINAM}$$

where the right-handside $e(\cdot)$ takes into account perturbations, or errors. Similarly, taking the temporal discretization as before gives

$$\frac{1}{h^2}(x_{k+1} - 2x_k + x_{k-1}) + \frac{\gamma}{h}(x_k - x_{k-1}) + \nabla f\left(x_k + \frac{\beta_f}{h}(x_k - x_{k-1})\right) \\ + B\left(x_{k+1} + \frac{\beta_b}{h}(x_{k+1} - x_k)\right) = e_k. \tag{4.53}$$

Hence, we obtain the following algorithm.

(iDINAAM-pert) :

Initialize : $x_0 \in \mathcal{H}, x_1 \in \mathcal{H}$

$\alpha = 1 + \frac{\beta_b}{h}$,

$\xi_k = x_k + \frac{\alpha}{1+\gamma h}(x_k - x_{k-1}) - \frac{\alpha h^2}{1+\gamma h}\nabla f\left(x_k + \frac{\beta_f}{h}(x_k - x_{k-1})\right) + \frac{\alpha h^2}{1+\gamma h}e_k$,

$x_{k+1} = \frac{\alpha-1}{\alpha}x_k + \frac{1}{\alpha}\left(\mathrm{Id} + \frac{\alpha h^2}{1+\gamma h}B\right)^{-1}(\xi_k)$.

Theorem 4.4.2 *Let us make the assumptions of Theorem 4.4.1, and suppose that the sequence* (e_k) *of perturbations, errors satisfies :*

$$\sum_{k=1}^{\infty}\|e_k\| < +\infty.$$

Then, the sequence (x_k) *generated by the algorithm* (iDINAAM-pert) *has the following properties (where* $p \in S$*) :*

(i) (x_k) *converges weakly to an element in* S *;*

(ii) $$\sum_{k=1}^{\infty}\|x_k - x_{k-1}\|^2 < +\infty, \quad \sum_{k=1}^{\infty}\left\|\nabla f\left(x_k + \frac{\beta_f}{h}(x_k - x_{k-1})\right) - \nabla f(p)\right\|^2 < +\infty,$$

$$\sum_{k=1}^{\infty}\left\|B\left(x_k + \frac{\beta_f}{h}(x_k - x_{k-1})\right) - B(p)\right\|^2 < +\infty, \sum_{k=1}^{\infty}\|\nabla f(x_k) - \nabla(p)\|^2 < +\infty,$$

$$\sum_{k=1}^{\infty}\left\|B\left(x_k + \frac{\beta_f}{h}(x_k - x_{k-1})\right) - B(p)\right\|^2 < +\infty, \sum_{k=1}^{\infty}\|B(x_k) - B(p)\|^2 < +\infty,$$

$$\sum_{k=1}^{\infty}\|\nabla f(x_k) - \nabla f(x_{k-1})\|^2 < +\infty, \ \textit{and} \ \sum_{k=1}^{\infty}\|B(x_k) - B(x_{k-1})\|^2 < +\infty;$$

(iii) $\lim_{k\to\infty}\|x_{k+1} - x_k\| = 0, \ \lim_{k\to\infty}\|B(x_k) - B(p)\| = 0, \ \lim_{k\to\infty}\|\nabla f(x_k) - \nabla f(p)\| = 0.$

Proof The proof's outline is analogus to that of Theorem 4.4.1. It uses the following sequence (E_k) as a discrete energy function

$$E_k := \frac{1}{2}\|x_k - p + \frac{\beta_f}{h}(x_k - x_{k-1})\|^2 + \frac{\delta}{2}\|x_k - p\|^2,$$

where δ are positive coefficient to adjust.
By setting

$$\begin{aligned}\delta &= \gamma\beta_f - 1,\\ Y_k &= \nabla f\left(x_k + \frac{\beta_f}{h}(x_k - x_{k-1})\right) - \nabla f(p),\\ Z_k &= B\left(x_{k+1} + \frac{\beta_b}{h}(x_{k+1} - x_k)\right) - B(p),\end{aligned}$$

for $k \geq 1$ and using the same argument as in the proof of Theorem 4.4.1, we have

$$E_{k+1} - E_k + S_k + \left(\frac{h\beta_f}{L} - \frac{1}{2}h^2\beta_f^2\right)\|Y_k\|^2 \leq \varepsilon_k, \tag{4.54}$$

where

$$\begin{aligned}S_k = &\left(\frac{1}{2}\delta^2 + \frac{\delta\beta_f}{h} + \frac{1}{2}\delta\right)\|x_{k+1} - x_k\|^2 + (h\beta_f\delta + \beta_f(\beta_f - \beta_b))\langle x_{k+1} - x_k, Z_k\rangle\\ &+ \left(\frac{1}{2}h^2\beta_f^2 + h\beta_f\lambda\right)\|Z_k\|^2,\end{aligned}$$

and

$$\begin{aligned}\varepsilon_k = &\, h\beta_f\langle x_{k+1} - p + \frac{\beta_f}{h}(x_{k+1} - x_k), e_k\rangle + h\beta_f\delta\langle x_{k+1} - x_k, e_k\rangle\\ &+ h^2\beta_f^2\langle\nabla f(y_k) + B(z_k), e_k\rangle - \frac{1}{2}h^2\beta_f^2\|e_k\|^2.\end{aligned}$$

As a result of an elementary inequality in Hilbert space, one has

$$\langle x_{k+1} - x_k, e_k\rangle \leq \frac{1}{2\eta}\|x_{k+1} - x_k\|^2 + \frac{\eta}{2}\|e_k\|^2, \tag{4.55}$$

holds for any $\eta > 0$. Likewise, the following inequality

$$\langle\nabla f(y_k) + B(z_k), e_k\rangle = \langle Y_k + Z_k, e_k\rangle \leq \frac{1}{2\eta_1}\|Y_k\|^2 + \frac{1}{2\eta_2}\|Z_k\|^2 + \frac{\eta_1 + \eta_2}{2}\|e_k\|^2 \tag{4.56}$$

is valid for any $\eta_1, \eta_2 > 0$.
Moreover, applying Cauchy-Schwarz inequality, then we have

$$\langle x_{k+1} - p, e_k\rangle \leq \|x_{k+1} - p\|\|e_k\|. \tag{4.57}$$

Combining these facts (4.54)-(4.57), we obtain

$$E_{k+1} - E_k + S_k + \left(\frac{h\beta_f}{L} - \frac{1}{2}h^2\beta_f^2 - \frac{h^2\beta_f^2}{2\eta_1} \right) \|Y_k\|^2 \leq \varepsilon_k', \tag{4.58}$$

where

$$S_k = \left(\frac{1}{2}\delta^2 + \frac{\delta\beta_f}{h} + \frac{1}{2}\delta - \frac{\beta_f^2 + \delta h \beta_f}{2\eta} \right) \|x_{k+1} - x_k\|^2 + (h\beta_f\delta + \beta_f(\beta_f - \beta_b))\langle x_{k+1} - x_k, Z_k \rangle$$
$$+ \left(\frac{1}{2}h^2\beta_f^2 + h\beta_f\lambda - \frac{h^2\beta_f^2}{2\eta_2} \right) \|Z_k\|^2,$$

and

$$\varepsilon_k' = \frac{1}{2}\left(\eta(\beta_f^2 + \delta h\beta_f) + \eta_1 + \eta_2 - h^2\beta_f^2\right)\|e_k\|^2 + h\beta_f\|x_{k+1} - p\|\|e_k\|.$$

Taking $\eta > 0$ such that $\frac{1}{2}\delta^2 + \frac{\delta\beta_f}{h} + \frac{1}{2}\delta - \frac{\beta_f^2 + \delta h\beta_f}{2\eta} > 0$ and $\eta(\beta_f^2 + \delta h\beta_f) + \eta_1 + \eta_2 - h^2\beta_f^2 > 0$. S_k is a quadratic form and thus $S_k > 0$ provided that

$$4\left(\frac{1}{2}h^2\beta_f^2 + h\beta_f\lambda \right)\left(\frac{1}{2}\delta^2 + \frac{\delta\beta_f}{h} + \frac{1}{2}\delta - \frac{\beta_f^2 + \delta h\beta_f}{2\eta} \right) - (h\beta_f\delta + \beta_f(\beta_f - \beta_b))^2 > 0. \tag{4.59}$$

Notice that

$$\lim_{h \to 0^+} 4\left(\frac{1}{2}h^2\beta_f^2 + h\beta_f\lambda \right)\left(\frac{1}{2}\delta^2 + \frac{\delta\beta_f}{h} + \frac{1}{2}\delta - \frac{\beta_f^2 + \delta h\beta_f}{2\eta} \right) - (h\beta_f\delta + \beta_f(\beta_f - \beta_b))^2$$
$$= 4\beta_f^2\delta\left[\lambda - \frac{(\beta_b - \beta_f)^2}{4\delta} \right] > 0 \tag{4.60}$$

due to the assumption on the parameters. This ensures the existence of $h \in \left(0, \frac{2}{L\beta_f}\right)$ satisfying (4.59). Hence, there exists a positive real number μ such that for any $k \geq 1$,

$$E_{k+1} - E_k + \mu\|x_{k+1} - x_k\|^2 + \mu\|B(z_k) - B(p)\|^2$$
$$+ \left(\frac{h\beta_f}{L} - \frac{1}{2}h^2\beta_f^2 - \frac{h^2\beta_f^2}{2\eta_1} \right)\|\nabla f(y_k) - \nabla f(p)\|^2 \leq \varepsilon_k'. \tag{4.61}$$

This implies that

$$E_{k+1} \leq E_1 + \sum_{1 \leq i < k+1} \varepsilon_i'.$$

Taking into account the form of the energy sequence (E_k), we obtain

$$\frac{\delta}{2}\|x_{k+1} - p\|^2 \leq E_1 + \sum_{1\leq i<k+1} \varepsilon'_i. \tag{4.62}$$

According to the assumption $\sum_{k=1}^{\infty} \|e_k\| < +\infty$, this implies that $\sum_{k=1}^{\infty} \|e_k\|^2 < +\infty$. Therefore, there exists $C > 0$ such that

$$\sum_{1\leq i<k+1} \varepsilon'_i \leq h\beta_f \sum_{1\leq i<k+1} \|x_{i+1} - p\|\|e_i\| + C. \tag{4.63}$$

From (4.62) and (4.63), we conclude that

$$\frac{\delta}{2}\|x_{k+1} - p\|^2 \leq E_1 + h\beta_f \sum_{1\leq i<k+1} \|x_i - p\|\|e_i\| + C.$$

More precisely, let us rewrite this estimate as below

$$\frac{1}{2}\|x_{k+1} - p\|^2 \leq \frac{1}{2}C_0^2 + c_0 \sum_{1\leq i<k+1} \|x_{i+1} - p\|\|e_i\|, \tag{4.64}$$

in which

$$C_0 = \sqrt{\frac{E_1 + C}{\delta}}, \quad c_0 = \frac{h\beta_f}{\delta}.$$

Now, by applying Lemma 1.3.7 to (4.64), we obtain

$$\|x_{k+1} - p\| \leq C_0 + c_0 \sum_{1\leq i<k+1} \|e_i\| < +\infty.$$

Therefore, $(\|x_k - p\|)$ and consequently $(\|x_k\|)$ is a bounded sequence.
Returning to (4.63), according to the boundedness of $(\|x_k - p\|)$ and the assumption of (e_k), we obtain

$$\sum_{k=1}^{\infty} \epsilon'_k < +\infty.$$

The rest of the proof is similar to that of Theorem 4.4.1, so we omit here.

4.4.3 A variant of the proximal-gradient algorithm

In this section, we will study a variant of the preceding proximal-gradient algorithm, in which we reverse the role of the two operators. We examine the following semi-implicit finite-difference scheme for (iDINAM) :

$$\frac{1}{h^2}(x_{k+1}-2x_k+x_{k-1})+\frac{\gamma}{h}(x_{k+1}-x_k)+\nabla f\left(x_{k+1}+\frac{\beta_f}{h}(x_{k+1}-x_k)\right)$$
$$+B\left(x_k+\frac{\beta_b}{h}(x_k-x_{k-1})\right)=0, \tag{4.65}$$

where $h>0$ is a fixed time step.
After expanding (4.65), we obtain the following algorithm.

(iDINAAM-var) :

Initialize : $x_0\in\mathcal{H}, x_1\in\mathcal{H}$
$\alpha=1+\frac{\beta_f}{h}$,
$y_k=x_k+(h^2-\gamma h)(x_k-x_{k-1})-h^2B\left(x_k+\frac{\beta_b}{h}(x_k-x_{k-1})\right)$,
$z_k=(\operatorname{Id}+\alpha h^2\nabla f)^{-1}(\alpha y_k-(\alpha-1)x_k)$,
$x_{k+1}=\frac{1}{\alpha}(\alpha-1)x_k+\frac{1}{\alpha}z_k$.

Theorem 4.4.3 *Let $B:\mathcal{H}\to\mathcal{H}$ be a λ-cocoercive operator and $f:\mathcal{H}\to\mathbb{R}$ be a convex differentiable function whose gradient is L-Lipschitz continuous. Suppose that the positive parameters $\lambda,\gamma,\beta_b,\beta_f$ satisfy*

$$\gamma\beta_f>1 \text{ and } \lambda>\frac{(\beta_b-\beta_f)^2}{4(\gamma\beta_f-1)}. \tag{4.66}$$

Then, there exists h^* such that for all $0<h<h^*$, the sequence (x_k) generated by the algorithm (iDINAAM-var) has the following properties (where $p\in S$) :

(i) (x_k) *converges weakly to an element in S ;*

(ii) $\sum_{k=1}^{\infty}\|x_k-x_{k-1}\|^2<+\infty,\quad \sum_{k=1}^{\infty}\left\|\nabla f\left(x_k+\frac{\beta_f}{h}(x_k-x_{k-1})\right)-\nabla f(p)\right\|^2<+\infty,$

$$\sum_{k=1}^{\infty}\left\|B\left(x_k+\frac{\beta_f}{h}(x_k-x_{k-1})\right)-B(p)\right\|^2<+\infty, \sum_{k=1}^{\infty}\|\nabla f(x_k)-\nabla(p)\|^2<+\infty,$$

$$\sum_{k=1}^{\infty}\|B\left(x_k+\frac{\beta_f}{h}(x_k-x_{k-1})\right)-B(p)\|^2<+\infty, \sum_{k=1}^{\infty}\|B\left(x_k\right)-B(p)\|^2<+\infty,$$

$$\sum_{k=1}^{\infty}\|\nabla f(x_k)-\nabla f(x_{k-1})\|^2<+\infty, \ \textit{and} \ \sum_{k=1}^{\infty}\|B(x_k)-B(x_{k-1})\|^2<+\infty;$$

(iii) $\lim_{k\to\infty}\|x_{k+1}-x_k\|=0, \ \lim_{k\to\infty}\|B(x_k)-B(p)\|=0, \ \lim_{k\to\infty}\|\nabla f(x_k)-\nabla f(p)\|=0.$

Proof **The discrete energy** Take $p\in S$. Consider the sequence (E_k) defined for all $k\geq 1$ by the formula

$$E_k:=\frac{1}{2}\|x_k-p+\frac{\beta_f}{h}(x_k-x_{k-1})\|^2+\frac{\delta}{2}\|x_k-p\|^2,$$

where δ is a positive coefficient to adjust.

For each $k\geq 1$, let us briefly write E_k as follows :

$$E_k=\frac{1}{2}\|v_k\|^2+\frac{\delta}{2}\|x_k-p\|^2,$$

with

$$v_k:=x_k-p+\frac{\beta_f}{h}(x_k-x_{k-1}).$$

By definition of v_k and the formula (4.65), we have

$$\begin{aligned}
&v_{k+1}-v_k=x_{k+1}-x_k+\frac{\beta_f}{h}(x_{k+1}-2x_k+x_{k-1})\\
&=(1-\gamma\beta_f)(x_{k+1}-x_k)-h\beta_f\nabla f\left(x_{k+1}+\frac{\beta_f}{h}(x_{k+1}-x_k)\right)-h\beta_f\left(B(x_k+\frac{\beta_b}{h}(x_k-x_{k-1})\right)\\
&=(1-\gamma\beta_f)(x_{k+1}-x_k)-h\beta_f\nabla f\left(y_k\right)-h\beta_f B(z_k),
\end{aligned}$$

in which

$$\begin{aligned}
y_k &=x_{k+1}+\frac{\beta_f}{h}(x_{k+1}-x_k),\\
z_k &=x_k+\frac{\beta_b}{h}(x_k-x_{k-1}).
\end{aligned}$$

Therefore, for $k\geq 1$, we have

$$\begin{aligned}
\frac{1}{2}\|v_{k+1}\|^2-\frac{1}{2}\|v_k\|^2&=-\frac{1}{2}\|v_{k+1}-v_k\|^2+\langle v_{k+1}-v_k,v_{k+1}\rangle\\
&\leq-(\gamma\beta_f-1)\langle x_{k+1}-p+\frac{\beta_f}{h}(x_{k+1}-x_k),x_{k+1}-x_k\rangle\\
&-\langle x_{k+1}-p+\frac{\beta_f}{h}(x_{k+1}-x_k),h\beta_f\nabla f\left(y_k\right)+h\beta_f B(z_k)\rangle. \qquad (4.67)
\end{aligned}$$

Using the elementary identity, one has

$$\frac{1}{2}\|x_{k+1}-p\|^2-\frac{1}{2}\|x_k-p\|^2=-\frac{1}{2}\|x_{k+1}-x_k\|^2+\langle x_{k+1}-x_k,x_{k+1}-p\rangle. \qquad (4.68)$$

Take $\delta = \gamma\beta_f - 1$. Then, from (4.67) and (4.68), we deduce that

$$\begin{aligned} E_{k+1} - E_k \le &- \left(\frac{\delta\beta_f}{h} + \frac{1}{2}\delta\right) \|x_{k+1} - x_k\|^2 \\ &- \langle x_{k+1} - p + \frac{\beta_f}{h}(x_{k+1} - x_k), h\beta_f \nabla f(y_k) + h\beta_f B(z_k)\rangle. \end{aligned}$$

Notice that $\nabla f(p) + B(p) = 0$. Thus, we can rewrite the previous relation as follows

$$E_{k+1} - E_k \le -\left(\frac{\delta\beta_f}{h} + \frac{1}{2}\delta\right) \|x_{k+1} - x_k\|^2 - h\beta_f \langle x_{k+1} - p + \frac{\beta_f}{h}(x_{k+1} - x_k), Y_k + Z_k\rangle,$$

where $Y_k = \nabla f(y_k) - \nabla f(p)$ and $Z_k = B(z_k) - B(p)$.
Since B is λ-cocoercive, we have

$$\begin{aligned} &\langle x_{k+1} - p + \frac{\beta_f}{h}(x_{k+1} - x_k), Z_k\rangle = \langle z_k - p + (1 + \frac{1}{h}(\beta_f - \beta_b))(x_{k+1} - x_k), B(z_k) - B(p)\rangle \\ &\ge \lambda\|B(z_k) - B(p)\|^2 + (1 + \frac{1}{h}(\beta_f - \beta_b))\langle (x_{k+1} - x_k), B(z_k) - B(p)\rangle \\ &= \lambda\|Z_k\|^2 + (1 + \frac{1}{h}(\beta_f - \beta_b))\langle (x_{k+1} - x_k), Z_k\rangle, \end{aligned}$$

Moreover, due to ∇f is $1/L$-cocoercive, we deduce that

$$\langle x_{k+1} - p + \frac{\beta_f}{h}(x_{k+1} - x_k), Y_k\rangle = \langle y_k - p, \nabla f(y_k) - \nabla f(p)\rangle \ge \frac{1}{L}\|\nabla f(y_k) - \nabla f(p)\|^2. \tag{4.69}$$

This implies

$$\begin{aligned} E_{k+1} - E_k \le &-\left(\frac{\delta\beta_f}{h} + \frac{1}{2}\delta\right) \|x_{k+1} - x_k\|^2 \\ &+ \left(-h\beta_f - \beta_f(\beta_f - \beta_b)\right) \langle x_{k+1} - x_k, Z_k\rangle - h\beta_f\lambda\|Z_k\|^2 - \frac{h\beta_f}{L}\|Y_k\|^2. \end{aligned}$$

Equivalently,

$$E_{k+1} - E_k + \frac{h\beta_f}{L}\|Y_k\|^2 + S_k \le 0, \tag{4.70}$$

where $S_k = \left(\frac{\delta\beta_f}{h} + \frac{1}{2}\delta\right) \|x_{k+1} - x_k\|^2 + \left(h\beta_f + \beta_f(\beta_f - \beta_b)\right) \langle x_{k+1} - x_k, Z_k\rangle + h\beta_f\lambda\|Z_k\|^2$.
Our aim is to seek $h > 0$ such that $S_k > 0$. Let us observe that $q : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$

is a quadratic form

$$q(u,v) := a\|u\|^2 + b\langle u,v\rangle + g\|v\|^2,$$

with

$$\begin{aligned} a &= \frac{\delta\beta_f}{h} + \frac{1}{2}\delta, \\ b &= h\beta_f + \beta_f(\beta_f - \beta_b), \\ g &= h\beta_f\lambda. \end{aligned}$$

Then, $S_k = q(x_{k+1} - x_k, Z_k) > 0$ if $4ag - b^2 > 0$. One has,

$$\begin{aligned} 4ag - b^2 &= 4\left(\frac{\delta\beta_f}{h} + \frac{1}{2}\delta\right) h\beta_f\lambda - (h\beta_f + \beta_f(\beta_f - \beta_b))^2 \\ &= 4\left(\delta\beta_f + \frac{1}{2}h\delta\right)\beta_f\lambda - (h\beta_f + \beta_f(\beta_f - \beta_b))^2. \end{aligned}$$

Hence, $\lim_{h\to 0^+}(4ag - b^2) = \beta_f^2(4\lambda\delta - (\beta_f - \beta_b)^2) > 0$ since $4\lambda\delta > (\beta_f - \beta_b)^2$. This implies there exists $h^* > 0$ such that for all $h \in (0, h^*)$, we have $S_k > 0$.

Therefore, under the above condition, and by taking h sufficiently small, there exist positive real numbers μ and η such that for all $k \geq 1$,

$$E_{k+1} - E_k + \mu\|x_{k+1} - x_k\|^2 + \mu\|B(z_k) - B(p)\|^2 + \eta\|\nabla f(y_k) - \nabla f(p)\|^2 \leq 0. \quad (4.71)$$

The remain of the proof is analogous to Theorem 4.4.1's one, so we omit it.

4.5 Numerical illustrations

The main purpose of this section is to implement our algorithms to numerically compute the trajectory of the dynamical system (iDINAM). For further applications, we refer the reader to [4], [6]. Before starting, we recall that a broad and successful method to generate monotone cocoercive operators even if they are not gradients of convex functions is to take Yosida approximation A_λ of a linear skew symmetric operator A. For more details, we refer the reader to Remark 3.6.1.

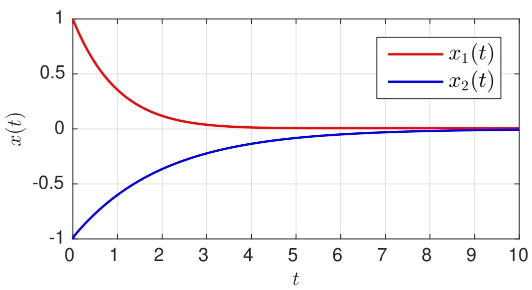
Example 4.5.1 We start with a simple illustrative example in $\mathbb{R}^2$. We take $\mathcal{H} = \mathbb{R}^2$ endowed with the usual Euclidean structure. Let B be a linear operator whose matrix in the canonical basis of $\mathbb{R}^2$ is given by $B = A_\lambda$ for $\lambda = 5$. Thanks to the previous remark, we can conclude that B is λ-cocoercive and nonpotential. To observe the classical oscillations, in the heavy ball with friction, we take $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x_1, x_2) = 10x_2^2.$$

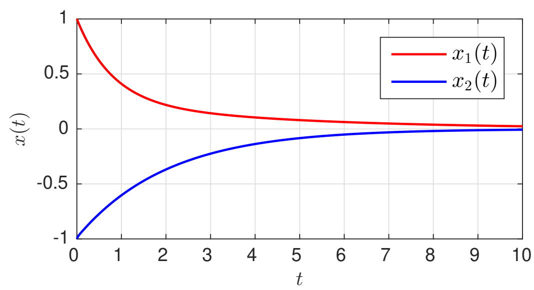
It is obvious that f is convex but not strongly convex. We set $\gamma = 0.9$ and consider the dynamical system (iDINAM) which γ, f, and B have already defined before. As a straightforward application of Theorem 4.3.1, we obtain that the trajectory $x(t)$ generated by (iDINAM) converges to x_∞, where $x_\infty \in S = (B + \nabla f)^{-1}(0) = \{0\}$ provided that the positive parameters β_b, β_f fulfill the following constraits

$$\gamma\beta_f > 1 \text{ and } \lambda > \frac{(\beta_b - \beta_f)^2}{4(\gamma\beta_f - 1)}.$$

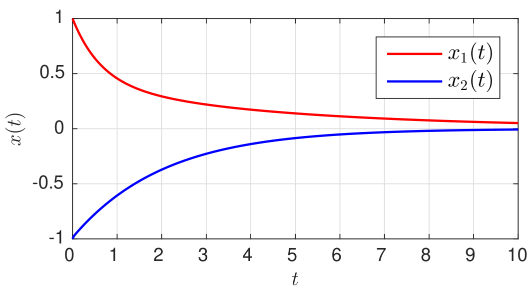
The trajectory obtained by using Matlab is depicted in Figure 4.1, where we represent the components $x_1(t)$ and $x_2(t)$ in red and blue respectively.



(a) Case $\beta_b = 1, \beta_f = 2$.



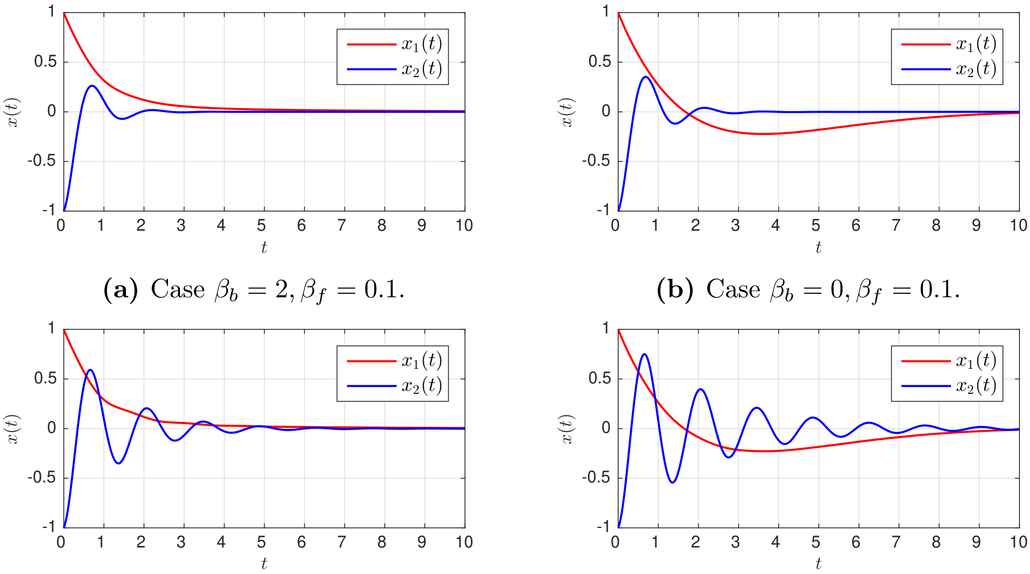
(b) Case $\beta_b = 2, \beta_f = 2$.



(c) Case $\beta_b = 3, \beta_f = 2$.

FIGURE 4.1 – Trajectories of (iDINAM) for variety values of the parameters β_b, β_f.

Now we investigate the behavior of the trajectories by examining several values of β_b and β_f. To do that, we study indepently four more disparate cases where their plots of the solutions have been depicted in Figure 4.2. Through Figures 4.1 and 4.2, we can observe that the presence of Hessian damping ($\beta_f > 0$) attenuates the oscillations of the



(a) Case $\beta_b = 2, \beta_f = 0.1$. **(b)** Case $\beta_b = 0, \beta_f = 0.1$.

(c) Case $\beta_b = 2, \beta_f = 0$. **(d)** Case $\beta_b = 0, \beta_f = 0$.

Figure 4.2 – Oscillation of the trajectories of (DINAM) for different values of β_b, β_f.

trajectories. These oscillations appear whenever β_f goes to 0, that is depicted obviously in Figure 4.3.

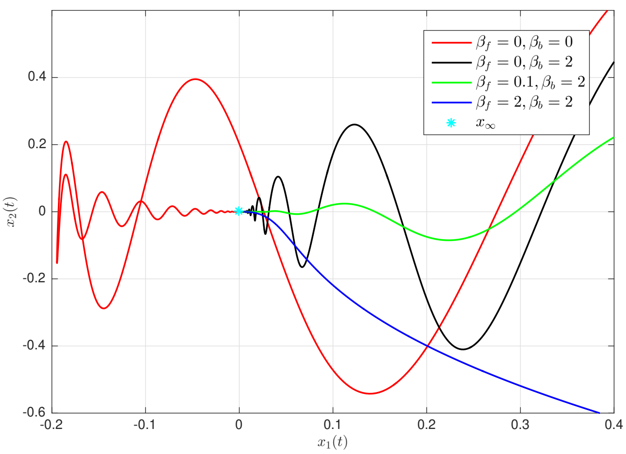
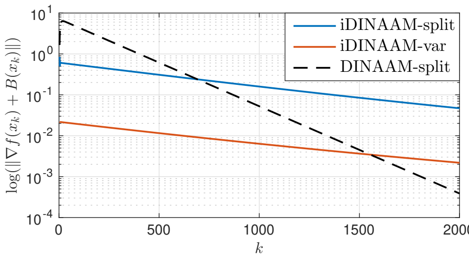


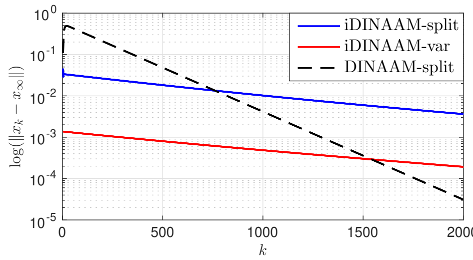
Figure 4.3 – The attenuation of the oscillations of (iDINAM) by introducing the Hessian damping ($\beta_f > 0$).

Example 4.5.2 In Chapter 2, we considered the dynamical system

$$\ddot{x}(t) + \gamma\dot{x}(t) + \nabla f(x(t)) + B(x(t)) + \beta_f \nabla^2 f(x(t))\dot{x}(t) + \beta_b B'(x(t))\dot{x}(t) = 0, \; t \geq 0. \tag{DINAM}$$



(a) The convergence rates of $(\|\nabla f(x_k) + B(x_k)\|)$ obtained by algorithms



(b) The convergence rates of (x_k) obtained by algorithms

Figure 4.4 – The numerical performance of algorithms (iDINAAM-split), (iDINAAM-var) and (DINAAM-split) to find the zeros of $\nabla f + B$ with the time step $h = 10^{-2}$.

It is shown that under certain conditions on the parameters, namely $\beta_f > 0$ and

$$4\lambda\gamma > \frac{(\beta_b - \beta_f)^2}{\beta_f} + 2\left(\beta_b + \frac{1}{\gamma}\right) + 2\sqrt{\left(\beta_b + \frac{1}{\gamma}\right)^2 + \frac{(\beta_b - \beta_f)^2}{\gamma\beta_f}}, \tag{4.72}$$

then any trajectory generated by (DINAM) converges weakly towards an element of the set $S = (\nabla f + B)^{-1}(0)$. Morover, in [6], the authors proposed some algorithms to find the zeros of $\nabla f + B$. Since our research is in line with that and provides similar results, it is relevant to compare these different types of algorithms. Following the same framework on B and γ as in the previous example and replacing f by $f(x) = 5x_1^2 + 10x_2^2$, let us make a comparison of their numerical performance.

Figure 4.4a shows the norm of the objective function $\nabla f(x_k)+B(x_k)$ on a logarithmic scale for each iteration k when we apply our algorithms, namely (iDINAAM-split), (iDINAAM-var) and (DINAAM-split) proposed in [6]. A numerical comparison among the norms of $x_k - x_\infty$ (x_∞ is a zero of $\nabla f + B$) is illustrated in Figure 4.4b as well. We can see that (iDINAAM-split) and (iDINAAM-var) gave the same numerical results while (DINAAM-split) did better in the long term in this case.

Example 4.5.3 Let us return to Example 4.5.1 and consider the effects of perturbations on numerical performance. With the same numerical values of the involved parameters, we just add the errors $e_k = \dfrac{1}{k^2}$ and $\bar{e}_k = \dfrac{1}{\sqrt{k}}$. Clearly, the errors (e_k) satisfy the assumptions of Theorem 4.4.2 while $(\bar{e}_k)$ does not. Implementing algorithm (iDINAAM-pert) in Matlab, the plot of $\|x_k - x_\infty\|$ according to k is depicted in Figure 4.5. It indicates that if the perturbed term (e_k) is "small" enough then the algorithm (iDINAAM-pert) behaves as well as the nonperturbed version.

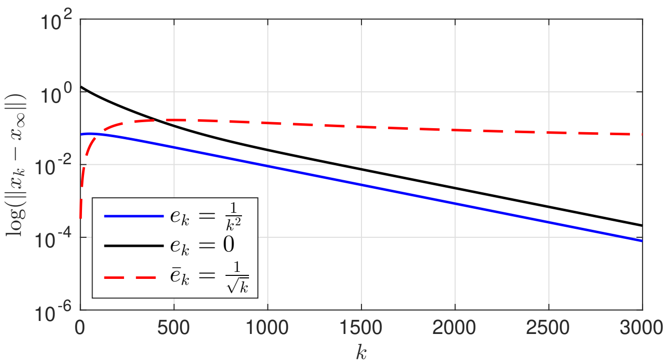


FIGURE 4.5 – The effects of pertubations on the algorithm (DINAAM-split).

4.6 Conclusion and perspectives

We devoted a significant amount of work to this thesis, namely in Chapters 2 - 4, finding and designing new algorithms to solve additively structured monotone problems of type

$$\text{Find } x \in \mathcal{H} \; : \; \underbrace{\nabla f(x)}_{potential} + \underbrace{B(x)}_{nonpotential} = 0$$

by examining temporal discretization of the associated dynamics (DINAM) and (iDINAM). The gradient of a continuously differentiable convex function f plays as the potential component, while the nonpotential one is a monotone and cocoercive operator B.
Specifically, the entire Chapter 4 was designed to address two aspects of (iDINAM) : continuous and discrete cases. The well-posedness of the Cauchy problem, as well as the asymptotic convergence characteristics of the trajectories generated by the continuous dynamic, were demonstrated in continuous analysis. Furthermore, the convergence was carried out using the parameters β_f and β_b related to the geometric dampings, as well as the parameters γ and λ. Lastly, in the algorithmic section, we propose efficient methods for solving structured monotone inclusions. The algorithm (iDINAAM-split) and its variations add to the library of algorithms for solving additively structured monotone problems.

5

Generalized accelerated Bregman proximal algorithms for composition convex optimization

The accelerated gradient method, developed by Nesterov in 1983 ([61], [62]), which reduces the theoretical convergence rate (for function values) from $\mathcal{O}(1/k)$ (of the standard gradient method) to $\mathcal{O}(1/k^2)$, is now recognized as one of the most powerful first-order methods for solving smooth convex optimization problems. This acceleration scheme was extensively developed for solving composition convex optimization problems of the form (5.1), in which the objective function is represented by the sum of a smooth convex function and a nonsmooth convex function (see [53, 62, 64, 65] and the references given therein).

In this chapter, we focus on studying the problem of minimizing the objective function including the sum of two convex functions f and Φ, in which f is differentiable and relatively smooth to convex function h, and Φ is possibly non-differentiable but simple to optimize. The generalized Nesterov's accelerated proximal gradient algorithm (GAPGA) proposed in [66] gives us a better convergence rate to solve the optimization problem when f is uniform smooth, *i.e.* ∇f is supposed to be Lipschitz continuous. While the uniform smoothness condition plays a central role in the development and analysis of first-order methods, there are many applications where the objective function does not have this property, despite being convex and differentiable [50]. Therefore, we aim to investigate the algorithms introduced in [66] in case of f is relatively smooth and propose a method that employs the Bregman distance of the reference function instead of euclidean distance.

This chapter constitutes the subject of the joint work in collaboration with S. Adly and H.V. Ngai.

5.1 Introduction and preliminary results

5.1.1 Composition convex optimization problem

Let $\mathbb{R}^n$ be the n-dimensional real euclidean space endowed with the scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We consider the *composition convex* optimization problem of the form

$$\min\{f(x) + \Phi(x) : x \in C\}, \tag{5.1}$$

where C is a closed convex set in $\mathbb{R}^n$, $\Phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper, lower-semicontinuous, and convex function and $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable, convex function whose gradient is L-Lipschitz continuous on dom Φ, for some $L > 0$, that is,

$$\|\nabla f(x) - \nabla f(y)\| \le L\|x - y\|, \quad \forall x, y \in \mathrm{dom}\,\Phi. \tag{5.2}$$

First-order methods for solving (5.1) are based on the idea of minimizing a simple approximation of the objective function for each iteration. Particularly, in the proximal gradient method, we begin with an initial point x_0 belonging to the relative interior of a closed convex set C in $\mathbb{R}^n$ and generate a sequence x_k for $k = 1, 2, \ldots$ with

$$x_{k+1} = \text{argmin}_{x \in C}\{f(x_k) + \langle \nabla f(x_k), x - x_k\rangle + \frac{L_k}{2}\|x - x_k\|^2 + \Phi(x)\}, \tag{5.3}$$

where $L_k > 0$ for all $k \geq 0$.
In the scope of our research, we restrict ourselves to the case $C = \mathbb{R}^n$. However, these theoretical results presented in this chapter can be extended for any closed convex set C in $\mathbb{R}^n$ provided that f is differentiable on an open set containing the relative interior of C (denoted as $\text{rint}C$ or $\text{rint}(C)$).

5.1.2 Related works and outline

The accelerated gradient method proposed by Nesterov in 1983 ([61], [62]) is truly a prior step to designing efficient first-order methods for solving smooth convex optimization problems. Based on this acceleration scheme, many algorithms were developed extensively for solving composition convex optimization of the form (5.1) in which the objective function is represented by the sum of two convex functions including a smooth and a nonsmooth one. Especially, by combining the forward-backward method with Nesterov's acceleration scheme, Beck-Teboulle ([38]) have proposed the *fast iterative shrinkage-thresholding* algorithm (FISTA) for solving (5.1) which has many applications, for example, image processing. Later, in [29] (see also [28]), Attouch-Peypouquet have shown that the convergence rate of the *accelerated forward-backward* method is actually $o(1/k^2)$ rather than $\mathcal{O}(1/k^2)$.
Recently, the authors in [66] continued to generalize the Nesterov's accelerated schemes (see [64]) and have proposed the new schemes with the convergence rate for the function values attaining the order $o(1/k^2)$ for the convex case. For the $p-$uniformly convex case with $p > 2$, the convergence rate is $\mathcal{O}\left(\ln k / k^{2p/(p-2)}\right)$ and when the objective function is strongly convex, the convergence is linear.
To obtain such good performance, the gradient of f is assumed to be uniformly Lipschitz, i.e., there exists a constant L satisfying (5.2). The uniform smoothness condition (5.2) plays a key role in the development of first-order methods, however, there are many problems where the objective function does not satisfy this property, even if it is convex and differentiable. For example, the gradient of the objective function in D-optimal experiment design (e.g., [51, 33]) involving the logarithm in the form of log-determinant might blow

up towards the boundary of the feasible region. To solve the familiy of such problems, the notion of relative smoothness was introduced in several recent works such as [35, 56, 74]. In those works, the Bregman distances have been used instead of the usual euclidian distance ; recently, some accelerated schemes with the Bregman distances have been studied in [50, 49]. In these papers, the authors considered problem (5.1) when the function f is relative smooth with respect to a Bregman distance satisfying the called *generalized triangle scaling property*, and with this property, the proposed accelerated algorithms attain a convergence rate of order $\mathcal{O}(1/k^{\gamma})$ with γ being the triangle scaling exponent of the respected Bregman distance (as $\gamma = 2$ for the euclidian distance, so the convergence rate established in the latter mentioned papers is of $\mathcal{O}(1/k^2)$ as the Nesterov-type accelerated algorithms, e.g., (FISTA) in the euclidean case). Thus, this chapter is a development of the accelerated algorithms proposed in [66] to the framework of the Bregman distances for problem (5.1).

Throughout this chapter, we consider the Bregman distances satisfying the *Hölderian triangle scaling property*. This property, a generalization of the triangle scaling property considered in [50] covers some important situations for which the latter property is not satisfied ; for example, while the gradient of the convex function h defining the Bregman distance is not Lipschitzian (on the interior of its domain) but merely Hölderian. The established convergence rates show the efficiency in the theoretical aspect of the accelerated schemes proposed in the present paper ; some initial numerical experiments are reported to demonstrate the efficiency in the computational aspect of the proposed algorithms.

The outline of the chapter is the following. In the introductory Section 5.1, we introduce our problem and recall some of preliminary results concerning the Bregman distance and relative smoothness of a function. The main contribution is presented in Section 5.2 and 5.3. In Section 5.2, we aim to analyzing the convergence properties of the generalized Nesterov's algorithm. We also highlight the convergent rate of our scheme by setting appropriate parameters and the smoothness, convexity of function f. In Section 5.3, we continue to analyze the convergence rate of the generalized accelerated forward-backward algorithm. Some numerical experiments will be shown in Section 5.4 to see how these new schemes to be applied in certain problems.

5.1.3 Divergence and relative smoothness

Definition 5.1.1 ([50]) *Let* $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a strictly convex function that is differentiable on* $\operatorname{rint}(\operatorname{dom} h)$*. The Bregman distance associated with* h *is defined as*

$$D_h(x, y) := h(x) - h(y) - \langle \nabla h(y), x - y \rangle, \forall x \in \operatorname{dom} h, y \in \operatorname{rint}(\operatorname{dom} h).$$

From the definition of D_h, we immediately deduce some basic properties.

Proposition 5.1.1 *Let D_h be the Bregman distance associated with a strictly convex function h. Then,*

(i) $D_h(x, y) \geq 0$ *for all* $x \in \operatorname{dom} h, y \in \operatorname{rint} \operatorname{dom} h$ *and* $D_h(x, y) = 0$ *if and only if* $x = y$ *;*

(ii) $D_h(x, y)$ *is convex in x for fixed y ;*

(iii) D_h *is not symmetric in general. Thus, in order to emphasize lack of symmetry, D_h is also called a directed distance or divergence.*

We can see that for $h(x) = \frac{1}{2}\|x\|^2$ one has $D_h(x, y) = \frac{1}{2}\|x - y\|^2$. Therefore, it is natural for us to replace the squared euclidean distance with a Bregman distance.

Below are some specific Bregman distances :

— *The generalized Kullback-Leibler (KL) divergence.* Let h be the negative Boltzmann-Shannon entropy given by

$$h(x) = \sum_{i=1}^{n} x^{(i)} \log x^{(i)},$$

which is defined on $\mathbb{R}^n_+$. The Bregman distance associated with h is

$$D_{KL}(x, y) = \sum_{i=1}^{n} \left(x^{(i)} \log \left(\frac{x^{(i)}}{y^{(i)}} \right) - x^{(i)} + y^{(i)} \right).$$

— *The Itakura-Saito (IS) distance.* The IS distance is the Bregman distance associated to Burg's entrophy $h(x) = -\sum_{i=1}^{n} \log(x^{(i)})$ with $\operatorname{dom} h = \mathbb{R}^n_{++}$. And

$$D_{IS}(x, y) = \sum_{i=1}^{n} \left(-\log \left(\frac{x^{(i)}}{y^{(i)}} \right) + \frac{x^{(i)}}{y^{(i)}} - 1 \right).$$

— *Logistic loss divergence.* By taking $h(x) = \sum_{i=1}^{n} (x^{(i)} \log x^{(i)} + (1 - x^{(i)}) \log(1 - x^{(i)}))$, then the Bregman distance is given by

$$D_{LL}(x, y) = \sum_{i=1}^{n} \left(x^{(i)} \log \frac{x^{(i)}}{y^{(i)}} + (1 - x^{(i)}) \log \frac{1 - x^{(i)}}{1 - y^{(i)}} \right).$$

Definition 5.1.2 ([50]) *Let $C \subseteq \operatorname{rint}(\operatorname{dom} h)$ be a closed convex set in $\mathbb{R}^n$.*

The function f is called L-smooth relative to h on C if there exists an $L > 0$ such that

$$f(x) \leq f(y) + \langle \nabla f(y), x - y\rangle + LD_h(x, y), \forall x \in C, y \in \text{rint}C.$$

The function f is said to be μ-strong convex relative to h on C if

$$f(x) \geq f(y) + \langle \nabla f(y), x - y\rangle + \mu D_h(x, y), \forall x \in C, y \in \text{rint}C,$$

for some $\mu > 0$.

As shown in [35] and [56], the relative smoothness is equivalent to the following statements :

— $Lh - f$ is a convex function on $\text{rint}C$.

— If both f and h are twice differentiable, then $\nabla^2 f(x) \preceq L\nabla^2 h(x)$ for all $x \in \text{rint}C$.

— $\langle \nabla f(x) - \nabla f(y), x - y\rangle \leq L\langle \nabla h(x) - \nabla h(y), x - y\rangle$ for all $x, y \in \text{rint}C$.

It is obviously that if ∇f is $L-$Lipschitz continous, then f is $L-$smooth relative to $h = \frac{1}{2}\|\cdot\|^2$ as well. That motivates us to replace the squared euclidean distance in (5.3) with a Bregman distance :

$$x_{k+1} = \text{argmin}_{x\in C}\{f(x_k) + \langle \nabla f(x_k), x - x_k\rangle + L_k D_h(x, x_k) + \Phi(x)\}. \tag{5.4}$$

In [50], the authors have proposed an accelerated proximal gradient method with Bregman distances which satisfy a called *triangle scaling property* : There is some $\gamma > 0$ such that for all $x, z, \bar{z} \in \text{rintdom}h$,

$$D_h((1-\theta)x + \theta z, (1-\theta)x + \theta\bar{z}) \leq \theta^\gamma D_h(z, \bar{z}). \tag{5.5}$$

We call γ a (uniform) triangle scaling exponent (TSE) of D_h. Let $h(x) = \dfrac{1}{2}\|x\|^2$ and $D_h(x, y) = \dfrac{1}{2}\|x - y\|^2$. It is easy to check that

$$\begin{aligned} D_h((1-\theta)x + \theta z, (1-\theta)x + \theta\bar{z}) &= \frac{1}{2}\|(1-\theta)x + \theta z - (1-\theta)x - \theta\bar{z}\|^2 \\ &= \frac{\theta^2}{2}\|z - \bar{z}\|^2 = \theta^2 D_h(z, \bar{z}). \end{aligned}$$

Thus, the squared euclidean distance has triangle scaling property with $\gamma = 2$.

Though the γ uniform triangle scaling exponent is a crucial property in our framework, it is not easy to determine γ TSE of the Bregman distance in general. If $D_h(x, y)$ is jointly convex in (x, y), then the inequality (5.5) holds with $\gamma = 1$ because

$$D_h\left((1-\theta)x + \theta z, (1-\theta)x + \theta \bar{z}\right) \leq (1-\theta)D_h(x, x) + \theta D_h(z, \bar{z}) = \theta^\gamma D_h(z, \bar{z}).$$

Hence, it is essential to study jointly convex Bregman distances. The following proposition gives us a criterion to check that if the Bregman distance is jointly convex.

Proposition 5.1.2 ([36]) *Suppose $h : \mathbb{R} \mapsto (-\infty, +\infty]$ is strictly convex and twice continuously differeniable on an open interval in $\mathbb{R}$. Then, the Bregman distance $D_h(\cdot, \cdot)$ is jointly convex if and only if $1/h''$ is concave. Specially, whenever h can be written separably as $h(x) = \sum_{i=1}^{n} h_i(x^{(i)})$ and $1/h_i''$ is concave for each $i = 1, ..., n$, then one has D_h has a uniform TSE of at least 1.*

In the view of Proposition 5.1.2, we can conlude that D_{KL} and D_{LL} has a uniform TSE $\gamma = 1$ while D_{IS} has a uniform TSE likely to be less than 1. For more examples, we refer the reader to [50].

Definition 5.1.3 ([50]) *The intrinsic TSE of D_h, denoted γ_{in}, is the largest γ such that for all $x, z, \bar{z} \in \operatorname{rint}\operatorname{dom} h$,*

$$\limsup_{\theta \to 0} \frac{D_h\left((1-\theta)x + \theta z, (1-\theta)x + \theta \bar{z}\right)}{\theta^\gamma} < +\infty.$$

In [50], the authors showed that a board family of Bregman distances share the same intrinsic $\gamma_{in} = 2$.

Our study will benefit from recent progress concerning Nesterov's accelerated proximal gradient algorithm linked to the Bregman distance of the reference function as the proximity measure. In this chapter, we will present a generalized variant of Nesterov's accelerated proximal gradient method for solving composition convex optimization in which the reference function $h : \mathbb{R}^n \to \mathbb{R}$ is convex but not neccessary strongly convex and the key estimate $f(x) \leq f(y) + \langle \nabla f(y), x - y\rangle + \frac{L}{2}\|x - y\|^2$ satisfied by the smoothness assumption on f is replaced with

$$f(x) \leq f(y) + \langle \nabla f(y), x - y\rangle + LD_h(x, y), \forall x \in \operatorname{dom} h, y \in \operatorname{rint}\operatorname{dom} h.$$

We aim to finding another generalized method to solve (5.1) in case f is smooth relative. To meet our expectation, we will investigate and promote the Generalized Nesterov's accelerated proximal gradient algorithm (GAPGA) introduced in [66] by replacing the squared euclidean distance in that algorithm with a Bregman one. After that we introduce a generalized accelerated forward-backward algorithm including Bregman distance.

In this work, we consider Bregman distances with a *Hölderian triangle scaling property* which is more general than (5.5), defined as follows.

Definition 5.1.4 *Let h be a strictly convex function that is differentiable on* $\operatorname{rint}(\operatorname{dom} h)$. *The Bregman distance D_h has the Hölderian relaxed triangle scaling property if there are some $\eta_1 \in (0,2]$, $\eta_2 \in (0,1]$, and $M > 0$ such that such that for all $x, z, \bar{z} \in \operatorname{rintdom} h$, $\theta \in [0,1]$,*

$$D_h\left((1-\theta)x + \theta z, (1-\theta)x + \theta \bar{z}\right) \le M\theta^{\eta_1} D_h(z,\bar{z})^{\eta_2}. \tag{5.6}$$

We shall call (η_1, η_2), the (Hölderian) triangle scaling exponent pair (TSE) of D_h.

The following lemma gives sufficient conditions to guarantee the property (5.6).

Lemma 5.1.1 *Let h be a convex function which is differentiable on* $\operatorname{rintdom} h$. *Suppose that the following two conditions are satisfied :*

(i) h is uniformly convex of order $p \ge 2$.

(ii) The gradient ∇h is Hölderian on $\operatorname{rintdom} h$ *with exponent $\nu \in (0,1]$:*

$$\|\nabla h(x_1) - \nabla h(x_2)\| \le a\|x_1 - x_2\|^{\nu},$$

for all $x_1, x_2 \in \operatorname{rintdom} h$, for some $a > 0$. Then the Bregman distance D_h associated to h verifies the Hölderian triangle scaling property with the exponent pair $\eta_1 = 1+\nu$, $\eta_2 = (1+\nu)/p$.

Proof. For $x, z, \bar{z} \in \operatorname{rintdom} h$, $\theta \in [0,1]$, by the mean value theorem, we can find $\xi \in [(1-\theta)x + \theta z, (1-\theta)x + \theta\bar{z}]$ such that

$$\begin{aligned}
&D_h\left((1-\theta)x + \theta z, (1-\theta)x + \theta \bar{z}\right) \\
&= h((1-\theta)x + \theta\bar{z}) - h(1-\theta)x + \theta z) - \theta\langle \nabla h(1-\theta)x + \theta z), \bar{z} - z\rangle \\
&= \theta\langle \nabla h(\xi) - \nabla h(1-\theta)x + \theta z), \bar{z} - z\rangle.
\end{aligned}$$

Using (ii), it implies that

$$\begin{aligned}
D_h\left((1-\theta)x + \theta z, (1-\theta)x + \theta \bar{z}\right) &\le a\theta\|\langle \nabla h(\xi) - \nabla h(1-\theta)x + \theta z)\|\|z - \bar{z}\| \\
&\le a\theta^{1+\nu}\|z - \bar{z}\|^{1+\nu}.
\end{aligned}$$

On the other hand, as h is uniformly convex of order p, there is $\rho > 0$ such that $D_h(z,\bar{z}) \ge \rho\|z - \bar{z}\|^p$, so one has

$$D_h\left((1-\theta)x + \theta z, (1-\theta)x + \theta \bar{z}\right) \le M\theta^{1+\nu} D_h(z,\bar{z})^{(1+\nu)/p},$$

where $M := a/\rho^{1/p}$. □

The following elementary inequality will be used in the sequel.

Lemma 5.1.2 *For all three positive reals α, β, t, one has*

$$t^\beta + \frac{\alpha}{t} \geq \frac{\beta+1}{\beta^{\beta/(\beta+1)}}\alpha^{\beta/(\beta+1)}.$$

Proof. Considering the function $\varphi(t) := t^\beta + \frac{\alpha}{t}$, $t \in (0, +\infty)$, one has $\varphi'(t) = -\frac{\alpha}{t} + \beta t^{\beta-1}$, so for $\bar{t} = (\alpha/\beta)^{1/(\beta+1)}$, $\varphi'(\bar{t}) = 0$; $\varphi'(t) > 0$ for all $t \in (\bar{t}, +\infty)$, and $\varphi'(t) < 0$ for all $t \in (0, \bar{t})$. Hence, φ attains minimum at $\bar{t}$, that is,

$$t^\beta + \frac{\alpha}{t} \geq \frac{\beta+1}{\beta^{\beta/(\beta+1)}}\alpha^{\beta/(\beta+1)}, \quad \forall t > 0.$$

□

The following corollary is straightforward from the preceding lemma and the definition of the Hölderian triangle scaling property.

Corollary 5.1.1 *Assume that the Bregman distance D_h has the Hölderian triangle scaling property with respect to $\eta_1 \in (0, 2]$, $\eta_2 \in (0, 1]$, and $M > 0$. Then for all $x, z, \bar{z} \in \operatorname{rint}\operatorname{dom} h$, $\theta \in [0, 1]$, all $t \in]0, 1]$, one has*

$$D_h\left((1-\theta)x + \theta z, (1-\theta)x + \theta\bar{z}\right) \leq M\theta^{\eta_1}\left(\sigma_1 D_h(z, \bar{z})t^{-1} + \sigma_2 t^\beta\right), \tag{5.7}$$

where,

- *If $\eta_2 = 1$, then $\sigma_1 := 1$ and $\sigma_2 := \beta = 0$;*
- *otherwise $\eta_2 \in]0, 1[$,*

$$\sigma_1 = \sigma_2 := \frac{\beta^{\beta/(\beta+1)}}{\beta+1}, \quad \beta := \frac{\eta_2}{1-\eta_2}.$$

5.2 Generalized Nesterov's Algorithm and convergence rates

Let us consider the composition convex optimization problem of the form

$$\min\{f(x) + \Phi(x) : x \in \mathbb{R}^n\}, \tag{5.8}$$

where $\Phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper, lower-semicontinuous, and convex function and $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable, convex function which is L-smooth relative to

h on dom Φ, for some $L > 0$ and a stricltly convex function h.

Firstly, let us recall the following notion of support functions of a convex function at a point.

Definition 5.2.1 ([66]) *Let $\Phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function and z be a point in $\mathbb{R}^n$. A convex function $\Psi_z := \Psi_{z,\Phi} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be a lower support function to Φ at z if $\Psi_z \leq \Phi$ and $\Psi_z(z) = \Phi(z)$.*

Obviously, Φ and the linear function

$$\Psi_z(x) := \Phi(z) + \langle z^*, x - z \rangle, \quad x \in \mathbb{R}^n,$$

where $z^* \in \partial\Phi(z)$, the subdifferential of Φ at z, are two usual lower support functions of a convex Φ, at a point z

Throughout in this part, we make the following assumptions :

(A1) The optimal solution set of problem (5.8) is nonempty.

(A2) $\Phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper, lower-semicontinuous, and convex function.

(A3) $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable, convex function which is L-smooth relative to h on dom Φ, for some $L > 0$ and a stricltly convex function h.

(A4) The Bregman distance D_h has the Höderian relaxed triangle scaling property for some $M > 0$ and $\eta_1 \in (0, 2]$, $\eta_2 \in (0, 1]$, i.e., for all $x, z, \bar{z} \in$ rint dom h, and $\theta \in [0, 1]$,

$$D_h\left((1-\theta)x + \theta z, (1-\theta)x + \theta\bar{z}\right) \leq M\theta^{\eta_1} D_h(z, \bar{z})^{\eta_2}.$$

Pick parameters $C, \kappa, \mu > 0$ and three sequences of positive reals $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$ that verify the condition

$$A_k := \sum_{i=0}^{k} \alpha_k \geq B_k := \sum_{i=0}^{k} \beta_k, \text{for all } k \in \mathbb{N}.$$

The algorithm is stated in the following scheme.

Algorithm 1.

Initialization : $y_0 \in$ dom Φ. Define the functions $G_{-1}(x) = 0$;

$$F_0(x) = G_0(x) := CD_h(x, y_0) + \alpha_0[f(y_0) + \langle\nabla f(y_0), x - y_0\rangle + \Phi(x) + \mu\gamma_0 D_h(x, y_0)], \; x \in \mathbb{R}^n.$$

Main loop : For $k = 0, 1, ...$

1. Find

$$x_k = \operatorname{argmin}\{\Phi(x) + \langle \nabla f(y_k), x - y_k\rangle + \tfrac{1}{\kappa} D_h(x, y_k) : x \in \mathbb{R}^n\} \tag{5.9}$$

2. Find

$$z_k = \operatorname{argmin}_{x\in\mathbb{R}^n} F_k(x).$$

3. Set Ψ_{z_k} is a lower support function to Φ at z_k such that

$$\min_{x\in\mathbb{R}^n} F_k(x) = \min_{x\in\mathbb{R}^n}\{G_{k-1}(x) + \alpha_k[f(y_k) + \langle \nabla f(y_k), x - y_k\rangle + \Psi_{z_k}(x) + \mu\gamma_i D_h(x, y_k)]\}.$$

4. Set

$$\tau_k = \frac{\alpha_{k+1}}{A_{k+1} - B_k}, \quad y_{k+1} = \tau_k z_k + (1 - \tau_k)x_k.$$

5. Set

$$G_k(x) = G_{k-1}(x) + \alpha_k[f(y_k) + \langle \nabla f(y_k), x - y_k\rangle + \Psi_{z_k}(x) + \mu\gamma_k D_h(x, y_k)], \quad x \in \mathbb{R}^n;$$

$$F_{k+1}(x) = G_k(x) + \alpha_k[f(y_k) + \langle \nabla f(y_k), x - y_k\rangle + \Phi(x) + \mu\gamma_k D_h(x, y_k)], \quad x \in \mathbb{R}^n.$$

By the definition of the functions F_k, G_k in the algorithm, one has for all $k \in \mathbb{N}$,

$$\begin{aligned} F_k(x) = CD_h(x, y_0) + \sum_{i=0}^{k-1} \alpha_i[f(y_i) + \langle \nabla f(y_i), x - y_i\rangle + \Psi_{z_i}(x) + \mu\gamma_i D_h(x, y_i)] \\ + \alpha_k[f(y_k) + \langle \nabla f(y_k), x - y_k\rangle + \Phi(x) + \mu\gamma_k D_h(x, y_k)], x \in \mathbb{R}^n, \end{aligned} \tag{5.10}$$

$$G_k(x) = CD_h(x, y_0) + \sum_{i=0}^{k} \alpha_i[f(y_i) + \langle \nabla f(y_i), x - y_i\rangle + \Psi_{z_i}(x) + \mu\gamma_i D_h(x, y_i)]. \tag{5.11}$$

Remark 5.2.1 *Obviously, the usual two ways to pick the lower support function to* Φ *at* z_k *which fulfills Step 3 of Algorithm 1 are either* $\Psi_{z_k} := \Phi$*, or*

$$\Psi_{z_k}(x) := \langle z_k^*, x - z_k \rangle + \Phi(z_k), \quad x \in \mathbb{R}^n, \tag{5.12}$$

where $z_k^* \in \partial\Phi(z_k)$ *such that*

$$\partial G_{k-1}(z_k) + \alpha_k[\nabla f(y_k) + z_k^*] + \mu\gamma_k[\nabla h(z_k) - \nabla h(y_k)] = 0. \tag{5.13}$$

The existence of such z_k^* *is guaranteed from Step 2 of the algorithm.*

This remark allows us to derive the two useful variants of Algorithm 1 as follows. Firstly, for all $k \in \mathbb{N}$, one takes $\Psi_{z_k} := \Phi$, then Algorithm 1 gives a Bregman accelerated dual averaging algorithm which generalizes the one by Nesterov ([63, 64]). Secondly, consider for all iterations $k \in \mathbb{N}$, Ψ_{z_k} is defined by (5.12). At the current iteration k, for the next iteration $k+1$, there is some $z_{k+1}^* \in \partial\Phi(z_{k+1})$ such that

$$\partial G_k(z_{k+1}) + \alpha_{k+1}[\nabla f(y_{k+1}) + z_{k+1}^*] + \mu\gamma_{k+1}[\nabla h(z_{k+1}) - \nabla h(y_{k+1})] = 0. \tag{5.14}$$

Since

$$\begin{aligned} \partial G_k(z_{k+1}) &= \partial G_{k-1}(z_{k+1}) + \alpha_k[\nabla f(y_k) + z_k^*] + \mu\gamma_k[\nabla h(z_{k+1}) - \nabla h(y_k)] \\ &= \partial G_{k-1}(z_k) + \left(C + \mu\sum_{i=0}^{k-1}\gamma_i\right)[\nabla h(z_{k+1}) - \nabla h(z_k)] \\ &+ \alpha_k[\nabla f(y_k) + z_k^*] + \mu\gamma_k[\nabla h(z_{k+1}) - \nabla h(y_k)] \end{aligned},$$

Relations (5.13), (5.14) imply

$$\alpha_{k+1}[\nabla f(y_{k+1}) + z_{k+1}^*] + C_k[\nabla h(z_{k+1}) - \nabla h(z_k)] + \mu\gamma_{k+1}[\nabla h(z_{k+1}) - \nabla h(y_{k+1})] = 0,$$

where $C_k := C + \mu\sum_{i=0}^{k-1}\gamma_i$, $k \in \mathbb{N}$. Obviously, this relation is equivalent to

$$z_{k+1} = \operatorname{argmin}_{x\in\mathbb{R}^n}\left\{\Phi(x) + \langle\nabla f(y_{k+1}), x\rangle + \frac{1}{\alpha_{k+1}}[C_k D_h(x, z_k) + \mu\gamma_{k+1}D_h(x, y_{k+1})]\right\}. \tag{5.15}$$

So one obtains the following algorithm as a particular case of Algorithm 1 :

Algorithm 1.1

For $k = 0, 1, 2, ...,$ setting $C_0 := C$,

1.
$$x_k = \operatorname{argmin}\{\Phi(x) + \langle \nabla f(y_k), x - y_k\rangle + \tfrac{1}{\kappa} D_h(x, y_k) : x \in \mathbb{R}^n\};$$

2.
$$C_{k+1} = C_k + \mu\gamma_{k+1},$$
$$z_{k+1} = \operatorname{argmin}_{x\in\mathbb{R}^n}\left\{\varPhi(x) + \langle \nabla f(y_{k+1}), x\rangle + \frac{1}{\alpha_{k+1}}[C_k D_h(x, z_k) + \mu\gamma_{k+1} D_h(x, y_{k+1})]\right\}.$$

3.
$$\tau_k := \frac{\alpha_{k+1}}{A_{k+1} - B_k}, \quad y_{k+1} = \tau_k z_k + (1 - \tau_k) x_k.$$

In particular, when $\mu = 0$, the sequence $\{z_k\}$ defined recurrently in step 2 of Algorithm 1.1 is given simply as

---

$$z_{k+1} = \operatorname{argmin}_{x\in\mathbb{R}^n}\left\{\varPhi(x) + \langle \nabla f(y_{k+1}), x\rangle + \frac{C}{\alpha_{k+1}} D_h(x, z_k)\right\}. \tag{5.16}$$

The following proposition is useful in our analysis.

Proposition 5.2.1 *Let assumptions $(A1) - (A4)$ hold. Suppose that $\{z_k\}$ is the sequence defined by Algorithm 1. Then for any $k \in \mathbb{N}$, one has*

$$G_k(x) \geq \min_{x\in\mathbb{R}^n} G_k(x) + s_k D_h(x, z_k) \text{ for all } x \in \mathbb{R}^n,$$

where G_k is given by (5.11), and

$$s_k = C + \sum_{i=0}^{k} \alpha_i \mu \gamma_i. \tag{5.17}$$

Proof. Let Γ_k be a function on $\mathbb{R}^n$ given by

$$\Gamma_k(x) = G_k(x) - G_k(z_k) - s_k D_h(x, z_k).$$

By the definition of the Bregman distance, it yields

$$\Gamma_k(x) = C\langle \nabla h(z_k) - \nabla h(y_0), x - z_k\rangle + \sum_{i=0}^{k} \alpha_i(\Psi_{z_i}(x) - \Psi_{z_i}(z_k)) \tag{5.18}$$

$$+ \left\langle \sum_{i=0}^{k} \alpha_i(\nabla f(y_i) + \mu\gamma_i(\nabla h(z_k) - \nabla h(y_i)), x - z_k \right\rangle. \tag{5.19}$$

Since z_k is a minimizer of G_k, one deduces that there are $u_i \in \partial\Psi_{z_i}(z_k)$, such that

$$C(\nabla h(z_k) - \nabla h(y_0)) + \sum_{i=0}^{k} \alpha_i[\nabla f(y_i) + u_i + \mu\gamma_i (\nabla h(z_k) - \nabla h(y_i))] = 0. \tag{5.20}$$

Substituting (5.20) into (5.18) and rearranging the terms, we obtain

$$\Gamma_k(x) = \sum_{i=0}^{k} \alpha_i(\Psi_{z_i}(x) - \Psi_{z_i}(z_k) - \langle u_i, x - z_k\rangle). \tag{5.21}$$

As $\Psi_{z_i}, i = 0, \ldots, k$ are convex functions, we conclude that $\Gamma_k(x) \geq 0$ for all $x \in \mathbb{R}^n$. Consequently,

$$G_k(x) \geq \min_{x\in\mathbb{R}^n} G_k(x) + s_k D(x, z_k) \text{ for all } x \in \mathbb{R}^n.$$

■

Theorem 5.2.1 *Suppose that the assumptions* $(A1) - (A4)$ *hold. Let* (x_k) *and* (y_k) *be the sequences generated by Algorithm 1. With respect to* $\eta_2 \in]0.1]$, *define the quantities* $\sigma_1, \sigma_2, \beta$ *as follows.*

— *If* $\eta_2 = 1$, *then* $\sigma_1 := 1$ *and* $\sigma_2 := \beta = 0$;

— *otherwise* $\eta_2 \in]0, 1[$,

$$\sigma_1 = \sigma_2 := \frac{\beta^{\beta/(\beta+1)}}{\beta + 1}, \; \beta := \frac{\eta_2}{1 - \eta_2}.$$

Suppose that $0 < \kappa < 1/L$ *and the sequences* $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$ *verify the following condition for a sequence of positive reals* $\{\varepsilon_k\}$ *with* $\varepsilon_k \in]0, 1]$, *and* $\varepsilon_0 < M\sigma_1$,

$$\left(C + \mu\sum_{i=0}^{k-1} \alpha_i\gamma_i\right) (A_k - B_{k-1})^{\eta_1 - 1} \geq M\sigma_1\kappa^{-1}\alpha_k^{\eta_1}\varepsilon_k^{-1}, \text{ for all } k \in \mathbb{N}. \tag{5.22}$$

Then, for all $k \in \mathbb{N}$, one has

$$\sum_{i=0}^{k} \beta_i[f(x_i) + \Phi(x_i)] + (A_k - B_k)[f(x_k) + \Phi(x_k)]$$
$$+(\kappa^{-1} - L)\sum_{i=0}^{k}(A_i - B_{i-1})D_h(x_i, y_i) \leq \min_{x\in\mathbb{R}^n} F_k(x) + M\kappa^{-1}\sigma_2 \sum_{i=0}^{k-1} \tau_i^{\eta_1}\varepsilon_i^{\beta}(A_{i+1} - B_i). \tag{5.23}$$

Here, we set $B_{-1} = 0$. Moreover, if f is μ-strongly convex relative to h for some $0 < \mu \leq \kappa^{-1}$, then (5.23) holds if $\gamma_k = 1, k \in \mathbb{N}$ and

$$\left(C + \mu\sum_{i=0}^{k-1}\alpha_i\right)(A_k - B_{k-1})^{\eta_1 - 1} \geq M\sigma_1(\kappa^{-1} - \mu)\alpha_k^{\eta_1}\varepsilon_k^{-1}, \text{ for all } k \in \mathbb{N}. \tag{5.24}$$

Proof.
Now, we prove (5.23) by induction on $k \in \mathbb{N}$. For $k = 0$, one has

$$\begin{aligned}
&\min_{x\in\mathbb{R}^n} F_0(x)\\
&= \min\{CD_h(x, y_0) + \alpha_0[f(y_0) + \langle\nabla f(y_0), x - y_0\rangle + \Phi(x) + \mu\gamma_0 D_h(x, y_0)] : x \in \mathbb{R}^n\}\\
&\geq \alpha_0 \min\{(C + \alpha_0\mu\gamma_0)\alpha_0^{-1}D_h(x, y_0) + f(y_0) + \langle\nabla f(y_0), x - y_0\rangle + \Phi(x) : x \in \mathbb{R}^n\}\\
&\geq \alpha_0 \min\{\kappa^{-1}M^{-1}D_h(x, y_0) + f(y_0) + \langle\nabla f(y_0), x - y_0\rangle + \Phi(x) : x \in \mathbb{R}^n\}\\
&= \alpha_0\left[\kappa^{-1}M^{-1}D_h(x_0, y_0) + f(y_0) + \langle\nabla f(y_0), x_0 - y_0\rangle + \Phi(x_0)\right]\\
&\geq (\kappa^{-1} - L)\alpha_0 D_h(x_0, y_0) + \alpha_0\left[f(x_0) + \Phi(x_0)\right].
\end{aligned}$$

That is, (5.23) holds for $k = 0$. Assuming (5.23) to hold for some $k \in \mathbb{N}$, we will prove it for $k + 1$. In fact, one has for $x \in \mathbb{R}^n$,

$$F_{k+1}(x) = G_k(x) + \alpha_{k+1}[f(y_{k+1}) + \langle\nabla f(y_{k+1}), x - y_{k+1}\rangle + \Phi(x) + \mu\gamma_{k+1}D_h(x, y_{k+1})].$$

According to Proposition 5.2.1, we have

$$G_k(x) \geq \min_{x\in\mathbb{R}^n} G_k(x) + s_k D_h(x, z_k).$$

Hence, we immediately deduce that

$$F_{k+1}(x) \geq \min_{x\in\mathbb{R}^n} G_k(x) + s_k D_h(x, z_k) + \alpha_{k+1}[f(y_{k+1}) + \langle \nabla f(y_{k+1}), x - y_{k+1}\rangle + \Phi(x) + \mu\gamma_{k+1} D_h(x, y_{k+1})].$$

Since

$$\min_{x\in\mathbb{R}^n} F_k(x) = \min_{x\in\mathbb{R}^n} G_k(x),$$

using the induction assumption, one has for $x \in \mathbb{R}^n$,

$$M\kappa^{-1}\sigma_2 \sum_{i=0}^{k-1} \tau_i^{\eta_1}\varepsilon_i^{\beta}(A_{i+1} - B_i) + F_{k+1}(x) \tag{5.25}$$

$$\geq \sum_{i=0}^{k} \beta_i[f(x_i) + \Phi(x_i)] + (A_k - B_k)[f(x_k) + \Phi(x_k)] + (\kappa^{-1} - L)\sum_{i=0}^{k}(A_i - B_{i-1})D_h(x_i, y_i)$$
$$+ s_k D(x, z_k) + \alpha_{k+1}[f(y_{k+1}) + \langle \nabla f(y_{k+1}), x - y_{k+1}\rangle + \Phi(x) + \mu\gamma_{k+1} D_h(x, y_{k+1})].$$

By the convexity of f and Φ, we have

$$f(x_k) \geq f(y_{k+1}) + \langle \nabla f(y_{k+1}), x_k - y_{k+1}\rangle, \tag{5.26}$$

and

$$(A_k - B_k)\Phi(x_k) + \alpha_{k+1}\Phi(x) \geq (A_{k+1} - B_k)\Phi(\tau_k x + (1 - \tau_k)x_k). \tag{5.27}$$

Hence, for all $x \in \mathbb{R}^n$,

$$(A_{k+1} - B_k)[f(x_k) + \Phi(x_k)] + s_k D_h(x, z_k) + \alpha_{k+1}[f(y_{k+1}) + \langle \nabla f(y_{k+1}), x - y_{k+1}\rangle + \Phi(x)]$$
$$\geq (A_{k+1} - B_k)[f(y_{k+1}) + s_k(A_{k+1} - B_k)^{-1}D_h(x, z_k) + \tau_k\langle \nabla f(y_{k+1}), x - z_k\rangle + \Phi(\tau_k x + (1 - \tau_k)x_k)].$$

In the view of (5.22), we have

$$s_k(A_{k+1} - B_k)^{-1} \geq M\sigma_1\varepsilon_k^{-1}\tau_k^{\eta_1}\kappa^{-1}.$$

By setting $y := \tau_k x + (1 - \tau_k)x_k$, due to the Höderian triangle scaling property of D_h, in view of Corollary 5.1.1, we have

$$\begin{aligned} D_h(y, y_{k+1}) &= D_h(\tau_k x + (1-\tau_k)x_k, \tau_k z_k + (1-\tau_k)x_k) \\ &\leq M\tau_k^{\eta_1}\left(\sigma_1 D_h(x, z_k)\varepsilon_k^{-1} + \sigma_2\varepsilon_k^{\beta}\right). \end{aligned}$$

Thus,

$$\kappa^{-1}D_h(y, y_{k+1}) \leq s_k(A_{k+1} - B_k)^{-1}D_h(x, z_k) + M\sigma_2\kappa^{-1}\tau_k^{\eta_1}\varepsilon_k^{\beta}.$$

Therefore, the previous relations imply

$$\begin{aligned} &(A_{k+1} - B_k)[M\sigma_2\kappa^{-1}\tau_k^{\eta_1}\varepsilon_k^{\beta} + f(x_k) + \Phi(x_k)] + s_k D_h(x, z_k) \\ &+ \alpha_{k+1}[f(y_{k+1}) + \langle \nabla f(y_{k+1}), x - y_{k+1}\rangle + \Phi(x)] \\ &\geq (A_{k+1} - B_k)[f(y_{k+1}) + \kappa^{-1}D_h(y, y_{k+1}) + \langle \nabla f(y_{k+1}), y - y_{k+1}\rangle + \Phi(y)]. \end{aligned} \tag{5.28}$$

By (5.9), we have

$$\begin{aligned} &(A_{k+1} - B_k)[f(y_{k+1}) + \kappa^{-1}D_h(y, y_{k+1}) + \langle \nabla f(y_{k+1}), y - y_{k+1}\rangle + \Phi(y)] \\ &\geq (A_{k+1} - B_k)[f(y_{k+1}) + \kappa^{-1}D_h(x_{k+1}, y_{k+1}) + \langle \nabla f(y_{k+1}), x_{k+1} - y_{k+1}\rangle + \Phi(x_{k+1})] \\ &\geq (A_{k+1} - B_k)[(\kappa^{-1} - L)D_h(x_{k+1}, y_{k+1}) + f(x_{k+1}) + \Phi(x_{k+1})]. \end{aligned} \tag{5.29}$$

Combining (5.25), (5.28) and (5.29) together, we get

$$\begin{aligned} &M\kappa^{-1}\sigma_2\sum_{i=0}^{k}\tau_i^{\eta_1}\varepsilon_i^{\beta}(A_{i+1} - B_i) + \min_{x\in\mathbb{R}^n} F_{k+1}(x) \\ &\geq \sum_{i=0}^{k}\beta_i[f(x_i) + \Phi(x_i)] + (\kappa^{-1} - L)\sum_{i=0}^{k}(A_i - B_{i-1})D_h(x_i, y_i) \\ &+ (A_{k+1} - B_k)[(\kappa^{-1} - L)D_h(x_{k+1}, y_{k+1}) + f(x_{k+1}) + \Phi(x_{k+1})] \\ &= \sum_{i=0}^{k+1}\beta_i[f(x_i) + \Phi(x_i)] + (A_{k+1} - B_{k+1})[f(x_{k+1}) + \Phi(x_{k+1})] \\ &+ (\kappa^{-1} - L)\sum_{i=0}^{k+1}(A_i - B_{i-1})D_h(x_i, y_i). \end{aligned}$$

That means (5.23) holds for $k+1$ and it completes the proof of the first part.

Suppose that now f is μ-strongly convex relative to h and $\gamma_k = 1$, $k \in \mathbb{N}$. The proof follows the same lines as the above. The different point is as follows. Instead of (5.26), by the strongly convex relative to h with the parameter μ,

$$f(x_k) \geq f(y_{k+1}) + \langle \nabla f(y_{k+1}), x_k - y_{k+1} \rangle + \mu D_h(x_k, y_{k+1}). \tag{5.30}$$

Since $D_h(\cdot,\cdot)$ is convex in its first argument, we have that

$$(A_k - B_k)D_h(x_k, y_{k+1}) + \alpha_{k+1} D_h(x, y_{k+1}) \geq (A_{k+1} - B_k)D_h(\tau_k x + (1-\tau_k)x_k, y_{k+1}). \tag{5.31}$$

By setting $y = \tau_k x + (1-\tau_k)x_k$, we write shortly

$$(A_{k+1} - B_k)D_h(x_k, y_{k+1}) + \alpha_{k+1} D_h(x, y_{k+1}) \geq (A_{k+1} - B_k)D_h(y, y_{k+1}). \tag{5.32}$$

Using this fact and the inequality (5.32), we obtain

$$\begin{aligned}
&(A_{k+1} - B_k)[f(x_k) + \Phi(x_k)] + s_k D_h(x, z_k)\\
&+ \alpha_{k+1}[f(y_{k+1}) + \langle \nabla f(y_{k+1}), x - y_{k+1} \rangle + \mu D_h(x, y_{k+1}) + \Phi(x)]\\
&\geq (A_{k+1} - B_k)[f(y_{k+1}) + s_k(A_{k+1} - B_k)^{-1} D_h(x, z_k) + \tau_k \langle \nabla f(y_{k+1}), x - z_k \rangle\\
&+ \Phi(\tau_k x + (1-\tau_k)x_k)] + \mu[(A_k - B_k)D_h(x_k, y_{k+1}) + \alpha_{k+1} D_h(x, y_{k+1})]\\
&\geq (A_{k+1} - B_k)[f(y_{k+1}) + s_k(A_{k+1} - B_k)^{-1} D_h(x, z_k) + \mu D_h(y, y_{k+1})\\
&+ \tau_k \langle \nabla f(y_{k+1}), x - z_k \rangle + \Phi(\tau_k x + (1-\tau_k)x_k)],
\end{aligned}$$

where $s_k := C + \mu \sum_{i=0}^{k} \alpha_i = C + \mu A_k$.
Once gain, in the view of (5.24) and Corollary 5.1.1, one has

$$\begin{aligned}
D_h(y, y_{k+1}) &= D_h(\tau_k x + (1-\tau_k)x_k, \tau_k z_k + (1-\tau_k)x_k)\\
&\leq M\tau_k^{\eta_1}\left(\sigma_1 D_h(x, z_k)\varepsilon_k^{-1} + \sigma_2 \varepsilon_k^{\beta}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&M\sigma_2 \kappa^{-1} \tau_k^{\eta_1} \varepsilon_k^{\beta} + s_k(A_{k+1} - B_k)^{-1} D_h(x, z_k) + \mu D_h(y, y_{k+1}) \\
&\geq (\kappa^{-1} - \mu) D_h(y, y_{k+1}) + \mu D_h(y, y_{k+1}) = \kappa^{-1} D_h(y, y_{k+1}).
\end{aligned} \tag{5.33}$$

Thus,

$$
\begin{aligned}
&M\sigma_2\kappa^{-1}\tau_k^{\eta_1}\varepsilon_k^{\beta}+(A_k-B_k)[f(x_k)+\Phi(x_k)]+s_kD_h(x,z_k)\\
&+\alpha_{k+1}[f(y_{k+1})+\langle\nabla f(y_{k+1}),x-y_{k+1}\rangle+\mu D_h(x,y_{k+1})+\Phi(x)]\\
&\geq(A_{k+1}-B_k)[f(y_{k+1})+\kappa^{-1}M^{-1}D_h(y,y_{k+1})+\langle\nabla f(y_{k+1}),y-y_{k+1}\rangle+\Phi(y)] \qquad (5.34)
\end{aligned}
$$

The remain of the proof is similar to the one in the first part, so we omit it. ■

From the preceding theorem, we will obtain the convergence rates of Algorithm 1 by picking sequences of parameter in special ways such that the assumptions of Theorem 5.2.1 are verified. Firstly, we consider the case where the Bregman distance verifies the triangle scaling property (5.5), that is assumption (A4) with $\eta_1=\gamma\in]0,2]$, $\eta_2=1, M=1$.

Theorem 5.2.2 *Assume that D_h satisfies the triangle scaling property (5.5) with $\gamma\in]0,2]$. In Algorithm 1, let us pick $C,\kappa>0$ such that $C\geq\kappa^{-1}\geq L$, and $\alpha_k=ak^{\gamma-1}$, $\beta_k=\dfrac{ak^{\gamma-1}}{2}$, $\mu=0$, where*

$$0<a\leq\frac{C\kappa}{(2\gamma)^{\gamma-1}}\quad \text{if}\quad \gamma\geq 1;\;\; 0<a\leq\frac{C\kappa(2^\gamma-1)^{\gamma-1}}{(2\gamma)^{\gamma-1}}\quad \text{if}\quad\gamma<1. \qquad (5.35)$$

Then, for a minimizer x^* of problem (5.8),

$$
\begin{aligned}
Ca^{-1}D_h(x^*,y_0)\geq&\frac{1}{2}\sum_{i=1}^{k}i^{\gamma-1}[f(x_i)+\Phi(x_i)-f(x^*)-\Phi(x^*)]\\
&+\frac{\delta}{2}k^\gamma[f(x_k)+\Phi(x_k)-f(x^*)-\Phi(x^*)],
\end{aligned}
$$

where, $\delta=\gamma^{-1}$ if $\gamma\geq1$; otherwise $\delta=(2^\gamma-1)\gamma^{-1}$. As a result,

$$\lim_{k\to\infty}\min_{i=\lfloor k/2\rfloor,\dots,k}k^\gamma[f(x_i)+\Phi(x_i)-f(x^*)-\Phi(x^*)]=0,$$

where $\lfloor k/2\rfloor$ stands for the integer part of $k/2$. Therefore, if $\{f(x_k)+\Phi(x_k)\}$ is a decreasing sequence, then

$$\lim_{k\to\infty}k^\gamma[f(x_k)+\Phi(x_k)-f(x^*)-\Phi(x^*)]=0.$$

Proof. Suppose that $\alpha_k=ak^{\gamma-1},\beta_k=\dfrac{1}{2}ak^{\gamma-1},\mu=0$, and $C\geq\kappa^{-1}\geq L$, where $a>0$ is choosen suitablely. By checking directly, we have for all $k \geq 0$

$$\begin{aligned} & C(A_k - B_{k-1})^{\gamma-1} - \alpha_k^\gamma/\kappa \\ & = C(\sum_{i=0}^{k} ai^{\gamma-1} - \frac{1}{2}\sum_{i=0}^{k-1} ai^{\gamma-1})^{\gamma-1} - a^\gamma k^{\gamma(\gamma-1)}/\kappa \\ & \geq C\left(\frac{1}{2}\sum_{i=1}^{k} ai^{\gamma-1}\right)^{\gamma-1} - a^\gamma k^{\gamma(\gamma-1)}/\kappa. \end{aligned}$$

For $\gamma \geq 1$, one has

$$\sum_{i=1}^{k} i^{\gamma-1} \geq \sum_{i=0}^{k-1} \int_i^{i+1} x^{\gamma-1} dx = \frac{1}{\gamma} k^\gamma; \tag{5.36}$$

otherwise $0 < \gamma < 1$,

$$\sum_{i=1}^{k} i^{\gamma-1} \geq \sum_{i=i}^{k+1} \int_i^{i+1} x^{\gamma-1} dx = \frac{1}{\gamma}((k+1)^\gamma - 1) \geq \frac{2^\gamma - 1}{\gamma} k^\gamma, \tag{5.37}$$

here the last inequality is due from the fact that for $\gamma < 1$, the function $\xi(t) := (1+t)^\gamma - t^\gamma$ is decreasing on $[0, 1]$, so for $k \geq 1$,

$$\xi(1/k) = (1 + 1/k)^\gamma - 1/k^\gamma \geq \xi(1) = 2^\gamma - 1.$$

Thus for a given in (5.35) (notice that $C \geq \kappa^{-1}$), one has

$$C\left(\frac{1}{2}\sum_{i=1}^{k} ai^{\gamma-1}\right)^{\gamma-1} \geq a^\gamma k^\gamma/\kappa \tag{5.38}$$

That is,

$$C(A_k - B_{k-1})^{\gamma-1} - \alpha_k^\gamma/\kappa \geq 0.$$

That means such α_k, β_k verify the condition (5.22) with $\varepsilon_k = 1$, for $k \in \mathbb{N}^*$. By setting $x = x^*$ in (5.23), then using the convexity of f and the definition of the support function Ψ_{z_i},

$$\min_{x \in \mathbb{R}^n} F_k(x) \leq F_k(x^*) \leq CD_h(x^*) + \sum_{i=0}^{k} \alpha_i [f(x^*) + \Phi(x^*)]. \tag{5.39}$$

Therefore, in view of relation (5.23) in the preceding,

$$CD_h(x^*, y_0) + \sum_{i=0}^{k} \alpha_i[f(x^*) + \Phi(x^*)] \geq \sum_{i=0}^{k} \beta_i[f(x_i) + \Phi(x_i)] + (A_k - B_k)[f(x_k) + \Phi(x_k)].$$

Equivalently,

$$\begin{aligned} CD_h(x^*, y_0) \geq &\sum_{i=0}^{k} \beta_i[f(x_i) + \Phi(x_i) - f(x^*) - \Phi(x^*)] \\ &+ (A_k - B_k)[f(x_k) + \Phi(x_k) - f(x^*) - \Phi(x^*)]. \end{aligned}$$

For $\alpha_k = ak^{\gamma-1}$ and $\beta_k = \dfrac{ak^{\gamma-1}}{2}$, by estimates (5.36), (5.37), one obtains

$$\begin{aligned} &\frac{a}{2}\sum_{i=1}^{k} i^{\gamma-1}[f(x_i) + \Phi(x_i) - f(x^*) - \Phi(x^*)] \\ &+ \frac{a\delta}{2} k^{\gamma}[f(x_k) + \Phi(x_k) - f(x^*) - \Phi(x^*)] \leq CD_h(x^*, y_0), \end{aligned}$$

where $\delta = 1/\gamma$, if $\gamma \geq 1$, and $\delta = (2^{\gamma} - 1)/\gamma$, otherwise. This relation implies

$$\sum_{i=0}^{\infty} i^{\gamma-1}[f(x_i) + \Phi(x_i) - f(x^*) - \Phi(x^*)] < +\infty.$$

Therefore,

$$\lim_{k\to\infty} \sum_{i=\lfloor k/2 \rfloor}^{k} i^{\gamma-1}[f(x_i) + \Phi(x_i) - f(x^*) - \Phi(x^*)] = 0.$$

One has

$$\begin{aligned} &\sum_{i=\lfloor k/2 \rfloor}^{k} i^{\gamma-1}[f(x_i) + \Phi(x_i) - f(x^*) - \Phi(x^*)] \\ &\geq \min_{i=\lfloor k/2 \rfloor,\ldots,k} k^{\gamma}[f(x_i) + \Phi(x_i) - f(x^*) - \Phi(x^*)] \sum_{i=[k/2]}^{k} \frac{i^{\gamma-1}}{k^{\gamma}} \\ &\geq \frac{1}{\gamma}\left(1 - \frac{1}{2^{\gamma}}\right) \min_{i=\lfloor k/2 \rfloor,\ldots,k} k^{\gamma}[f(x_i) + \Phi(x_i) - f(x^*) - \Phi(x^*)]. \end{aligned}$$

The last inequality holds thanks to Lemma 5.2.1. From that we deduce

$$\lim_{k\to\infty} \min_{i=[k/2],\ldots,k} k^{\gamma}[f(x_i) + \Phi(x_i) - f(x^*) - \Phi(x^*)] = 0.$$

■

Lemma 5.2.1 *For all $\gamma \in]0,2]$, one has*

$$\sum_{i=\lfloor k/2 \rfloor}^{k} \frac{i^{\gamma-1}}{k^\gamma} \geq \frac{1}{\gamma}\left(1 - \frac{1}{2^\gamma}\right),$$

where $\lfloor k/2 \rfloor$ stands for the integer part of $k/2$.

Proof. Firstly, for $\gamma \geq 1$,

$$\begin{aligned}\sum_{i=\lfloor k/2 \rfloor}^{k} \frac{i^{\gamma-1}}{k^\gamma} &\geq \frac{1}{k^\gamma}\sum_{i=\lfloor k/2 \rfloor-1}^{k-1}\int_i^{i+1} x^{\gamma-1}dx \\ &= \frac{1}{\gamma k^\gamma}\left[k^\gamma - (\lfloor k/2 \rfloor - 1)^\gamma\right] \\ &= \frac{1}{\gamma}\left[1 - \frac{(\lfloor k/2 \rfloor - 1)^\gamma}{k^\gamma}\right].\end{aligned}$$

For $k = 2m$, $m \in \mathbb{N}^*$,

$$\frac{1}{\gamma}\left[1 - \frac{(\lfloor k/2 \rfloor - 1)^\gamma}{k^\gamma}\right] = \frac{1}{\gamma}\left[1 - \left(\frac{1}{2} - \frac{1}{2n}\right)^\gamma\right] \geq \frac{1}{\gamma}\left(1 - \frac{1}{2^\gamma}\right).$$

For $k = 2m + 1$, then

$$\frac{1}{\gamma}\left[1 - \frac{(\lfloor k/2 \rfloor - 1)^\gamma}{k^\gamma}\right] = \frac{1}{\gamma}\left[1 - \left(\frac{n-1}{2n+1}\right)^\gamma\right] \geq \frac{1}{\gamma}\left(1 - \frac{1}{2^\gamma}\right).$$

In summary, for $k \geq 0$,

$$\sum_{i=\lfloor k/2 \rfloor}^{k} \frac{i^{\gamma-1}}{k^\gamma} \geq \frac{1}{\gamma}\left(1 - \frac{1}{2^\gamma}\right).$$

For $0 < \gamma < 1$,

$$\begin{aligned}\sum_{i=\lfloor k/2 \rfloor}^{k} \frac{i^{\gamma-1}}{k^\gamma} &\geq \frac{1}{k^\gamma}\sum_{i=\lfloor k/2 \rfloor}^{k}\int_i^{i+1} x^{\gamma-1}dx \\ &= \frac{1}{\gamma k^\gamma}\left[(k+1)^\gamma - (\lfloor k/2 \rfloor)^\gamma\right] \\ &= \frac{1}{\gamma}\left[\frac{(k+1)^\gamma}{k^\gamma} - \frac{(\lfloor k/2 \rfloor)^\gamma}{k^\gamma}\right].\end{aligned}$$

Similarly as above, we derive the desired inequality. ■

Let h be a strictly convex function defined on $\mathbb{R}^n$. A function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be p-uniformly convex ralative to h with parameter μ, for some $\mu \geq 0, p \geq 2$, or shortly called (μ, p)-uniformly convex if for all $x, y \in \mathbb{R}^n$, $\lambda \in [0,1]$ one has

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y) - \frac{2^{p/2}\mu}{p}\lambda(1-\lambda)D_h(x,y)^{\frac{p}{2}}. \tag{5.40}$$

For $p = 2$, the function φ is said to be strongly convex relative to h with parameter μ. Note that if φ is (μ, p)-uniformly convex, then for all $x, y \in \mathbb{R}^n$, all $x^* \in \partial\varphi(x)$,

$$\langle x^*, y - x\rangle \leq \varphi(y) - \varphi(x) - \frac{2^{p/2}\mu}{p}D_h(x^*, y_i)^{p/2}. \tag{5.41}$$

Theorem 5.2.3 *Assume that D_h satisfies the triangle scaling property (5.5) with $\gamma \in]0, 2]$. Let f be $(\mu, 2p/\gamma)$-uniformly convex relative to h with $p > \gamma, \mu > 0$. Let $0 < \kappa \leq L^{-1}$, and $C, m > 0$ such that*

$$m\mu\kappa \geq \begin{cases} 2^{\frac{2\gamma}{p-\gamma}}\frac{p^{\gamma-1}((2-\gamma)p+\gamma^2)}{(p-\gamma)^\gamma}, if\ \gamma < p < \frac{\gamma+\gamma^2}{\gamma-1}, \\ 2^{\gamma-1}\frac{p^{\gamma-1}((2-\gamma)p+\gamma^2)}{(p-\gamma)^\gamma}, if\ p \geq \frac{\gamma+\gamma^2}{\gamma-1}, \end{cases}$$

$$C \geq \begin{cases} 2^{\frac{2\gamma}{p-\gamma}}\kappa^{-1}\left(\frac{p}{p-\gamma}\right)^{\gamma-1}, if\ \gamma < p < \frac{\gamma+\gamma^2}{\gamma-1}, \\ \frac{p-\gamma}{(2-\gamma)p+\gamma^2}m\mu, if\ p \geq \frac{\gamma+\gamma^2}{\gamma-1}. \end{cases}$$

In Algorithm 1, pick $\alpha_k = k^{\frac{p+\gamma}{p-\gamma}}, \beta_k = 0, \gamma_0 = 0$ and $\gamma_k = mk^{-\gamma}$ for $k \geq 1$. Then, for a minimizer x^* of problem (5.8), one has

$$f(x_k) + \Phi(x_k) - f(x^*) - \Phi(x^*) \leq \frac{2p}{p-\gamma}\left(CD_h(x^*, y_0) + \frac{1}{2}(p/2)^{\frac{2}{p-\gamma}}m^{\frac{p}{p-\gamma}}\ln(k+1)\right)k^{-\frac{2p}{p-\gamma}},$$

for all $k \in \mathbb{N}$.

Proof. The proof is considered as a generalized of the one of Corollary 2 in [66]. Let us start with the inequalities

$$\sum_{i=1}^{k} i^\alpha \geq \sum_{i=0}^{k-1}\int_i^{i+1} x^\alpha dx = \frac{1}{\alpha+1}k^{\alpha+1}, \tag{5.42}$$

if $\alpha > 0$ and

$$\sum_{i=1}^{k} i^{\alpha} \geq \sum_{i=1}^{k} \int_{i}^{i+1} x^{\alpha} dx = \frac{1}{\alpha+1}[(k+1)^{\alpha+1} - 1], \tag{5.43}$$

if $-1 < \alpha \leq 0$. Thus, for $k \geq 1$, we have

$$\left(C + \mu \sum_{i=0}^{k-1} \alpha_i \gamma_i\right) A_k^{\gamma-1} \geq (\frac{p-\gamma}{2p})^{\gamma-1} k^{\frac{2p(\gamma-1)}{p-\gamma}} [C + \frac{p-\gamma}{(2-\gamma)p+\gamma^2} m\mu (k-1)^{\frac{(2-\gamma)p+\gamma^2}{p-\gamma}}],$$

if $\gamma < p < \frac{\gamma+\gamma^2}{\gamma-1}$, and if $p \geq \frac{\gamma+\gamma^2}{\gamma-1}$,

$$\left(C + \mu \sum_{i=0}^{k-1} \alpha_i \gamma_i\right) A_k^{\gamma-1} \geq \left(\frac{p-\gamma}{2p}\right)^{\gamma-1} k^{\frac{2p(\gamma-1)}{p-\gamma}} [C + \frac{p-\gamma}{(2-\gamma)p+\gamma^2} m\mu (k^{\frac{(2-\gamma)p+\gamma^2}{p-\gamma}} - 1)].$$

In the view of these two inequalities, we will show the valid of (5.22) in Theorem 5.2.1. That means

$$(\frac{p-\gamma}{2p})^{\gamma-1} k^{\frac{2p(\gamma-1)}{p-\gamma}} [C + \frac{p-\gamma}{(2-\gamma)p+\gamma^2} m\mu (k-1)^{\frac{(2-\gamma)p+\gamma^2}{p-\gamma}}] \geq \kappa^{-1} k^{\frac{(p+\gamma)\gamma}{p-\gamma}} \tag{5.44}$$

and

$$\left(\frac{p-\gamma}{2p}\right)^{\gamma-1} k^{\frac{2p(\gamma-1)}{p-\gamma}} [C + \frac{p-\gamma}{(2-\gamma)p+\gamma^2} m\mu (k^{\frac{(2-\gamma)p+\gamma^2}{p-\gamma}} - 1)] \geq \kappa^{-1} k^{\frac{(p+\gamma)\gamma}{p-\gamma}}. \tag{5.45}$$

In fact, the inequality (5.44) can be written as

$$(\frac{p-\gamma}{2p})^{\gamma-1} k^{-\frac{(2-\gamma)p+\gamma^2}{p-\gamma}} [C\kappa + \frac{p-\gamma}{(2-\gamma)p+\gamma^2} m\mu\kappa (k-1)^{\frac{(2-\gamma)p+\gamma^2}{p-\gamma}}] \geq 1. \tag{5.46}$$

Or equivalently,

$$C\kappa(\frac{p-\gamma}{2p})^{\gamma-1} k^{-\frac{(2-\gamma)p+\gamma^2}{p-\gamma}} + (\frac{p-\gamma}{2p})^{\gamma-1} \frac{p-\gamma}{(2-\gamma)p+\gamma^2} m\mu\kappa (1-k^{-1})^{\frac{(2-\gamma)p+\gamma^2}{p-\gamma}} \geq 1. \tag{5.47}$$

From the conditions on the parameters, we deduce that

$$C\kappa(\frac{p-\gamma}{2p})^{\gamma-1} \geq 2^{\frac{(2-\gamma)p+\gamma^2}{p-\gamma}-1},$$

$$(\frac{p-\gamma}{2p})^{\gamma-1} \frac{p-\gamma}{(2-\gamma)p+\gamma^2} m\mu\kappa \geq 2^{\frac{(2-\gamma)p+\gamma^2}{p-\gamma}-1}.$$

Thus,

$$C\rho\kappa(\frac{p-\gamma}{2p})^{\gamma-1}k^{-\frac{(2-\gamma)p+\gamma^2}{p-\gamma}} + (\frac{p-\gamma}{2p})^{\gamma-1}\frac{p-\gamma}{(2-\gamma)p+\gamma^2}m\mu\kappa(1-k^{-1})^{\frac{(2-\gamma)p+\gamma^2}{p-\gamma}}$$
$$\geq 2^{\frac{(2-\gamma)p+\gamma^2}{p-\gamma}-1}[k^{-\frac{(2-\gamma)p+\gamma^2}{p-\gamma}} + (1-k^{-1})^{\frac{(2-\gamma)p+\gamma^2}{p-\gamma}}] \geq 1.$$

Now we will deal with (5.45). It can be rewritten as

$$[C\kappa - \frac{p-\gamma}{(2-\gamma)p+\gamma^2}m\mu\kappa](\frac{p-\gamma}{2p})^{\gamma-1}k^{-\frac{(2-\gamma)p+\gamma^2}{p-\gamma}} + (\frac{p-\gamma}{2p})^{\gamma-1}\frac{p-\gamma}{(2-\gamma)p+\gamma^2}m\mu\kappa \geq 1. \tag{5.48}$$

The validation of (5.48) is guaranteed by the conditions on parameters.
For $k \geq 1$, let us define

$$J_k := \{i \in \{1,\dots,k\} : D_h(x^*, y_i) \leq \frac{1}{2}(mp/2)^{\frac{\gamma}{p-\gamma}} i^{\gamma-2-\frac{2\gamma}{p-\gamma}}\}.$$

Then

$$\sum_{i\in J_k}\alpha_i\gamma_i D_h(x^*, y_i) \leq \frac{1}{2}(p/2)^{\frac{\gamma}{p-\gamma}} m^{\frac{p}{p-\gamma}}\sum_{i=1}^{k} i^{\frac{p+\gamma}{p-\gamma}} i^{-\gamma} i^{\gamma-2-\frac{2\gamma}{p-\gamma}} \tag{5.49}$$

$$= \frac{1}{2}(p/2)^{\frac{\gamma}{p-\gamma}} m^{\frac{p}{p-\gamma}}\sum_{i=1}^{k} i^{-1} \tag{5.50}$$

$$\leq \frac{1}{2}(p/2)^{\frac{\gamma}{p-\gamma}} m^{\frac{p}{p-\gamma}}(\ln k + 1), \tag{5.51}$$

where the last inequality follows from the one

$$\sum_{i=1}^{k} i^{-1} \leq 1 + \sum_{i=2}^{k}\int_{i-1}^{i} x^{-1}dx = 1 + \ln k.$$

For $i \in \{1,\dots,k\}\setminus J_k$, then $D_h(x^*, y_i) > \frac{1}{2}(mp/2)^{\frac{\gamma}{p-\gamma}} i^{\gamma-2-\frac{2\gamma}{p-\gamma}}$, therefore

$$\frac{2^{p/\gamma}\alpha_i}{p}D_h(x^*, y_i)^{\frac{p}{\gamma}} = \frac{2^{p/\gamma}\alpha_i}{p}D_h(x^*, y_i)^{\frac{p}{\gamma}-1}D_h(x^*, y_i)$$
$$\geq \frac{2}{p}\alpha_i(mp/2)k^{-\gamma}D_h(x^*, y_i)$$
$$= \alpha_i\gamma_i D_h(x^*, y_i). \tag{5.52}$$

Returning to (5.23), by setting $x = x^*$, we can derive that

$$A_k[f(x_k) + \Phi(x_k)] \le \min_{x\in\mathbb{R}^n} F_k(x) \le F_k(x^*). \tag{5.53}$$

Moreover, we also have that

$$\begin{aligned}
F_k(x^*) &\le CD_h(x^*, y_0) + \sum_{i=0}^{k} \alpha_i[f(y_i) + \langle\nabla f(y_i), x^* - y_i\rangle + \Phi(x^*) + \mu\gamma_i D_h(x^*, y_i)] \\
&= CD_h(x^*, y_0) + \sum_{i\in J_k}^{k} \alpha_i\mu\gamma_i D_h(x^*, y_i) + \sum_{i\in J_k}^{k} \alpha_i[f(y_i) + \langle\nabla f(y_i), x^* - y_i\rangle + \Phi(x^*)] \\
&+ \sum_{i=0, i\notin J_k}^{k} \alpha_i[f(y_i) + \langle\nabla f(y_i), x^* - y_i\rangle + \Phi(x^*) + \mu\gamma_i D_h(x^*, y_i)] \\
&\le CD_h(x^*, y_0) + \frac{1}{2}(p/2)^{\frac{\gamma}{p-\gamma}} m^{\frac{p}{p-\gamma}}(\ln k + 1) + \sum_{i\in J_k}^{k} \alpha_i[f(x^*) + \Phi(x^*)] \\
&+ \sum_{i=0, i\notin J_k}^{k} \alpha_i[f(y_i) + \langle\nabla f(y_i), x^* - y_i\rangle + \Phi(x^*) + \frac{2^{p/\gamma}\mu}{p} D_h(x^*, y_i)^{p/\gamma}].
\end{aligned}$$

Since f is $(\mu, 2p/\gamma)$- uniformly convex relative to h,

$$\sum_{i=0, i\notin J_k}^{k} \alpha_i[f(y_i) + \langle\nabla f(y_i), x^* - y_i\rangle + \Phi(x^*) + \frac{2^{p/\gamma}\mu}{p} D_h(x^*, y_i)^{p/\gamma}] \le \sum_{i=0, i\notin J_k}^{k} \alpha_i[f(x^*) + \Phi(x^*)].$$

Thus,

$$\begin{aligned}
F_k(x^*) &\le CD_h(x^*, y_0) + \frac{1}{2}(p/2)^{\frac{\gamma}{p-\gamma}} m^{\frac{p}{p-\gamma}}(\ln k + 1) + \sum_{i\in J_k}^{k} \alpha_i[f(x^*) + \Phi(x^*)] \\
&+ \sum_{i=0, i\notin J_k}^{k} \alpha_i[f(y_i) + \langle\nabla f(y_i), x^* - y_i\rangle + \Phi(x^*) + \frac{2^{p/\gamma}\mu}{p} D_h(x^*, y_i)^{p/\gamma}] \\
&\le CD_h(x^*, y_0) + \frac{1}{2}(p/2)^{\frac{\gamma}{p-\gamma}} m^{\frac{p}{p-\gamma}}(\ln k + 1) + A_k[f(x^*) + \Phi(x^*)].
\end{aligned} \tag{5.54}$$

From (5.53) and (5.54), we have

$$A_k[f(x_k) + \Phi(x_k)] \le CD_h(x^*, y_0) + \frac{1}{2}(p/2)^{\frac{\gamma}{p-\gamma}} m^{\frac{p}{p-\gamma}}(\ln k + 1) + A_k[f(x^*) + \Phi(x^*)]. \tag{5.55}$$

Again, from (5.42), one has

$$A_k = \sum_{i=1}^{k} i^{\frac{p+\gamma}{p-\gamma}} \geq \frac{p-\gamma}{2p} k^{\frac{2p}{p-\gamma}}. \tag{5.56}$$

Combing (5.55) and (5.56), we obtain for all $k \in \mathbb{N}$

$$f(x_k) + \Phi(x_k) - f(x^*) - \Phi(x^*) \leq \frac{2p}{p-\gamma} \left(CD_h(x^*, y_0) + \frac{1}{2}(p/2)^{\frac{\gamma}{p-\gamma}} m^{\frac{p}{p-\gamma}} \ln(k+1) \right) k^{-\frac{2p}{p-\gamma}}.$$

■

We showed that with a suitable choice of parameters, the convergence rate for the function valued of this proposed algorithm is of order $o(1/k^{\gamma})$ when the objective function is convex. Moreover, for the case of (μ, p)-uniform convexity of the objective function, we attains an $\mathcal{O}(\ln k / k^{2p/(p-\gamma)})$ convergence rate for some $\mu > 0, p > \gamma$.

Next we consider the case where f is μ-strongly convex relative to h for $\mu > 0$. The following theorem for the linear convergance of Algorithm 1 in the case of strong convexity.

Theorem 5.2.4 *Assume that D_h satisfies the triangle scaling property (5.5) with $\gamma \in [1, 2]$. Let f be μ-strongly convex relative to h for some $0 < \mu < \kappa^{-1}$ and let $q, C > 0$ such that*

$$q = 1 + \frac{\mu}{\kappa^{-1} - \mu} \quad \text{and} \quad C \geq \frac{\mu}{q-1}. \tag{5.57}$$

Then the sequence $\{x_k\}$ generarted by Algorithm 1 with sequences $\alpha_k = q^k, \beta_k = 0$, and $\gamma_k = 1, k \in \mathbb{N}$ satisfies

$$f(x_k) + \Phi(x_k) - f(x^*) - \Phi(x^*) \leq \frac{q-1}{q^{k+1}-1} CD_h(x^*, y_0), \quad \text{for all } k \in \mathbb{N}. \tag{5.58}$$

Here, x^* is a minimizer of problem (5.8).

Proof. For $\alpha_k = q^k$ with $q > 1$ and $\beta_k = 0$, relation (5.24) becomes

$$\left(C + \mu \frac{q^k - 1}{q-1} \right) \left(\frac{q^{k+1}-1}{q-1} \right)^{\gamma-1} \geq q^{\gamma k}(\kappa^{-1} - \mu), \forall k \in \mathbb{N}.$$

Equivalently,

$$\left(C(q-1) + \mu(q^k - 1) \right) \left(q^{k+1} - 1 \right)^{\gamma-1} \geq q^{\gamma k}(\kappa^{-1} - \mu)(q-1)^{\gamma}, \forall k \in \mathbb{N}. \tag{5.59}$$

Since $C \geq \frac{\mu}{q-1}$, for all $k \geq 0$

$$\left(C(q-1)+\mu(q^k-1)\right)\left(q^{k+1}-1\right)^{\gamma-1} \geq \mu q^k\left(q^{k+1}-1\right)^{\gamma-1}.$$

Thus, (5.59) holds true if

$$\mu q^k\left(q^{k+1}-1\right)^{\gamma-1} \geq q^{\gamma k}(\kappa^{-1}-\mu)(q-1)^{\gamma}, \quad \forall k \in \mathbb{N}.$$

It can be simplified to

$$\left(q^{k+1}-1\right)^{\gamma-1} \geq q^{(\gamma-1)k}(q-1)^{\gamma-1}, \quad \forall k \in \mathbb{N}.$$

Due to $q > 1$, the previous condition reduces to

$$\left(q^{k+1}-1\right) \geq q^k(q-1), \quad \forall k \in \mathbb{N}.$$

The last inequation holds as $q = 1 + \frac{\mu}{\kappa^{-1}-\mu} > 1$.
According to Theorem 5.2.1, one deduces that

$$A_k[f(x_k)+\Phi(x_k)] \leq \min_{x\in\mathbb{R}^n} F_k(x).$$

Note that f is μ-strongly convex relative to h, for all $i = 1, \ldots, k$

$$f(y_i) + \langle \nabla f(y_i), x-y_i\rangle + \mu\gamma_i D_h(x,y_i) \leq f(x), \quad \forall x \in \mathbb{R}^n,$$

therefore, $F_k(x) \leq CD_h(x,y_0) + A_k[f(x)+\Phi(x)]$. This implies

$$A_k[f(x_k)+\Phi(x_k)] \leq \min_{x\in\mathbb{R}^n} F_k(x) \leq F_k(x^*) \leq CD_h(x^*,y_0) + A_k[f(x^*)+\Phi(x^*)].$$

That shows

$$f(x_k)+\Phi(x_k)-f(x^*)-\Phi(x^*) \leq \frac{q-1}{q^{k+1}-1}CD_h(x^*,y_0), \text{ for all } k \in \mathbb{N}.$$

■

We consider now the case when the Bregman distance D_h satisfies the Höderian triangle property with respect to the pair of parameters (η_1,η_2) with $\eta_1 \in]0,2]$, $\eta_2 \in]0,1]$, and $\eta_1+\eta_2 > 1$. In algorithm 1, set $\alpha_k = ak^{\gamma-1}$, $\beta_k = 0$, $k \in \mathbb{N}^*$, $\alpha_0 = \beta_0 = 0$, where

$\gamma = \eta_1 + \eta_2 - 1$ and $a > 0$ will be chosen such that condition (5.22) in Theorem 5.2.1 is satisfied with a suitable sequence $\{\varepsilon_k\}$. Define the parameters as in Theorem 5.2.1 :

$$\sigma_1 = \sigma_2 := \sigma = \frac{\beta^{\beta/(\beta+1)}}{\beta+1}, \ \beta := \frac{\eta_2}{1-\eta_2}.$$

By taking $\varepsilon_k = k^{\eta_2 - 1}$, (5.22) becomes

$$Ca^{\eta_1 - 1}\left[\sum_{i=1}^{k} i^{\gamma-1}\right]^{\eta_1 - 1} \geq M\sigma_1\kappa^{-1}a^{\eta_1}k^{(\gamma-1)\eta_1+1-\eta_2},$$

or equivalently,

$$C\left[\sum_{i=1}^{k} i^{\gamma-1}\right]^{\eta_1 - 1} \geq M\sigma_1\kappa^{-1}ak^{(\gamma-1)\eta_1+1-\eta_2} \tag{5.60}$$

By using the estimates (5.36), (5.37) in the proof of Theorem 5.2.2 (noticing that $\gamma(\eta_1 - 1) = (\gamma-1)\eta_1 + 1 - \eta_2$), the above inequality, so (5.22), holds provided that

$$0 < a \leq \frac{C\delta\kappa}{M\sigma}, \ \text{where,} \ \delta = 1/\gamma^{\gamma-1}, \ \text{if} \ \gamma \geq 1, \ \delta = (2^\gamma - 1)^{\gamma-1}/\gamma^{\gamma-1}, \ \text{otherwise.} \tag{5.61}$$

Then, relation (5.23) in Theorem 5.2.1 gives

$$(A_k - B_k)[f(x_k) + \Phi(x_k)] \leq \min_{x\in\mathbb{R}^n} F_k(x) + M\kappa^{-1}\sigma_2\sum_{i=0}^{k-1}\tau_i^{\eta_1}i^{\beta}A_{i+1},$$

which implies that for a minimizer x^* of problem (5.8),

$$A_k[f(x_k)+\Phi(x_k)-f(x^*)-\Phi(x^*)] \leq CD_h(x^*, y_0)+M\kappa^{-1}\sigma_2 a^{\eta_1}\sum_{i=0}^{k-1}(i+1)^{(\gamma-1)\eta_1}i^{\beta(\eta_2-1)}A_{i+1}^{1-\eta_1}.$$

As $A_k = \mathcal{O}(k^\gamma)$, then

$$\begin{aligned}\sum_{i=0}^{k-1}(i+1)^{(\gamma-1)\eta_1}i^{\beta(\eta_2-1)}A_{i+1}^{1-(\gamma-1)\eta_1} &= \sum_{i=1}^{k}\mathcal{O}\left(i^{(\gamma-1)\eta_1+\beta(\eta_2-1)+\gamma(1-\eta_1)}\right)\\ &= \sum_{i=1}^{k}\mathcal{O}\left(i^{-1}\right) = \mathcal{O}(\ln k),\end{aligned}$$

therefore the preceding inequality implies that there is some constant $\rho > 0$ such that for all $k \in \mathbb{N}^*$,

$$f(x_k) + \Phi(x_k) - f(x^*) - \Phi(x^*) \leq \frac{\rho(D_h(x^*, y_0) + \ln k)}{k^{\eta_1+\eta_2-1}}.$$

So one obtains the following convergence result.

Theorem 5.2.5 *Assume that D_h satisfies the Höderian relaxed triangle scaling property (A4) with respect to parameters $M > 0$, $\eta_1 \in]0, 2]$, $\eta_2 \in]0, 1[$ with $\eta_1 + \eta_2 > 1$. In Algorithm 1, let us pick $C, \kappa > 0$ such that $C \geq \kappa^{-1} \geq L$; and $\alpha_k = ak^{\gamma-1}$, $\beta_k = 0$, $\mu = 0$, with $0 < a \leq \frac{C\delta\kappa}{M\sigma}$, where*

$$\sigma = \frac{\beta^{\beta/(\beta+1)}}{\beta+1}; \;\; \delta = 1/\gamma^{\gamma-1}, \; \textit{if}\; \gamma \geq 1, \; \delta = (2^\gamma - 1)^{\gamma-1}/\gamma^{\gamma-1}, \; \textit{otherwise.} \tag{5.62}$$

Then, for a minimizer x^* of problem (5.8), there is $\rho > 0$ such that

$$f(x_k) + \Phi(x_k) - f(x^*) - \Phi(x^*) \leq \frac{\rho(D_h(x^*, y_0) + \ln k)}{k^{\eta_1+\eta_2-1}}. \tag{5.63}$$

5.3 Generalized accelerated forward-backward algorithm

In [29], the authors have studied the following accelerated forward-backward scheme for solving (5.8) :

$$\begin{cases} y_k = x_k + \dfrac{k-1}{k+\alpha-1}(x_k - x_{k-1}), \\ x_{k+1} = \text{prox}_{\kappa\phi}(y_k - \kappa\nabla f(y_k)), \end{cases} \tag{5.64}$$

in which $\alpha > 0, \kappa > 0$. It was shown that the convergence rate of the order $o(1/k^2)$ when $\alpha > 3$, $\kappa \leq 1/L$ and f is supposed to be convex differential whose gradient is L-Lipschitz on the whole space $\mathbb{R}^n$.

Recently, in [66], the authors have developed that scheme by introducing a strongly convex function h in their algorithm and obtain the convergence rate of order $o(1/k^2)$. In this section we consider to generalize the algorithm given in [66] when the squared norm is replaced by the Bregman distance, under the assumptions (B1), (B2), (B4) (identical to (A1), (A2), (A4)), while (A3) is replaced by a stronger one (B3) :

(B1) The optimal solution set of problem (5.8) is nonempty.

(B2) $\Phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper lower-semicontinuous, and convex function.

(B3) $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable, convex function which is L-smooth relative to h on $\mathbb{R}^n$, for some $L > 0$ and a stricltly convex function h.

(B4) The Bregman distance D_h has the Höderian relaxed triangle scaling property for some $M > 0$ and $\eta_1 \in (0, 2]$, $\eta_2 \in (0, 1]$, i.e., for all $x, z, \bar{z} \in \operatorname{rint} \operatorname{dom} h$, and $\theta \in [0, 1]$,

$$D_h\left((1-\theta)x + \theta z, (1-\theta)x + \theta \bar{z}\right) \leq M\theta^{\eta_1} D_h(z, \bar{z})^{\eta_2}.$$

Before we start, let us recall some conditions on choosing parameters. For any parameters $C, \kappa, \mu > 0$ and three sequences of positive reals $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$ that verify the condition

$$A_k := \sum_{i=0}^{k} \alpha_k \geq B_k := \sum_{i=0}^{k} \beta_k, \text{for all } k \in \mathbb{N}.$$

The algorithm is stated in the following scheme.

Algorithm 2.

Initialization : $x_0 = z_0 = y_0 \in \operatorname{dom} \Phi$. Set $k = 0$.

Main loop : For $k = 0, 1, \ldots$

1. Set
$$\tau_k = \frac{\alpha_k}{A_k - B_{k-1}}, y_k = \tau_k z_k + (1-\tau_k)x_k.$$

2. Find
$$x_{k+1} = \operatorname{argmin}\{\Phi(x) + \langle \nabla f(y_k), x - y_k\rangle + \frac{1}{\kappa} D_h(x, y_k) : x \in \mathbb{R}^n\}.$$

3. Find
$$z_{k+1} = \operatorname{argmin}_{x\in\mathbb{R}^n} \Big\{ C D_h(x, y_0) + \sum_{i=0}^{k} \alpha_i [-\frac{1}{\kappa}\langle \nabla h(x_{i+1}) - \nabla h(y_i), x - x_{i+1}\rangle + \mu\gamma_i D_h(x, y_i)] \Big\}.$$

Let us define the operator G_k which plays a key roles in the proof of the convergence result.

$$G_\kappa(y_k) = \frac{1}{\kappa}(y_k - \text{argmin}\{\Phi(x) + \langle \nabla f(y_k), x - y_k \rangle + \tfrac{1}{\kappa} D_h(x, y_k) : x \in \mathbb{R}^n\}).$$

The following property of the operator G_κ is useful in our argument.

Proposition 5.3.1 *Suppose that $h : \mathbb{R}^n \to \mathbb{R}$ is strictly convex. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a constinuous differentiable function which is L-smooth relative to h on $\mathbb{R}^n$ and $0 < \kappa \leq 1/L$. Then, for $\bar{y} = y - \kappa G_\kappa(y)$, one has the following inequality*

$$(f + \Phi)(x) \geq (f + \Phi)(\bar{y}) - \frac{1}{\kappa} D_h(\bar{y}, y) - \frac{1}{\kappa}\langle \nabla h(\bar{y}) - \nabla h(y), x - \bar{y}\rangle \tag{5.65}$$

is valid for all $x, y \in \mathbb{R}^n$.

Proof. We have

$$\bar{y} = y - \kappa G_\kappa(y) = \text{argmin}\{\Phi(x) + \langle \nabla f(y), x - y\rangle + \tfrac{1}{\kappa} D_h(x, y) : x \in \mathbb{R}^n\}.$$

Thus,

$$0 \in \nabla f(y) + \frac{1}{k}(\nabla h(\bar{y}) - \nabla h(y)) + \partial \Phi(\bar{y}).$$

Since f, Φ are convex we have

$$f(x) \geq f(y) + \langle \nabla f(y), x - y\rangle, \tag{5.66}$$

$$\Phi(x) \geq \Phi(\bar{y}) + \langle x - \bar{y}, \partial \Phi(\bar{y})\rangle. \tag{5.67}$$

Moreover, due to f is L-smooth relative to h, then

$$f(\bar{y}) \leq f(y) + \langle \nabla f(y), \bar{y} - y\rangle + L D_h(\bar{y}, y). \tag{5.68}$$

Summing the above inequalities yields

$$(f + \Phi)(x) - (f + \Phi)(\bar{y}) \geq \langle \nabla f(y) + \partial\Phi(\bar{y}), x - \bar{y}\rangle - L D_h(\bar{y}, y).$$

Equivalently,

$$(f + \Phi)(x) - (f + \Phi)(\bar{y}) \geq -\frac{1}{\kappa}\langle \nabla h(\bar{y}) - \nabla h(y), x - \bar{y}\rangle - L D_h(\bar{y}, y).$$

This completes the proof. ■

Now, we define the following functions $E_k, k \in \mathbb{N}$ which play as "estimating functions" for Algorithm 2 given by

$$E_k(x) = CD_h(x, y_0) + \sum_{i=0}^{k} \alpha_i[f(x_{i+1}) + \Phi(x_{i+1}) - \frac{1}{\kappa}\langle \nabla h(x_{i+1}) - \nabla h(y_i), x - x_{i+1}\rangle$$
$$- \frac{1}{\kappa} D_h(x_{i+1}, y_i) + \mu\gamma_i D_h(x, y_i)]. \tag{5.69}$$

Proposition 5.3.2 *Assume that assumptions (B1)-(B4) hold. Let E_k be function given by (5.69). Then, for any $k \in \mathbb{N}$, one has*

$$E_k(x) \geq \min_{x\in\mathbb{R}^n} E_k(x) + s_k D_h(x, z_{k+1}),$$

for $x \in \mathbb{R}^n$, where $z_{k+1} = \operatorname{argmin}_{x\in\mathbb{R}^n} E_k(x)$ and $s_k = C + \mu \sum_{i=0}^{k} \alpha_i\gamma_i$.

Proof. Set

$$\Gamma_k(x) := E_k(x) - s_k D_h(x, z_{k+1}) - E_k(z_{k+1}).$$

Let $\bar{x}_k$ be a minimizer of Γ_k. Without confusion, we write $\bar{x}$ instead of $\bar{x}_k$. Then,

$$\sum_{i=0}^{k} \alpha_i[-\kappa^{-1}\nabla h(x_{i+1}) + \kappa^{-1}\nabla h(y_i) + \mu\gamma_i \nabla h(\bar{x}) - \mu\gamma_i \nabla h(y_i)]$$
$$+C[\nabla h(\bar{x}) - \nabla h(y_0)] - s_k[\nabla h(\bar{x}) - \nabla h(z_{k+1})] = 0.$$

Hence,

$$\sum_{i=0}^{k} \alpha_i[-\kappa^{-1}\nabla h(x_{i+1}) + \kappa^{-1}\nabla h(y_i)] = -\sum_{i=0}^{k} \alpha_i(\mu\gamma_i \nabla h(\bar{x}) - \mu\gamma_i \nabla h(y_i))$$
$$-C[\nabla h(\bar{x}) - \nabla h(y_0)] + s_k[\nabla h(\bar{x}) - \nabla h(z_{k+1})]. \tag{5.70}$$

In the other hand,

$$
\begin{aligned}
\Gamma_k(\bar{x}) &= E_k(\bar{x}) - s_k D_h(\bar{x}, z_{k+1}) - E_k(z_{k+1}) \\
&= C(h(\bar{x}) - h(z_{k+1}) - \langle \nabla h(y_0), \bar{x} - z_{k+1}\rangle) + \sum_{i=0}^{k} \alpha_i[-\frac{1}{\kappa}\langle \nabla h(x_{i+1}) - \nabla h(y_i), \bar{x} - z_{k+1}\rangle] \\
&+ \sum_{i=0}^{k} \alpha_i \mu \gamma_i [h(\bar{x}) - h(z_{k+1}) - \langle \nabla h(y_i), \bar{x} - z_{k+1}\rangle] \\
&- s_k[h(\bar{x}) - h(z_{k+1}) - \langle \nabla h(z_{k+1}), \bar{x} - z_{k+1}\rangle]. \qquad (5.71)
\end{aligned}
$$

Substitute (5.70) into (5.71), it yields

$$\Gamma_k(x) \geq \Gamma_k(\bar{x}) = 0, \text{ for all } x \in \mathbb{R}^n.$$

Therefore, for all $x \in \mathbb{R}^n$, one has

$$E_k(x) \geq \min_{x \in \mathbb{R}^n} E_k(x) + s_k D_h(x, z_{k+1}),$$

■

Now we are at the position to state the main result in this part.

Theorem 5.3.1 *Suppose that the assumptions $(B1) - (B4)$ hold. Let (x_k) and (y_k) be the sequences generated by Algorithm 2. With respect to $\eta_2 \in]0,1]$, we define the quantities $\sigma_1, \sigma_2, \beta$ as follows.*

— *If $\eta_2 = 1$, then $\sigma_1 := 1$ and $\sigma_2 := \beta = 0$;*

— *otherwise $\eta_2 \in]0,1[$,*

$$\sigma_1 = \sigma_2 := \frac{\beta^{\beta/(\beta+1)}}{\beta+1}, \ \beta := \frac{\eta_2}{1-\eta_2}.$$

Suppose that $0 < \kappa < 1/L$ and the sequences $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$ verify the following condition for a sequence of positive reals $\{\varepsilon_k\}$ with $\varepsilon_k \in]0,1]$, and $\varepsilon_0 < M\sigma_1$,

$$\left(C + \mu \sum_{i=0}^{k-1} \alpha_i \gamma_i\right)(A_k - B_{k-1})^{\eta_1 - 1} \geq M\sigma_1 \kappa^{-1} \alpha_k^{\eta_1} \varepsilon_k^{-1}, \text{ for all } k \in \mathbb{N}. \qquad (5.72)$$

Then for all $k \in \mathbb{N}$,

$$\sum_{i=0}^{k} \beta_i[f(x_{i+1}) + \Phi(x_{i+1})] + (A_k - B_k)[f(x_{k+1}) + \Phi(x_{k+1})]$$
$$\leq \min_{x\in\mathbb{R}^n} E_k(x) + M\kappa^{-1}\sigma_2 \sum_{i=0}^{k} \tau_i^{\eta_1}\varepsilon_i^{\beta}(A_i - B_{i-1}), \tag{5.73}$$

for all $k \in \mathbb{N}$. *Here, we set* $B_{-1} = 0$. *Furthermore, if* f *is* μ*-strong convex relative to* h*, then the condition* (5.72) *holds if* $\gamma_k = 1, k \in \mathbb{N}$*, and the sequences* $\{\alpha_k\}, \{\beta_k\}$ *fulfill the condition*

$$\left(C + \mu \sum_{i=0}^{k-1} \alpha_i\gamma_i\right)(A_k - B_{k-1})^{\eta_1 - 1} \geq M\sigma_1(\kappa^{-1} - \mu)\alpha_k^{\eta_1}\varepsilon_k^{-1}, \text{ for all } k \in \mathbb{N}. \tag{5.74}$$

Proof. We will prove (5.73) by induction on $k \in \mathbb{N}$. For $k = 0$, one has

$$\begin{aligned}
E_0(x) &= CD_h(x, y_0) + \mu\alpha_0\gamma_0 D_h(x, y_0) \\
&\quad + \alpha_0[f(x_1) + \Phi(x_1) - \frac{1}{\kappa}\langle\nabla h(x_1) - \nabla h(y_0), x - x_1\rangle - \frac{1}{\kappa}D_h(x_1, y_0)] \\
&= (C + \mu\alpha_0\gamma_0)D_h(x, y_0) \\
&\quad + \alpha_0[f(x_1) + \Phi(x_1) - \frac{1}{\kappa}\langle\nabla h(x_1) - \nabla h(y_0), x - x_1\rangle - \frac{1}{\kappa}D_h(x_1, y_0)] \\
&\geq \alpha_0[f(x_1) + \Phi(x_1)] + \alpha_0\kappa^{-1}[D_h(x, y_0) - \langle\nabla h(x_1) - \nabla h(y_0), x - x_1\rangle - D_h(x_1, y_0)] \\
&= \alpha_0[f(x_1) + \Phi(x_1)] + \alpha_0\kappa^{-1}[h(x) - h(x_1) - \langle\nabla h(x_1), x - x_1\rangle] \\
&\geq \alpha_0[f(x_1) + \Phi(x_1)],
\end{aligned}$$

for all $x \in \mathbb{R}^n$. The last inequality holds since h is convex. That shows (5.73) holds for $k = 0$. Suppose that (5.73) is true for $k - 1 \in \mathbb{N}$, we will show that it holds for k as well. In fact, since $z_k = \operatorname{argmin}_{x\in\mathbb{R}^n} E_{k-1}(x)$, according to Proposition 5.3.2, one has

$$E_{k-1}(x) \geq \min_{x\in\mathbb{R}^n} E_{k-1}(x) + s_{k-1}D_h(x, z_k),$$

for $x \in \mathbb{R}^n$, where $s_{k-1} = C + \mu \sum_{i=0}^{k-1} \alpha_i \gamma_i$. This leads us

$$\begin{aligned}
&E_k(x) + M\kappa^{-1}\sigma_2 \sum_{i=0}^{k-1} \tau_i^{\eta_1} \varepsilon_i^{\beta}(A_i - B_{i-1}) \\
&= M\kappa^{-1}\sigma_2 \sum_{i=0}^{k-1} \tau_i^{\eta_1} \varepsilon_i^{\beta}(A_i - B_{i-1}) + E_{k-1}(x) \\
&+ \alpha_k[f(x_{k+1}) + \Phi(x_{k+1}) - \frac{1}{\kappa}\langle \nabla h(x_{k+1}) - \nabla h(y_k), x - x_{k+1}\rangle - \frac{1}{\kappa} D_h(x_{k+1}, y_k) + \mu\gamma_k D_h(x, y_k)] \\
&\geq \sum_{i=0}^{k-1} \beta_i[f(x_{i+1}) + \Phi(x_{i+1})] + (A_{k-1} - B_{k-1})[f(x_k) + \Phi(x_k)] + s_{k-1} D_h(x, z_k) \\
&+ \alpha_k[f(x_{k+1}) + \Phi(x_{k+1}) - \frac{1}{\kappa}\langle \nabla h(x_{k+1}) - \nabla h(y_k), x - x_{k+1}\rangle - \frac{1}{\kappa} D_h(x_{k+1}, y_k) + \mu\gamma_k D_h(x, y_k)].
\end{aligned}$$

In the view of Proposition 5.3.1, we have

$$(f + \Phi)(x_k) \geq (f + \Phi)(x_{k+1}) - \frac{1}{\kappa} D_h(x_{k+1}, y_k) - \frac{1}{\kappa}\langle \nabla h(x_{k+1}) - \nabla h(y_k), x_k - x_{k+1}\rangle.$$

Therefore,

$$\begin{aligned}
&M\kappa^{-1}\sigma_2 \sum_{i=0}^{k-1} \tau_i^{\eta_1} \varepsilon_i^{\beta}(A_i - B_{i-1}) + E_k(x) \\
&\geq \sum_{i=0}^{k} \beta_i[f(x_{i+1}) + \Phi(x_{i+1})] + (A_k - B_k)[f(x_{k+1}) + \Phi(x_{k+1})] + \omega_k(x),
\end{aligned}$$

where

$$\begin{aligned}
\omega_k(x) := {} & s_{k-1} D_h(x, z_k) - \frac{A_{k-1} - B_{k-1}}{\kappa} D_h(x_{k+1}, y_k) \\
& - \frac{A_{k-1} - B_{k-1}}{\kappa}\langle \nabla h(x_{k+1}) - \nabla h(y_k), x_k - x_{k+1}\rangle \\
& - \frac{\alpha_k}{\kappa}\langle \nabla h(x_{k+1}) - \nabla h(y_k), x - x_{k+1}\rangle - \frac{\alpha_k}{\kappa} D_h(x_{k+1}, y_k).
\end{aligned}$$

We shall show that $\omega_k(x) \geq -M\kappa^{-1}\tau_k^{\eta_1}\sigma_2\varepsilon_k^{\beta}(A_k - B_{k-1})$ for all $x \in \mathbb{R}^n$. Indeed, by the definition of τ_k, we have that

$$\begin{aligned}
(A_k - B_{k-1})^{-1}\omega_k(x) = {} & \frac{M s_{k-1}}{A_k - B_{k-1}} D_h(x, z_k) - \kappa^{-1} D_h(x_{k+1}, y_k) \\
& - \kappa^{-1}\langle \nabla h(x_{k+1}) - \nabla h(y_k), (1 - \tau_k)(x_k - x_{k+1}) + \tau_k(x - x_{k+1})\rangle.
\end{aligned}$$

From (5.72), we deduce that $\dfrac{s_{k-1}}{A_k - B_{k-1}} \geq M\sigma_1\varepsilon_k^{-1}\tau_k^{\eta_1}\kappa^{-1}$. Hence,

$$\begin{aligned}\kappa(A_k - B_{k-1})^{-1}\omega_k(x) \geq\ & M\sigma_1\varepsilon_k^{-1}\tau_k^{\eta_1}\kappa^{-1}D_h(x, z_k) - D_h(x_{k+1}, y_k) \\ & - \langle \nabla h(x_{k+1}) - \nabla h(y_k), (1-\tau_k)(x_k - x_{k+1}) + \tau_k(x - x_{k+1})\rangle.\end{aligned}$$

By setting $y = \tau_k x + (1-\tau_k)x_k$, and thanks to the triangle scaling property of D_h, we have

$$D_h(y, y_k) = D_h(\tau_k x + (1-\tau_k)x_k, \tau_k z_k + (1-\tau_k)x_k) \leq M\tau_k^{\eta_1}\left(\sigma_1 D_h(x, z_k)\varepsilon_k^{-1} + \sigma_2\varepsilon_k^{\beta}\right).$$

Thus,

$$\begin{aligned}\kappa(A_k - B_{k-1})^{-1}\omega_k(x) \geq\ & D_h(y, y_k) - D_h(x_{k+1}, y_k) \\ & - \langle \nabla h(x_{k+1}) - \nabla h(y_k), y - x_{k+1}\rangle - M\tau_k^{\eta_1}\sigma_2\varepsilon_k^{\beta}.\end{aligned}$$

Now, using the definition of D_h and the fact that h is convex, we deduce that

$$\begin{aligned}&\kappa(A_k - B_{k-1})^{-1}\omega_k(x) \\ &\geq h(y) - h(x_{k+1}) - \langle \nabla h(y_k), y - x_{k+1}\rangle - \langle \nabla h(x_{k+1}) - \nabla h(y_k), y - x_{k+1}\rangle - M\tau_k^{\eta_1}\sigma_2\varepsilon_k^{\beta} \\ &= h(y) - h(x_{k+1}) - \langle \nabla h(x_{k+1}), y - x_{k+1}\rangle - M\tau_k^{\eta_1}\sigma_2\varepsilon_k^{\beta} \\ &\geq -M\tau_k^{\eta_1}\sigma_2\varepsilon_k^{\beta}.\end{aligned}$$

From that, we can conclude that $\omega_k(x) \geq -M\kappa^{-1}\tau_k^{\eta_1}\sigma_2\varepsilon_k^{\beta}(A_k - B_{k-1})$ for all $x \in \mathbb{R}^n$. This implies

$$E_k(x) + M\kappa^{-1}\sigma_2\sum_{i=0}^{k}\tau_i^{\eta_1}\varepsilon_i^{\beta}(A_i - B_{i-1}) \geq \sum_{i=0}^{k}\beta_i[f(x_{i+1}) + \Phi(x_{i+1})] + (A_k - B_k)[f(x_{k+1}) + \Phi(x_{k+1})]$$

for all $x \in \mathbb{R}^n$. This completes the first part of the proof. The remain part is simimilar to the one in Theorem 5.2.1, so we omit here. ■

One again, from the preceding theorem, by picking sequences of parameter in special ways such that the assumptions of Theorem 5.3.1 are verified, we obtain the convergence rates of Algorithm 2. First of all, let us consider the case where the Bregman distance verifies the triangle scaling property (5.5), that is assumption (B4) with $\eta_1 = \gamma \in]0, 2]$, $\eta_2 = 1, M = 1$.

Theorem 5.3.2 *Suppose that D_h satisfies the triangle scaling property (5.5) with $\gamma \in]0, 2]$. In Algorithm 2, let us pick $\mu = 0$ and any two sequences of positive reals $\{\alpha_k\}$ and $\{\beta_k\}$*

with $\alpha_k \geq \beta_k$ *for* $k \in \mathbb{N}$ *and*

$$0 < \liminf_{k\to\infty} \frac{\beta_k}{k^{\gamma-1}} \leq \limsup_{k\to\infty} \frac{\alpha_k}{k^{\gamma-1}} < +\infty, \quad \limsup_{k\to\infty} \frac{\beta_k}{\alpha_k} < 1. \tag{5.75}$$

Then, there exists $C_0 > 0$ *satisfying the condition*

$$C_0(A_k - B_{k-1})^{\gamma-1} \geq \alpha_k^\gamma \kappa^{-1}, \forall k \in \mathbb{N} \tag{5.76}$$

and for any $C \geq C_0$*, the sequence* $\{x_k\}$ *defined by Algorithm 2 has the following property*

$$\sum_{i=0}^{\infty} \beta_i[f(x_{i+1}) + \Phi(x_{i+1}) - f(x^*) - \Phi(x^*)] < +\infty.$$

Proof. With the assumptions on $\{\alpha_k\}, \{\beta_k\}$ there are $0 < a_1 < a_2 < a_3 < a_4$ and $b_1, b_2, b_3, b_4 \in \mathbb{R}$ such that for k large enough, we have

$$a_1 k^{\gamma-1} + b_1 \leq \beta_k \leq a_2 k^{\gamma-1} + b_2, \quad a_3 k^{\gamma-1} + b_3 \leq \alpha_k \leq a_4 k^{\gamma-1} + b_4.$$

Hence, for k large enough,

$$(A_k - B_{k-1})^{\gamma-1} \geq \left(\sum_{i=0}^{k} (a_3 i^{\gamma-1} + b_3) - \sum_{i=0}^{k-1} (a_2 i^{\gamma-1} + b_2) \right)^{\gamma-1} = \mathcal{O}(k^{\gamma(\gamma-1)}),$$

and

$$\alpha_k^\gamma \leq (a_4 k^{\gamma-1} + b_4)^\gamma = \mathcal{O}(k^{\gamma(\gamma-1)}).$$

So there exists $C_0 > 0$ such that

$$C_0(A_k - B_{k-1})^{\gamma-1} \geq \alpha_k^\gamma \kappa^{-1}.$$

That means the condition (5.72) is satisfied for all $C \geq C_0$.
For x^* being a minimizer of problem (5.8), in the view of Proposition 5.3.1, one has

$$\begin{aligned} E_k(x^*) &= CD_h(x^*, y_0) \\ &\quad + \sum_{i=0}^{k} \alpha_i [f(x_{i+1}) + \Phi(x_{i+1}) - \frac{1}{\kappa}\langle \nabla h(x_{i+1}) - \nabla h(y_i), x^* - x_{i+1}\rangle - \frac{1}{\kappa} D_h(x_{i+1}, y_i)] \\ &\leq CD_h(x^*, y_0) + A_k[f(x^*) + \Phi(x^*)]. \end{aligned} \tag{5.77}$$

According to Theorem 5.3.1, we have

$$\sum_{i=0}^{k} \beta_i[f(x_{i+1}) + \Phi(x_{i+1})] + (A_k - B_k)[f(x_{k+1}) + \Phi(x_{k+1})] \leq E_k(x^*). \tag{5.78}$$

From (5.77) and (5.78), we obtain

$$\begin{aligned} CD_h(x^*, y_0) \geq & \sum_{i=0}^{k} \beta_i[f(x_{i+1}) + \Phi(x_{i+1}) - f(x^*) - \Phi(x^*)] \\ & + (A_k - B_k)[f(x_{k+1}) + \Phi(x_{k+1}) - f(x^*) - \Phi(x^*)]. \end{aligned}$$

Hence,

$$\sum_{i=0}^{\infty} \beta_i[f(x_{i+1}) + \Phi(x_{i+1}) - f(x^*) - \Phi(x^*)] < +\infty.$$

■

Let us return to Theorem 5.3.2. If we pick $\alpha_k = ak^{\gamma-1}$, $\beta_k = \frac{ak^{\gamma-1}}{2}$, which a is an appropriate positive real satifying the condition (5.75), then, by using the same argument as in the proof of Theorem 5.2.2, we obtain the following theorem.

Theorem 5.3.3 *Suppose that D_h satisfies the triangle scaling property (5.5) with $\gamma \in]0, 2]$. In Algorithm 2, let us pick $C, \kappa > 0$ such that $C \geq \kappa^{-1} \geq L$; and $\alpha_k = ak^{\gamma-1}$, $\beta_k = \frac{ak^{\gamma-1}}{2}$, $\mu = 0$, where*

$$0 < a \leq \frac{C\kappa}{(2\gamma)^{\gamma-1}} \quad \text{if} \quad \gamma \geq 1; \quad 0 < a \leq \frac{C\kappa(2^\gamma - 1)^{\gamma-1}}{(2\gamma)^{\gamma-1}} \quad \text{if} \quad \gamma < 1. \tag{5.79}$$

Then, for a minimizer x^* of problem (5.8),

$$\begin{aligned} Ca^{-1}D_h(x^*, y_0) \geq & \frac{1}{2}\sum_{i=1}^{k} i^{\gamma-1}[f(x_i) + \Phi(x_i) - f(x^*) - \Phi(x^*)] \\ & + \frac{\delta}{2}k^\gamma[f(x_k) + \Phi(x_k) - f(x^*) - \Phi(x^*)], \end{aligned}$$

where, $\delta = \gamma^{-1}$ if $\gamma \geq 1$; otherwise $\delta = (2^\gamma - 1)\gamma^{-1}$. As a result,

$$\lim_{k\to\infty} \min_{i=\lfloor k/2 \rfloor,\ldots,k} k^\gamma[f(x_i) + \Phi(x_i) - f(x^*) - \Phi(x^*)] = 0,$$

where $\lfloor k/2 \rfloor$ stands for the integer part of $k/2$. Therefore, if $\{f(x_k) + \Phi(x_k)\}$ is a decreasing *sequence, then*

$$\lim_{k\to\infty} k^{\gamma}[f(x_k)+\Phi(x_k)-f(x^*)-\Phi(x^*)]=0.$$

$$\lim_{k\to\infty} k^{\gamma} \min_{i=\lfloor k/2\rfloor,\ldots,k}[f(x_i)+\Phi(x_i)-f(x^*)-\Phi(x^*)]=0,$$

for x^* being a minimizer of problem (5.8).

Theorem 5.3.4 *Suppose that f is (μ,p)-uniformly convex with $\mu>0$, $p>2$. Then with the same conditions as in Theorem 5.2.3, for any sequence $\{x_k\}$ defined by Algorithm 2, one has*

$$f(x_k)+\Phi(x_k)-f(x^*)-\Phi(x^*)=\mathcal{O}(k^{-\frac{2p}{p-\gamma}}\ln k). \tag{5.80}$$

Theorem 5.3.5 *Let f be μ-strongly convex relative to h for some $0<\mu<\kappa^{-1}$ and let $q,C>0$ such that*

$$q=1+\frac{\mu}{\kappa^{-1}-\mu} \quad \text{and} \quad C\rho\geq\frac{\mu}{q-1}. \tag{5.81}$$

Then the sequence $\{x_k\}$ generated by Algorithm 2 with sequences $\alpha_k=q^k,\beta_k=0$, and $\gamma_k=1,k\in\mathbb{N}$ satisfies

$$f(x_k)+\Phi(x_k)-f(x^*)-\Phi(x^*)=\mathcal{O}(q^{-k}). \tag{5.82}$$

Here, x^* is a minimizer of problem (5.8).

The proof of Theorems 5.3.4 and 5.3.5 are similar to the ones of Theorems 5.2.3 and 5.2.4, respectively. In these proofs, we use the following proposition.

Proposition 5.3.3 *Suppose that $h:\mathbb{R}^n\to\mathbb{R}$ is strictly convex. Let $f:\mathbb{R}^n\to\mathbb{R}$ be a constinuous differentiable function which is L-smooth relative to h on $\mathbb{R}^n$ and $0<\kappa\leq 1/L$. Then, for $\bar{y}=y-\kappa G_\kappa(y)$, one has*

(i) If f is (μ,p)-uniformly convex, then for any $x,y\in\mathbb{R}^n$

$$(f+\Phi)(x)\geq(f+\Phi)(\bar{y})-\frac{1}{\kappa}D_h(\bar{y},y)-\frac{1}{\kappa}\langle\nabla h(\bar{y})-\nabla h(y),x-\bar{y}\rangle+\frac{2^{p/2}\mu}{p}D_h(x,y)^{\frac{p}{2}}. \tag{5.83}$$

(ii) If Φ is (μ,p)-uniformly convex, then for any $x,y\in\mathbb{R}^n$

$$(f+\Phi)(x)\geq(f+\Phi)(\bar{y})-\frac{1}{\kappa}D_h(\bar{y},y)-\frac{1}{\kappa}\langle\nabla h(\bar{y})-\nabla h(y),x-\bar{y}\rangle+\frac{2^{p/2}\mu}{p}D_h(x,\bar{y})^{\frac{p}{2}}. \tag{5.84}$$

Proof. The proof is the same as in Proposition 5.3.1, just a diffent point is as follows.

Instead of (5.66), (5.67), since f or Φ is (μ, p)-uniformly convex, we have

$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{2^{p/2}\mu}{p} D_h(x, y)^{\frac{p}{2}}, \tag{5.85}$$

$$\Phi(x) \geq \Phi(\bar{y}) + \langle x - \bar{y}, \partial\Phi(\bar{y}) \rangle + \frac{2^{p/2}\mu}{p} D_h(x, \bar{y})^{\frac{p}{2}}. \tag{5.86}$$

■

Before finishing this part, it is worth to considering the case when the Bregman distance D_h satisfies the Höderian triangle property with respect to the pair of parameters (η_1, η_2) with $\eta_1 \in]0, 2]$, $\eta_2 \in]0, 1]$, and $\eta_1 + \eta_2 > 1$. Similarly, in algorithm 2, we set $\alpha_k = ak^{\gamma-1}$, $\beta_k = 0$, $k \in \mathbb{N}^*$, $\alpha_0 = \beta_0 = 0$, where $\gamma = \eta_1 + \eta_2 - 1$ and $a > 0$ will be chosen such that condition (5.22) in Theorem 5.3.2 is satisfied with a suitable sequence $\{\varepsilon_k\}$. With the same argument as in Theorem 5.2.5's, we claim the following result.

Theorem 5.3.6 *Suppose that D_h satisfies the Höderian relaxed triangle scaling property (B4) with respect to parameters $M > 0$, $\eta_1 \in]0, 2]$, $\eta_2 \in]0, 1[$ with $\eta_1 + \eta_2 > 1$. In Algorithm 2, let us pick $C, \kappa > 0$ such that $C \geq \kappa^{-1} \geq L$, and $\alpha_k = ak^{\gamma-1}$, $\beta_k = 0$, $\mu = 0$, with $0 < a \leq \dfrac{C\delta\kappa}{M\sigma}$, where*

$$\sigma = \frac{\beta^{\beta/(\beta+1)}}{\beta+1}, \quad \delta = 1/\gamma^{\gamma-1}, \text{ if } \gamma \geq 1, \quad \delta = (2^\gamma - 1)^{\gamma-1}/\gamma^{\gamma-1}, \text{ otherwise.} \tag{5.87}$$

Then, for a minimizer x^* of problem (5.8), there is $\rho > 0$ such that

$$f(x_k) + \Phi(x_k) - f(x^*) - \Phi(x^*) \leq \frac{\rho(D_h(x^*, y_0) + \ln k)}{k^{\eta_1+\eta_2-1}}. \tag{5.88}$$

5.4 Numerical experiments

In this section, we start to apply our schemes to solve these poptimal problems and study numerical performce of them.

5.4.1 A simple problem

Let us consider the following problem

$$\min_{x\in\mathbb{R}^n} \{\frac{1}{2}\|Ax - b\|^2 + c^\top x + r\|x\|\}, \tag{5.89}$$

where $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ and $r > 0$. In the form of problem (5.8) we have $f(x) = \dfrac{1}{2}\|Ax - b\|^2 + c^\top x$ and $\Phi(x) = r\|x\|$. It is clear that f is convex and its gradient is Lipschitz. In our experiment, for example, we take $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, c = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, r = 0.001$. In order to apply our algorithms to solve this problem, we pick C, ρ, κ, μ and three positive sequences $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$ as in Corollary 5.2.2 and take $h = 1/2\|.\|^2$ as a referece function. We notice that if x^* is a minimizer of (5.89), then

$$0 \in A^\top(Ax^* - b) + c + \partial\Phi(x^*).$$

Thus, we compute $\|A^\top(Ax_k - b) + c + \partial\Phi(x_k)\|$ at each iteration to observe the convergence rate order of our algorithms. Though both Algorithm 1 and 2 have the same convergence

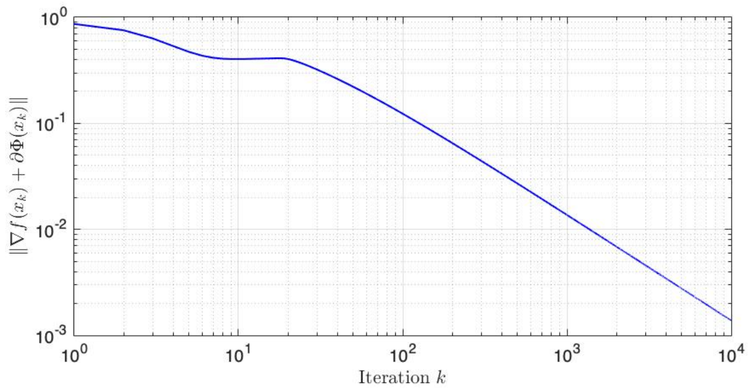


FIGURE 5.1 – $\|A^\top(Ax_k - b) + c + \partial\Phi(x_k)\|$ in k obtained by using Algorithm 1 in log-log plot.

order in theory, from Figure 5.1 and 5.2, we can see that Algorithm 2 gave better numerical result in this problem.

Now, we consider some applications of relatively smooth convex optimization : D-optimal experiment design, Poisson linear inverse problem, and relative-entropy nonnegative regression.

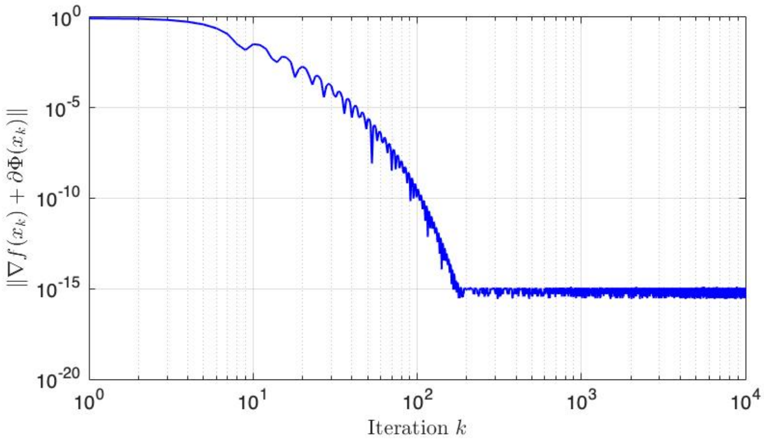


FIGURE 5.2 – $\|A^\top(Ax_k - b) + c + \partial\Phi(x_k)\|$ in k obtained by using Algorithm 2 in log-log plot.

5.4.2 D-optimal experiment design

Given n vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^m$ where $n \geq m+1$, the D-optimal design problem has the following form

$$\begin{aligned} \text{minimize} \quad & f(x) := -\log\det\Big(\sum_{i=1}^{n} x^{(i)} v_i v_i^\top\Big) \\ \text{subject to} \quad & \sum_{i=1}^{n} x^{(i)} = 1, \\ & x^{(i)} \geq 0, i = 1, \ldots, n. \end{aligned} \tag{5.90}$$

D-optimal designs are one form of design provided by a computer algorithm. These types of computer-aided designs are particularly useful when classical designs do not apply. It was shown in [56] that the function $f(x) = -\log\det\big(\sum_{i=1}^{n} x^{(i)} v_i v_i^\top\big)$ is 1-smooth relative to the Burg entropy $h(x) = -\sum_{i=1}^{n} \log(x^{(i)})$. In this case, D_h is the IS-distance. We notice that (5.90) can be reformulated as

$$\begin{aligned} \text{minimize} \quad & f(x) := -\log\det(HXH^\top) \\ \text{subject to} \quad & \langle e, x\rangle = 1, \\ & x \geq 0. \end{aligned} \tag{5.91}$$

Here, $X = \text{Diag}(x)$, $e = (1, 1, ..., 1)^\top \in \mathbb{R}^n$ and $H = [v_1, v_2, ..., v_n]$.
In order to apply our Algorithms to compupte the solution, we should solve the fol-

lowing subproblem

$$\begin{array}{ll} \text{minimize} & \langle c, x\rangle - \sum_{i=1}^{n} \log(x^{(i)}) \\ \text{subject to} & \langle e, x\rangle = 1, \\ & x \geq 0. \end{array} \tag{5.92}$$

The first-order optimality conditions read

$$\langle e, x\rangle = 1, x \geq 0, \text{ and } c - X^{-1}e = -\theta e$$

for some scalar multiplier θ. Given θ, it then follows that $x^{(i)} = 1/(c^{(i)} + \theta), i = 1, ..., n$. Notice that θ must satisfy

$$d(\theta) := \sum_{i=1}^{n} \frac{1}{c^{(i)} + \theta} - 1 = 0$$

for some $\theta \in (-\min_i\{c^{(i)}\}, +\infty)$. Since $d(\cdot)$ is strictly decreasing on $(-\min_i\{c^{(i)}\}, +\infty)$, $d(\theta) \to +\infty$ as $\theta \to -\min_i\{c^{(i)}\}^+$, and $d(\theta) \to -1$ as $\theta \to +\infty$. That ensure the existence and uniqueness of the solution of $d(\cdot)$ on $(-\min_i\{c^{(i)}\}, +\infty)$.

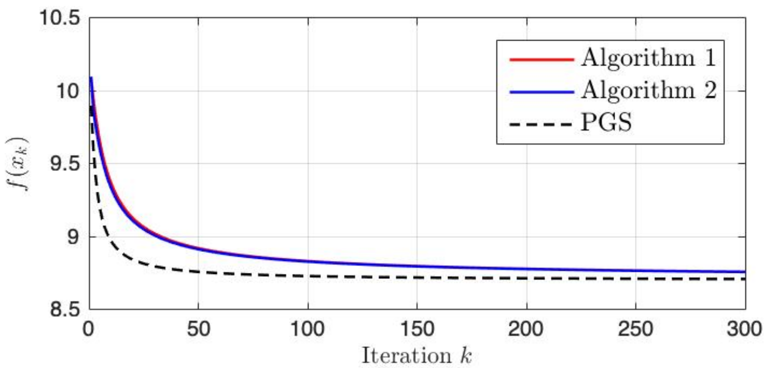


FIGURE 5.3 – Performance of three algorithms for a random D-optimal designs.

In our this experiment, we take $m = 80$ and $n = 200$ and generate n random vectors in $\mathbb{R}^m$, where the vector entries were generated using independent Gaussian distributions with a zero mean and unit variance. Figure 5.3 shows the comparison of different algorithms on another random problem. All algorithms we used converge. From this we can see that our two methods give equivalently numerical results while Primal Gradient Scheme (for short PGS) which was proposed in [56] gives a better numerical experiment.

To finish this section, we continue to consider two problems in which we can apply our algorithms to solve them numerically.

5.4.3 Poisson linear inverse problem

Given a nonnegative observation matrix $A \in \mathbb{R}_+^{m\times n}$ and a noisy measurement vector $b \in \mathbb{R}_{++}^m$. Our aim is to reconstruct the signal $x \in \mathbb{R}_+^n$ such that $Ax \approx b$. We consider problems of the form

$$\text{minimize } \{D_{KL}(b, Ax) + \Phi(x) : x \in \mathbb{R}_+^n\} \tag{5.93}$$

where $\Phi(x)$ is a simple regularization function, for example, we take $\Phi(x) = r\|x\|_1$, with $r = 0.001$. It was shown that $f(x) = D_{KL}(b, Ax)$ is L-smooth relative to $h(x) = -\sum_{i=1}^{n} \log(x^{(i)})$ on $\mathbb{R}_+^n$ for any $L \geq \|b\|_1$.

Let us denote by a_i the i-th row of matrix A for $i = 1, ..., m$. Then,

$$f(x) = D_{KL}(b, Ax) = \sum_{i=1}^{m} \left(b^{(i)} \log\left(\frac{b^{(i)}}{a_i x}\right) - b^{(i)} + a_i x \right).$$

Hence,

$$\nabla f(x) = \begin{pmatrix} \sum_{i=1}^{m} \left(-\frac{b^{(i)} a_i^{(1)}}{a_i x} + a_i^{(1)} \right) \\ \vdots \\ \sum_{i=1}^{m} \left(-\frac{b^{(i)} a_i^{(n)}}{a_i x} + a_i^{(n)} \right) \end{pmatrix}.$$

In our algorithms, we should find

$$x_{k+1} = \text{argmin}\{\Phi(x) + \langle \nabla f(y_k), x - y_k \rangle + \tfrac{1}{\kappa} D_h(x, y_k) : x \in \mathbb{R}_+^n\}.$$

The optimality condition leads us

$$re + \nabla f(y_k) + \tfrac{1}{\kappa}(\nabla h(x_{k+1}) - \nabla h(y_k)) = 0.$$

Then, we can compute x_{k+1} effectively. In our experiment, we take $m = 40$ and $n = 100$ servered as the size of the problem and generated n random vectors in $\mathbb{R}^m$, where the entries of the vectors were generated following independent Gaussian distributions with zero mean and unit variance. Using Matlab to implement our algorithms we obtain the numerical results as in Figure 5.4. Though Corollary 5.2.2 and 5.3.3 only give us general choices on three sequences $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$, the numerical performance of algorithms, *i.e.*, Algorithm 1 and 2, depends on them in certain problems.

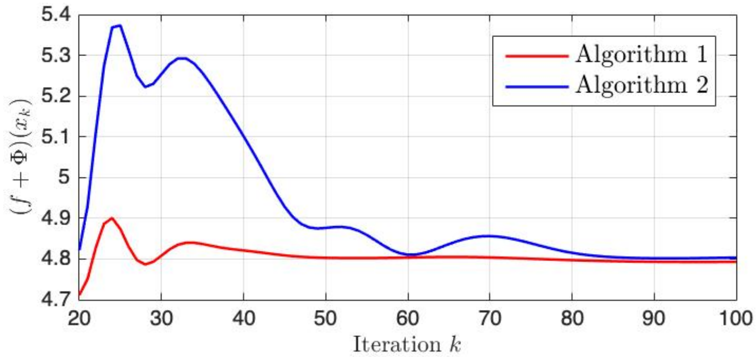


Figure 5.4 – Applying generalized Nesterov's accelerated Bregman proximal algorithms to solve Poisson linear inverse problem.

5.4.4 Relative-entropy nonnegative regression

An alternative approach to what was developed in the previous section consists in minimizing

$$\text{minimize } \{D_{KL}(Ax, b) + \Phi(x) : x \in \mathbb{R}^n_+\} \tag{5.94}$$

In [35], it was shown that $f(x) = D_{KL}(Ax, b)$ is L-smooth relative to $h(x) = \sum_{i=1}^{n} x^{(i)} \log(x^{(i)})$ on $\mathbb{R}_+$ for any $L \geq \max_{1\leq j\leq n} \sum_{i=1}^{m} A_{ij}$. Therefore, in our experiment, we use the KL-divergence D_{KL} as the proximity measure and apply l_1-regularization $\Phi(x) = \lambda\|x\|_1$ with $\lambda = 0.001$. Then, f reads

$$f(x) = D_{KL}(Ax, b) = \sum_{i=1}^{m}\{a_i^\top x \log(a_i^\top x) - (\log(b_i^\top x) + 1)a_i^\top x + b_i\}.$$

Hence,

$$\nabla f(x) = \begin{pmatrix} \sum_{i=1}^{m} a_i^{(1)} \log\left(\frac{a_i^\top x}{b_i}\right) \\ \vdots \\ \sum_{i=1}^{m} a_i^{(n)} \log\left(\frac{a_i^\top x}{b_i}\right) \end{pmatrix}.$$

Figure 5.5 shows the results for a randomly generated instance with $n = 100$, $m = 40$. This time, we vary the parameter μ while applying our methods, namely we took $\mu = 0, \mu = 1$ and $\mu = 10$. From this, it can be concluded that the choice of parameters also effects the numerical results. Figure 5.5b indicates that $\mu = 0$ is not a good choice in that instance problem.

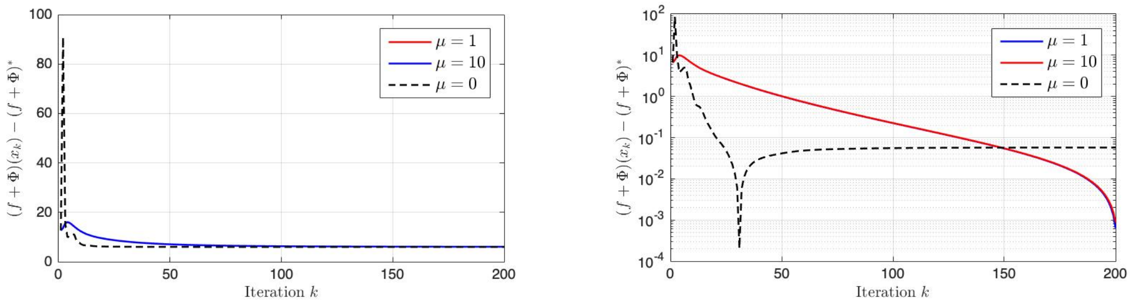


FIGURE 5.5 – One random example of relative entropy nonnegative regression.

5.5 Conclusion and perspectives

Until now, we have proposed the two generalized accelerated proximal gradient schemes in the framework of the Bregman distances for solving the composition convex problem (5.1). The convergence rate and some initial computational results demonstrate the efficiency of the proposed algorithms in the theoretical and computational aspects. However, we have not yet considered adaptive versions of the algorithms when the parameter of the relative smoothness and the triangle scaling exponents of the associated Bregman distance are not known priorly. This issue and the applications of the algorithms for solving large-scale problems in practice will be challenging topics in future works.

6

Conclusions & Perspectives

This thesis aimed to analyze inertial dynamics and associated algorithms for first-order optimization. Overall, we devoted ourselves to studying the continuous inertial dynamics, which involve dampings controlled by the Hessian of f and a Newton-type correction term attached to B.

The continuous dynamics part reported in chapters 2 - 4 dealt with two different second-order dynamics, known as (DINAM) and (iDINAM). For each model, we have shown the well-posedness of the solution and its weak convergence. These results relied on the Lyapunov analysis and the appropriate setting of damping parameters. The proofs and techniques are original due to the presence of the nonpotential term.

For the algorithmic part, we have proposed several brand-new algorithms that aim to find the zeros of an operator $A = \nabla f + B$, where ∇f is the gradient of a differentiable convex function f, and B is a nonpotential monotone and cocoercive operator. This part is a continuation and enhanced version of the continuous case. Our contribution is to combine these two aspects within the same algorithms and design inertial algorithms for structured monotone inclusions involving potential and nonpotential terms (skew-symmetric operators as a typical instance). As a sequel, this is fundamental for numerical reasons and modeling in engineering and decision sciences, whose processes involve cooperative and noncooperative aspects. Furthermore, our Lyapunov analysis emphasized the nonsymmetrical role played by the two operators. That is a significant step forward from preceding studies where we treated the two operators globally.

In parallel, Chapter 5 dealt with the two generalized accelerated proximal gradient schemes in the framework of the Bregman distances for solving the composition convex optimization problem (5.1). The convergence rate and some initial computational results demonstrate the efficiency of the proposed algorithms in the theoretical and computational aspects. We also made some numerical performances on our algorithms to solve some problems in practice.

Apart from what has been done in the thesis, we raise some open questions :

- It would be interesting to extend the analysis for both the continuous dynamic and its discretization to the case of an asymptotic vanishing damping $\gamma(t) = \dfrac{\alpha}{t}$ for both explicit and implicit models.

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \nabla f(x(t)) + B(x(t)) + \beta_f \nabla^2 f(x(t))\dot{x}(t) + \beta_b B'(x(t))\dot{x}(t) = 0.$$

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \nabla f\Big(x(t) + \beta_f \dot{x}(t)\Big) + B\Big(x(t) + \beta_b \dot{x}(t)\Big) = 0.$$

- Taking the coefficients $\beta_f(t)$ and $\beta_b(t)$ time-dependent could help to accelerate the convergence in both the discrete and continuous case.
- The limit dynamic when α goes to $+\infty$: $\alpha \mapsto x_\alpha(t)$.

All these questions will be the subject of a forthcoming research project.

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Publications

[1] Samir Adly, Hedy Attouch, Van Nam Vo : Asymptotic behavior of Newton-like inertial dynamics involving the sum of potential and nonpotential terms. *Fixed Point Theory Algorithms Sci Eng* **2021**, 17 (2021)

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Analyse des dynamiques inertielles et algorithmes associés pour l'optimisation du premier ordre

Résumé : Cette thèse est divisée en deux grandes parties. La première est consacrée à l'étude d'une classe d'algorithmes du premier ordre visant à résoudre des équations monotones structurées impliquant la somme de deux opérateurs : un opérateur potentiel ∇f (le gradient d'une fonction convexe différentiable f) et un autre non potentiel B (monotone et cocoercif). Le caractère bien posé et le comportement asymptotique des trajectoires des solution,s générées par la dynamique inertielle du second ordre impliquant ces deux opérateurs, sont analysés en détail. La discrétisation temporelle de ces dynamiques fournit des algorithmes de gradient proximal de type splitting ou décomposition. Leurs propriétés de convergence sont prouvées en utilisant l'analyse de Lyapunov. La seconde partie est dédiée à l'étude et à l'extension des algorithmes introduits par Nesterov dans le cas où f est relativement lisse. Une méthode, utilisant la distance de Bregman de la fonction à minimiser, est proposée. L'analyse de convergence des algorithmes associés est aussi étudiée et quelques simulations numériques sont proposées pour illustrer la partie théorique.

Mots clés : Algorithme de gradient proximal ; méthode inertielle ; amortissement piloté par le Hessien ; opérateur non potentiel ; opérateur cocoercif ; équation monotone structurée ; optimisation convexe ; algorithmes proximaux avec distance de Bregman.

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Analysis of inertial dynamics and associated algorithms for first-order optimization

Abstract : This thesis is divided into two main parts. The first one is devoted to the study of a class of first-order algorithms aiming at solving structured monotone equations involving the sum of two operators : a potential operator ∇f (the gradient of a differentiable convex function f) and a nonpotential one B (monotone and cocoercive). The well-posedness and the asymptotic behavior of the solution trajectories generated by the second-order inertial dynamics involving these two operators are analyzed in detail. The temporal discretization of these dynamics provides fully split proximal gradient algorithms. Their convergence properties are proved using Lyapunov analysis.
The second part is dedicated to the study and extension of the algorithms introduced by Nesterov in the case where f is relatively smooth. A method, using the Bregman distance of the function to be minimized, is proposed. The convergence analysis of the associated algorithms is also studied and some numerical simulations are proposed to illustrate the theoretical part.

Keywords : Proximal-gradient algorithm ; Inertial method ; Hessian-driven damping ; Nonpotential operator ; Cocoercive operator ; Structured monotone equation ; convex optimization ; Bregman proximal algorithms.

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